International Mathematical Forum, Vol. 9, 2014, no. 2, 51 - 63
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/imf.2014.311215

On p-Compact Sets in Classical Banach Spaces

Juan Manuel Delgado

Departamento de Matemática Aplicada I
Escuela Técnia Superior de Arquitectura
Avenida Reina Mercedes, 2
41012 Seville, Spain

Cándido Piñeiro

Departamento de Matemáticas
Facultad de Ciencias Experimentales
Campus Universitario del Carmen
21071 Huelva, Spain

Abstract

Given $p \geq 1$, we denote by $\mathcal{C}_p$ the class of all Banach spaces X satisfying the equality $\mathcal{K}_p(Y, X) = \Pi_p^d(Y, X)$ for every Banach space Y, $\mathcal{K}_p$ (respectively, Π_p^d) being the operator ideal of p-compact operators (respectively, of operators with p-summing adjoint). If X belongs to $\mathcal{C}_p$, a bounded set $A \subset X$ is relatively p-compact if and only if the evaluation map $U_A^*\colon X^* \longrightarrow \ell_\infty(A)$ is p-summing. We obtain p-compactness criteria valid for Banach spaces in $\mathcal{C}_p$.

Mathematics Subject Classification: 46B50, 47B07, 47B10

Keywords: p-compact set, p-nuclear operator, p-summing operator, (p, q)-summing sequence

1 Introduction

By a well known characterization due to Grothendieck [11], a subset A of a Banach space X is relatively compact if and only if there exists (x_n) in $c_0(X)$ (the space of norm-null sequences in X) such that $A \subset \{\sum_n a_n x_n : \sum_n |a_n| \leq 1\}$. Several authors have dealt with stronger forms of compactness studying sets sitting inside the convex hulls of special types of null sequences. For instance, it was observed in [20] (see also [5]) that if one considers, instead of $c_0(X)$, the space of q-summable sequences $\ell_q(X)$, for some fixed $q \geq 1$, then this stronger form of compactness characterizes the Reinov's approximation property of order p, $0 < p < 1$. This latter form of compactness was recently further strengthened by Sinha and Karn [21] as follows. Let $1 \leq p \leq \infty$ and let p' be the conjugate index of p (i.e., $1/p + 1/p' = 1$). The *p-convex hull* of a sequence $(x_n) \in \ell_p(X)$ is defined as $p\text{-co}\,(x_n) = \{\sum_n a_n x_n : \sum_n |a_n|^{p'} \leq 1\}$ ($\sup |a_n| \leq 1$ if $p = 1$). A set $A \subset X$ is said to be *relatively p-compact* if there exists $(x_n) \in \ell_p(X)$ ($(x_n) \in c_0(X)$ if $p = \infty$) such that $A \subset p\text{-co}\,(x_n)$. This nice notion has provoked the interest of several authors (see, for instance, [2], [6], [8] and [14]), whose contributions have made possible a deeper acknowledge of p-compactness in arbitrary Banach spaces. Anyway, there is no much information or examples of relative p-compact sets in concrete Banach spaces.

In [8], it is proved that a bounded subset A of an arbitrary Banach space X is relatively p-compact if and only if the corresponding evaluation map $U_A^*\colon x^* \in X^* \longmapsto (\langle x^*, a\rangle)_{a\in A} \in \ell_\infty(A)$ is p-nuclear ([8, Proposition 3.5]). However, for a wide class, say $\mathcal{C}_p$, of Banach spaces, the relatively p-compactness of any bounded set A occurs whenever U_A^* is just p-summing. For instance, reflexive spaces or separable dual spaces belong to $\mathcal{C}_p$ for all $p \geq 1$. In Section 2, a characterization of relatively p-compact sets in Banach spaces belonging to $\mathcal{C}_p$ is given; as an application, we obtain a characterization of p-compact sets in ℓ_1. Section 3 is devoted mainly to show some ways to produce relatively p-compact sets in Banach spaces not belonging to $\mathcal{C}_p$.

A Banach space X will be regarded as a subspace of its bidual X^{**} under the canonical embedding $i_X\colon X \to X^{**}$. We denote the closed unit ball of X by B_X. For Banach spaces X and Y, the Banach space of all bounded linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$. If $\mathcal{A}$ is a Banach ideal, then $\mathcal{A}^d$ denotes its dual ideal, that is, $\mathcal{A}^d(X, Y) = \{T \in \mathcal{L}(X, Y)\colon\ T^* \in \mathcal{A}(Y^*, X^*)\}$. We deal with the following operator ideals: $\mathcal{N}_p-$ p-nuclear operators, $\mathcal{QN}_p-$ quasi p-nuclear operators, $\mathcal{I}_p-$ p-integral operators and Π_p- p-summing operators. We refer to Pietsch's book [18] for operator ideals (see also [9] by Diestel, Jarchow, and Tonge for common operator ideals as $\mathcal{N}_p$ and Π_p, and [17] by Persson and Pietsch for $\mathcal{QN}_p$).

As usual, the space of all weakly p-summable sequences (respectively, p-

summable sequences) in X is denoted by $\ell_p^w(X)$ (respectively, $\ell_p(X)$) endowed with its norm

$$\|(x_n)\|_p^w = \sup_{x^*\in B_{X^*}} \left(\sum_n |\langle x^*, x_n\rangle|^p \right)^{1/p}.$$

$$\left(\text{respectively, } \|(x_n)\|_p = \left(\sum_n \|x_n\|^p \right)^{1/p} \right).$$

Relying on the notion of p-compactness, the notion of p-compact operator is defined in an obvious way (see [21]): an operator $T \in \mathcal{L}(X,Y)$ is said to be *p-compact* if $T(B_X)$ is relatively p-compact in Y. The space of all p-compact operators from X into Y is denoted by $\mathcal{K}_p(X,Y)$. It is shown in [21] that $\mathcal{K}_p$ is an operator ideal. We list some properties related to p-compactness:

- If $1 \le q \le p \le \infty$, every relatively q-compact set is relatively p-compact.
- An operator T belongs to $\mathcal{K}_p(X,Y)$ (respectively, $\mathcal{QN}_p(X,Y)$) if and only T^* belongs to $\mathcal{QN}_p(Y^*,X^*)$ (respectively, $\mathcal{K}_p(Y^*,X^*)$) [8, Corollary 3.4 and Proposition 3.8].

2 p-Compactness and p-summing evaluation maps

A bounded subset A of a Banach space X is relatively p-compact if and only if the corresponding evaluation map $U_A^*\colon x^* \in X^* \longmapsto (\langle x^*, a\rangle)_{a\in A} \in \ell_\infty(A)$ is (quasi) p-nuclear [8, Proposition 3.5]. Nevertheless, for a wide class of Banach spaces, the relative p-compactness of a set is characterized just by the p-summability of its evaluation map. For the time being, let us focus our attention on this type of spaces.

Definition 2.1. Let $1 \le p < \infty$. A Banach space X belongs to the class $\mathcal{C}_p$ if for every bounded subset A of X, A is relatively p-compact if and only if the evaluation map $U_A^*\colon x^* \in X^* \longmapsto (\langle x^*, a\rangle)_{a\in A} \in \ell_\infty(A)$ is p-summing.

Recall that $\mathcal{K}_p(Y,X) \subset \Pi_p^d(Y,X)$ [21, Proposition 5.3]. Related to this, the following are reformulations of the definition of the class $\mathcal{C}_p$.

Proposition 2.1. Let $1 \le p < \infty$. The following statements are equivalent for a Banach space X:

a) $X \in \mathcal{C}_p$.

b) $\mathcal{K}_p(Y,X) = \Pi_p^d(Y,X)$ for every Banach space Y.

c) $\mathcal{K}_p(\ell_1(\Gamma), X) = \Pi_p^d(\ell_1(\Gamma), X)$ for any set Γ.

d) $\mathcal{K}_p(\ell_1, X) = \Pi_p^d(\ell_1, X)$.

Proof. a)⇒b) For a given Banach space Y, consider $T \in \Pi_p^d(Y, X)$ and put $A := T(B_Y)$. Since $\|U_A^* x^*\|_\infty = \|T^* x^*\|$, we have that U_A^* is p-summing so, by hypothesis, $A = T(B_Y)$ is relatively p-compact.

b)⇒c) and c)⇒d) are obvious.

d)⇒a) Suppose $A \subset X$ is a bounded set such that U_A^* is p-summing. To see that A is relatively p-compact, it suffices to show that each countably subset of A is relatively p-compact. So consider $\{x_n\} \subset A$ and define $J\colon (\alpha_n) \in \ell_1 \longmapsto J(\alpha_n) \in \ell_1(A)$, where $J(\alpha_n)(x) = \alpha_n$ if $x = x_n$ and $J(\alpha_n)(x) = 0$ otherwise. From d), it follows that $U_A \circ J\colon \ell_1 \longrightarrow X$ is p-compact. Thus, $\{x_n\} = \{U_A \circ J(e_n)\}$ is relatively p-compact. □

Remark 2.2. Since $\ell_\infty(\Gamma)$ is an injective space, Π_p^d may be replaced with $\mathcal{I}_p^d$ in c) and d) of the above proposition ([9, Corollary 5.7]). In the same direction, $\mathcal{K}_p$ may be replaced with $\mathcal{N}_p^d$ in the mentioned statements since $\mathcal{K}_p(\ell_1(\Gamma), X) = \mathcal{N}_p^d(\ell_1(\Gamma), X)$ for every Banach space X ([8, Proposition 3.8] and [17, Theorem 38]). In particular, we have that X belongs to $\mathcal{C}_p$ if and only if $\mathcal{N}_p^d(\ell_1, X) = \mathcal{I}_p^d(\ell_1, X)$.

The preceding remark reveals that the equality $\mathcal{N}_p(Y, Z) = \mathcal{I}_p(Y, Z)$ becomes of great use to provide examples of Banach spaces belonging to $\mathcal{C}_p$.

Proposition 2.2. Let X be a Banach space and $1 \leq p < \infty$. Then

1. If X^{**} has the Radon–Nikodym property then $X \in \mathcal{C}_p$. In particular, every reflexive Banach space belongs to $\mathcal{C}_p$.

2. If $X^{**} \in \mathcal{C}_p$ then $X \in \mathcal{C}_p$.

3. $c_0, \ell_\infty \notin \mathcal{C}_p$.

4. If μ is a finite measure, then $L_1(\mu) \notin \mathcal{C}_p$.

Proof. According to [1, Proposition 1.1], we have that $\mathcal{N}_p(X^*, \ell_\infty(A)) = \mathcal{I}_p(X^*, \ell_\infty(A))$ whenever X^{**} has the Radon–Nykodim property.

To see 2, consider $A \subset X$ such that $U_A^* \in \Pi_p(X^*, \ell_\infty(A))$, that is,

$$\left(\sum_{n=1}^N |\langle x_n^*, x_n\rangle|^p\right)^{1/p} \leq \pi_p(U_A^*) \sup_{x\in B_X} \left(\sum_{n=1}^N |\langle x_n^*, x\rangle|^p\right)^{1/p} \tag{1}$$

for all finite subsets $\{x_1, \ldots, x_N\}$ in A and $\{x_1^*, \ldots, x_N^*\}$ in X^*. It suffices to show that $i_X(A)$ is relatively p-compact in X^{**} ([8, Corollary 3.6]). Given finite subsets $\{x_1, \dots, x_N\}$ in A and $\{x_1^{***}, \dots, x_N^{***}\}$ in X^{***}, we have from (1)

$$\left(\sum_{n=1}^{N} |\langle x_n^{***}, i_X(x_n)\rangle|^p\right)^{1/p} = \left(\sum_{n=1}^{N} |\langle i_X^*(x_n^{***}), x_n\rangle|^p\right)^{1/p}$$
$$\leq \pi_p(U_A^*) \sup_{x\in B_X} \left(\sum_{n=1}^{N} |\langle i_X^*(x_n^{***}), x\rangle|^p\right)^{1/p}$$
$$\leq \pi_p(U_A^*) \sup_{x^{**}\in B_X^{**}} \left(\sum_{n=1}^{N} |\langle x_n^{***}, x^{**}\rangle|^p\right)^{1/p}$$

It follows from the above reasoning that the evaluation map of $i_X(A)$ is p-summing and, by hypothesis, $i_X(A)$ is relatively p-compact in X^{**}.

Grothendieck's Theorem ensures that the natural embedding $i\colon \ell_1 \longrightarrow c_0$ has p-summing adjoint since i^* factors through ℓ_2. So, if $c_0 \in \mathcal{C}_p$ then $i \in \mathcal{K}_p(\ell_1, c_0)$ (Proposition 2.1) which is a contradiction because i is not even compact. Finally, 2 guarantees that ℓ_∞ does not belong to $\mathcal{C}_p$.

Finally, the formal identity $i_1\colon L_\infty(\mu) \longrightarrow L_1(\mu)$ is 1-integral, so i_1^* is [9, Theorem 5.15]. Then, i_1 is p-summing for all $p \geq 1$. Nevertheless, i_1 is not p-compact for any $p \geq 1$ (in fact, it is not even compact). In view of Proposition 2.1b, $L_1(\mu) \notin \mathcal{C}_p$. □

By definition, a 2-compact set A in $X = \ell_2$ is that for which there exists a 2-summable sequence (x_n) in X such that $A \subset \{\sum_n \alpha_n x_n\colon\ (\alpha_n) \in B_{\ell_2}\}$. The sequence (x_n) yields the Hilbert–Schmidt operator $\phi\colon e_n \in \ell_2 \longmapsto x_n \in X$ and we have $A \subset \phi(B_{\ell_2})$. This idea establishes a way to obtain p-compact sets $(1 \leq p \leq 2)$ in Hilbert spaces:

Corollary 2.3. Let X be a Hilbert space and $1 \leq p \leq 2$. A subset A of X is relatively p-compact if and only if there exists a Hilbert–Schmidt operator $\phi\colon \ell_2 \longrightarrow X$ such that $A \subset \phi(B_{\ell_2})$.

Proof. Since X^* has cotype 2, it suffices to deal with $p = 2$ ([19, Proposition 3.6]). Suppose $A \subset X$ is such that $A \subset \phi(B_{\ell_2})$ for a given Hilbert–Schmidt operator $\phi\colon \ell_2 \longrightarrow X$. Now, $\phi^* \in \Pi_2(X^*, \ell_2)$ [9, Theorem 4.10] and, by Proposition 2.1, $\phi \in \mathcal{K}_2(\ell_2, X)$. So $A \subset \phi(B_{\ell_2})$ must be relatively 2-compact. □

In order to show that $\ell_1(\Gamma) \in \mathcal{C}_p$ for any set Γ, we need the following

Lemma 2.4. Let Y and Z be Banach spaces. If $T\colon Y \longrightarrow Z^*$ is a weakly compact operator and $R := T^*_{|Z}$, then $R^{**} = T^*$.

Proof. Let $z_0^{**} \in B_{Z^{**}}$ and choose a net $(z_\delta)_\delta$ in B_Z such that

$$z_0^{**} = \sigma(Z^{**}, Z^*)\text{-}\lim_\delta z_\delta.$$

Since T^* is $\sigma(Z^{**}, Z^*)$-$\sigma(Y^*, Y^{**})$-continuous, we have

$$T^* z_0^{**} = \sigma(Y^*, Y^{**})\text{-}\lim_\delta T^* z_\delta = \sigma(Y^*, Y^{**})\text{-}\lim_\delta R z_\delta.$$

On the other hand, since $R = T^*_{|Z}$ is also a weakly compact operator, it follows that $R^{**}(Z^{**}) \subset Y^*$ and R^{**} is $\sigma(Z^{**}, Z^*)$-$\sigma(Y^*, Y^{**})$-continuous. Hence

$$R^{**} z_0^{**} = \sigma(Y^*, Y^{**})\text{-}\lim_\delta R^{**} z_\delta = \sigma(Y^*, Y^{**})\text{-}\lim_\delta R z_\delta T^* z_0^{**}.$$

□

Corollary 2.5. Every separable dual space belongs to $\mathcal{C}_p$.

Proof. Let $X = Z^*$ be a separable Banach space. It suffices to show that $\mathcal{I}_p^d(\ell_1, X) \subset \mathcal{N}_p^d(\ell_1, X)$ (Remark 2.2). Consider $T\colon \ell_1 \longrightarrow X$ such that $T^* \in \mathcal{I}_p(X^*, \ell_\infty)$. Now, $R = T^*_{|Z}$ is also p-integral and, according to [16, Theorem 5], p-nuclear. From this and Lemma 2.4, we have $R^{**} = T^*$ is p-nuclear. □

Arguing as in the proof of d)⇒a) in Proposition 2.1, Corollary 2.5 yields

Corollary 2.6. $\ell_1(\Gamma) \in \mathcal{C}_p$ for any set Γ.

Now, we deal with the problem of characterizing relatively p-compact sets in ℓ_1. A necessary condition for a bounded subset $A \subset \ell_1$ to be relatively p-compact is that U_A^* maps the weakly p-summable sequence (e_k) in ℓ_∞ to a p-summable sequence in $\ell_\infty(A)$. In this case, given $a = (a(k)) \in A$ we have

$$|a(k)| = |\langle a, e_k\rangle| \le \sup_{a\in A} |\langle a, e_k\rangle| = \|U_A^* e_k\|.$$

In other words, if $A \subset \ell_1$ is relatively p-compact then there exists $\gamma = (\gamma(k)) \in \ell_p$ such that $|a(k)| \le \gamma(k)$ for all $k \in \mathbb{N}$ and $a \in A$. Of course, the converse is not true when $p > 1$: if $a_n = (1/n, \overset{n)}{\ldots}, 1/n, 0, \ldots)$, the sequence (a_n) is "dominated" by $\gamma = (1/k)$ but it is not even relatively compact.

Corollary 2.7. A bounded subset $A \subset \ell_1$ is relatively 1-compact if and only if it is order bounded.

Proof. Suppose that $A \subset \ell_1$ is order bounded. In view of [9, Theorem 5.19], U_A is 1-integral, so U_A^* is. In particular, U_A^* is 1-summing and, according to Corollary 2.6, A is relatively 1-compact. □

The criterion of p-compactness in ℓ_1 ($p > 1$) will need the following result that characterizes bounded sets with p-summing evaluation map. Recall that a sequence (x_n) in X is *strongly p-summable* if $\sum_n |\langle x_n^*, x_n\rangle| < \infty$ for all $(x_n^*) \in \ell_{p'}^w(X^*)$ ([7]). This notion has been extended and studied later by several authors in a natural way: $(x_n) \subset X$ is said to be *(p,q)-summing* if $\sum_n |\langle x_n^*, x_n\rangle|^p < \infty$ for all $(x_n^*) \in \ell_q^w(X^*)$ (see, for instance, [3], [4] and [12]).

Theorem 2.8. Let X be a Banach space and $p \geq 1$. The following statements are equivalent for a bounded set $A \subset X$:

a) The evaluation map $U_A^*\colon X^* \longrightarrow \ell_\infty(A)$ is p-summing.

b) For all $(x_n) \in A^{\mathbb{N}}$ and $\beta = (\beta_n) \in \ell_{p'}$ ($\beta \in c_0$ if $p = 1$), the operator $\phi\colon \ell_p \longrightarrow X$ defined by $\phi(e_n) = \beta_n x_n$ is nuclear.

c) For all $(x_n) \in A^{\mathbb{N}}$ and $\beta = (\beta_n) \in \ell_{p'}$ ($\beta \in c_0$ if $p = 1$), the sequence $(\beta_n x_n)$ is strongly p'-summable.

d) For all $(x_n) \in A^{\mathbb{N}}$, the sequence (x_n) is (p,p)-summing.

Proof. a)⇒b) Fixed $(x_n) \in A^{\mathbb{N}}$ and $\beta = (\beta_n) \in \ell_{p'}$, consider the operators

$$D_\beta\colon \begin{array}{ccc} \ell_p & \longrightarrow & \ell_1 \\ (\alpha_n) & \longmapsto & (\beta_n \alpha_n) \end{array} \qquad P\colon \begin{array}{ccc} \ell_\infty(A) & \longrightarrow & \ell_\infty \\ \xi & \longmapsto & (\xi(x_n)) \end{array}$$

The adjoint of ϕ factors as follows:

$$\begin{array}{ccc} X^* & \xrightarrow{\ \phi^*\ } & \ell_{p'} \\ {\scriptstyle U_A^*}\downarrow & & \uparrow{\scriptstyle D_\beta^*} \\ \ell_\infty(A) & \xrightarrow[\ P\ ]{} & \ell_\infty \end{array}$$

It is easy to check that $D_\beta^* = \sum_n \beta_n e_n^* \otimes e_n$ where (e_n) and (e_n^*) denote the unit vector basis of $\ell_{p'}$ and ℓ_1, respectively. Thus, D_β is p'-nuclear and, since U_A^* is p-summing, we conclude that $\phi^* = D_\beta^* \circ P \circ U_A^* \in \mathcal{N}_1(X^*, \ell_{p'})$ ([17, Theorem 48]). According to [10, Theorem VIII.3.7], ϕ is a nuclear operator.

b)⇒c) According to [3, Theorem 2], the space $\mathcal{I}_1(\ell_p, X)$ is isometrically isomorphic to the space of all strongly p'-summable sequences in X and the isometry is given by $\phi \in \mathcal{I}_1(\ell_p, X) \longmapsto (\phi e_n)$. Now, c) is concluded since every nuclear operator is, in particular, integral.

c)⇒d) It is straightforward.

d)⇒a) By contradiction, suppose U_A^* is not p-summing. Then, for each $k \in \mathbb{N}$ there exist sequences $(x_{n,k})_n \in A^{\mathbb{N}}$ and $(x_{n,k}^*)_n \in B_{\ell_p^w(X^*)}$ such that $\sum_n |\langle x_{n,k}^*, x_{n,k}\rangle|^p \geq k^{2p}$. If $x \in X$,

$$\sum_k \sum_n \left| \langle \frac{1}{k^2} x_{n,k}^*, x\rangle \right|^p \leq \sum_k \frac{1}{k^{2p}},$$

that is to say, $(k^{-2}x^*_{n,k})_{n,k}$ is weakly p-summable in X^*. Nevertheless,

$$\sum_k \sum_n \left| \langle \frac{1}{k^2} x^*_{n,k}, x_{n,k} \rangle \right|^p \geq \sum_k \frac{1}{k^{2p}} k^{2p} = \infty$$

in contradiction to d). $\square$

Given a nuclear operator $\phi\colon \ell_p \longrightarrow \ell_1$, let us denote $(\sigma_n(k))_k = \phi(e_n)$. Then ϕ^* is also nuclear and, in particular, 1-summing. Hence,

$$\infty > \sum_k \|\phi^*(e^*_k)\|_{p'} = \sum_k \left(\sum_n |\sigma_n(k)|^{p'} \right)^{1/p'} \tag{2}$$

where $(e_k)^*$ denotes the canonical vector sequence in ℓ_∞. Conversely, if the matrix $(\sigma_n(k))_{n,k}$ verifies (2), then ϕ admits the nuclear representation $\sum_n (\sigma_n(k))_k \otimes e_k$.

Corollary 2.9. Let $p > 1$. A bounded subset $A \subset \ell_1$ is relatively p-compact if and only if

$$\sum_k \left(\sum_n |\beta_n x_n(k)|^{p'} \right)^{1/p'} < \infty$$

for all $(x_n) \in A^{\mathbb{N}}$ and $\beta = (\beta_n) \in \ell_{p'}$.

3 Final notes

In Proposition 2.2, we have mentioned that neither c_0 nor ℓ_∞ belong to $\mathcal{C}_p$. Anyway, we have the following way to generate 2-compact sets in c_0: if $A \subset \ell_2$ is relatively compact, then A is relatively 2-compact as a subset of c_0. In fact, the identity map from ℓ_2 to c_0 has 1-summing (hence, 2-summing) adjoint, so that operator maps relatively compact sets in ℓ_2 to relatively 2-compact sets in c_0 [8, Theorem 3.14]. This example inspires the following lemma:

Lemma 3.1. Let X be a $\mathcal{L}_\infty$-space and $1 \leq p \leq 2$. Then $A \subset X$ is relatively p-compact if and only if there exist a relatively compact set $K \subset \ell_2$ and an operator $\phi\colon \ell_2 \longrightarrow X$ such that $A \subset \phi(K)$.

Proof. The dual space X^* is a $\mathcal{L}_1$-space. Hence, X^* has cotype 2, so it suffices to deal with $p = 2$ ([19, Proposition 3.6]). If $A \subset X$ is relatively 2-compact, there exists $(x_n) \in \ell_2(X)$ such that $A \subset 2\text{-co}\,(x_n)$. Choose $(\alpha_n) \searrow 0$ so that $(\alpha_n^{-1} x_n)$ remains to be 2-summable. Now consider the operators $D\colon (e_n) \in \ell_2 \longmapsto (\alpha_n e_n) \in \ell_2$ and $\phi\colon e_n \in \ell_2 \longmapsto (\alpha_n^{-1} x_n) \in X$. It is clear that $A \subset \phi(K)$, K being the relatively compact set $D(B_{\ell_2})$. Conversely, suppose $A \subset X$ is such that there exist a relatively compact set $K \subset \ell_2$ and an operator $\phi\colon \ell_2 \longrightarrow X$ verifying $A \subset \phi(K)$. According to [9, Theorem 3.1], ϕ^* is 2-summing, so ϕ map relatively compact sets in ℓ_2 to relatively 2-compact sets in X [8, Theorem 3.14]. □

Given an absolutely convex and weakly compact set $B \subset X$, span(B) is denoted by X_B. This space is normed by the Minkowski's functional of B:

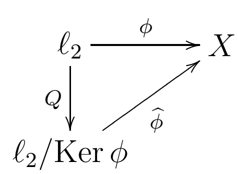
$$\rho_B(x) = \inf\{t > 0\colon\ x \in tB\}.$$

It is well known that (X_B, ρ_B) is complete and B is its closed unit ball. The canonical inclusion map from X_B into X is denoted by j_B.

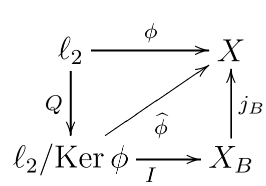
Proposition 3.1. Let X be a $\mathcal{L}_\infty$-space and $1 \leq p \leq 2$. Then $A \subset X$ is relatively p-compact if and only there exists $(x_n) \in \ell_2^w(X)$ such that the following conditions are satisfied:

1. $A \subset B := 2\text{-co}\,(x_n)$;
2. A is relatively compact in X_B.

Proof. As in the previous proof, it suffices to deal with the case $p = 2$. If $A \subset X$ is relatively 2-compact, Lemma 3.1 guarantees the existence of a relatively compact set $K \subset \ell_2$ and $\phi\colon \ell_2 \longrightarrow X$ such that $A \subset \phi(K)$. Put $x_n = \phi(e_n)$ and and $B := 2\text{-co}\,(x_n)$. To prove that A is relatively compact in X_B, let us consider the quotient map $Q\colon \ell_2 \longrightarrow \ell_2/\mathrm{Ker}\,\phi$ and the operator $\widehat{\phi}\colon \ell_2/\mathrm{Ker}\,\phi \longrightarrow X$ defined so that $\widehat{\phi}(Q(\beta_n)) = \phi(\beta_n)$ for every $(\beta_n) \in \ell_2$. Then, the following diagram is conmutative:



On the other side, it is not difficult to see that the operator $I\colon \ell_2/\mathrm{Ker}\,\phi \longrightarrow X_B$ defined by $I([(\alpha_n)]) = \sum_n \alpha_n x_n$ is an isomorphism between Banach spaces satisfying $\widehat{\phi} = j_B \circ I$:



Now, since $j_B(A) = A \subset \phi(K)$, it is clear that $\widehat{\phi}\big(I^{-1}(A)\big) \subset \widehat{\phi}\big(Q(K)\big)$. From the injectivity of $\widehat{\phi}$, it follows that $A \subset I\big(Q(K)\big)$.

Conversely, assume that $A \subset X$ verifies (1) and (2). If ϕ is the operator induced by the sequence (x_n), then the isomorphism $I\colon \ell_2/\operatorname{Ker}\phi \longrightarrow X_B$ defined as above enables to see X_B as a Hilbert space. According to [22, Theorem 10.8], j_B^* is 2-summing and, since A is relatively compact in X_B, $A = j_B(A)$ is relatively 2-compact in X [8, Theorem 3.14]. □

As an application, we show a relatively compact set in c_0 inside of the 2-convex hull of (e_k) but failing to be relatively 2-compact (here, (e_k) denotes the unit vector basis of c_0).

Example 3.2. For each $n \in \mathbb{N}$, put $x_n = \left(\frac{1}{\sqrt{n}}, \overset{n)}{\ldots}, \frac{1}{\sqrt{n}}, 0 \ldots\right) \in c_0$ and consider $A = \{x_n\colon\ n \in \mathbb{N}\} \subset B := 2\text{-co}\,(e_k)$. Then A is relatively compact; in fact,

$$\lim_n \|x_n\|_\infty = 0. \tag{3}$$

In order to see that A is not relatively ρ_B-compact, we first prove that $\rho_B(x_n) = 1$ for all $n \in \mathbb{N}$. By contradiction, assume that there exists $n \in \mathbb{N}$ so that $\rho_B(x_n) < 1$ and choose $t \in [\rho_B(x_n), 1)$ such that $x_n \in tB$. Then

$$x_n = \sum_n t\alpha_k e_k$$

for a fixed $(\alpha_k)_k \in B_{\ell_2}$. Thus $\langle x^*, x_n\rangle = \sum_n t\alpha_k \langle x^*, e_k\rangle$ for all $x^* \in \ell_1$. In particular,

$$t\alpha_k = \frac{1}{\sqrt{n}} \quad \text{if } k \le n$$

$$t\alpha_k = 0 \quad \text{if } k > n.$$

From this

$$1 \ge \sum_k \alpha_k^2 = \frac{1}{t^2},$$

which is a contradiction to $t < 1$. Now, if A is relatively ρ_B-compact, then there exists a subsequence $(x_{k(n)})$ of (x_n) ρ_B-convergent to $x \neq 0$. Since j_B is continuous, $(x_{k(n)})$ is $\|\cdot\|_\infty$-convergent to $x \neq 0$, a contradiction to (3).

In the previous section, we have also showed that $L_1(\mu)$ fails to be in $\mathcal{C}_p$ if $p \ge 1$. Anyway, a criterion of 1-compactness in $L_1(\mu)$ can be deduced using the characterization of nuclear operators into $L_1(\mu)$ due to Grothendieck (see [10, p. 258]):

Proposition 3.2. A bounded subset A of $L_1(\mu)$ is relatively 1-compact if and only if

1. A is order bounded, i.e., there exist $g \in L_1(\mu)$ such that $|f| \le g$ μ-almost everywhere for each $f \in A$, and

2. A is equimeasurable, i.e., given $\varepsilon > 0$, there is a measurable set Ω_ε such that $\mu(\Omega \backslash \Omega_\varepsilon) < \varepsilon$ and $\{f\chi_{\Omega_\varepsilon}\colon\ f \in A\}$ is relatively compact in $L_\infty(\mu)$.

Proof. If $A \subset L_1(\mu)$ is relatively 1-compact, then U_A^* is nuclear. According to [10, Theorem VIII.3.7], U_A is itself nuclear and this leads up to conclude that $A \subset U_A(B_{\ell_1(A)})$ is order bounded and equimeasurable [10, p. 258]. Conversely, let us see that U_A^* is nuclear whenever A is order bounded and equimeasurable in $L_1(\mu)$. For if, notice that $U_A(B_{\ell_1(A)}) \subset \operatorname{co}(A)$ is also order bounded and equimeasurable (here, $\operatorname{co}(A)$ denotes the closed absolutely convex hull of A). Then, U_A is nuclear, as well as U_A^*. □

Since operators from any $\mathcal{L}_\infty$-space to any space with cotype 2 are 2-summing [9, Theorem 11.14], we can reproduce the proof of Lemma 3.1 to obtain 2-compact sets in $\mathcal{L}_1$-spaces.

Proposition 3.3. Let X be a $\mathcal{L}_1$-space. Then $A \subset X$ is relatively 2-compact if and only if there exist a relatively compact set $K \subset \ell_2$ and an operator $\phi\colon \ell_2 \longrightarrow X$ such that $A \subset \phi(K)$.

We finish with some results concerning to the equality $\mathcal{L}(Y, \ell_q) = \mathcal{K}_p(Y, \ell_q)$. The following is a consequence of the equality $\mathcal{K}_p(Y, \ell_1) = \Pi_p^d(Y, \ell_1)$ and [9, Theorem 11.14].

Proposition 3.4. Let Y be a Banach space such that Y^* has cotype $s \geq 2$. We have:

1. If $s = 2$, then $\mathcal{L}(Y, \ell_1) = \mathcal{K}_2(Y, \ell_1)$.
2. If $s > 2$, then $\mathcal{L}(Y, \ell_1) = \mathcal{K}_p(Y, \ell_1)$ for every $p > s$.

Corollary 3.3. Let $p \geq 2$. We have:

1. $\mathcal{L}(\ell_r, \ell_1) = \mathcal{K}_2(\ell_r, \ell_1)$ for every $r \geq 2$.
2. If $p > 2$, $\mathcal{L}(\ell_r, \ell_1) = \mathcal{K}_p(\ell_r, \ell_1)$ for every $r > p'$.

Remark 3.4. Notice that $\mathcal{L}(\ell_r, \ell_1) \neq \mathcal{K}_2(\ell_r, \ell_1)$ whenever $r < 2$. For if, consider an operator $T \in \mathcal{L}(c_0, \ell_{r'})$ failing to be r'-summing [13, Theorem 7]. Thus, $T^* \notin \Pi_2^d(\ell_r, \ell_1) = \mathcal{K}_2(\ell_r, \ell_1)$. If $p > 2$, the same argument can be used to explain that $\mathcal{L}(\ell_r, \ell_1) \neq \mathcal{K}_p(\ell_r, \ell_1)$ whenever $r \leq p'$.

If $p < 2$, the equality $\mathcal{L}(Y, \ell_1) = \mathcal{K}_p(Y, \ell_1)$ implies that Y is finite dimensional. Indeed, if $\mathcal{L}(Y, \ell_1) = \Pi_p^d(Y, \ell_1)$ holds, it follows that the identity map on Y^* is $(p, 1)$-summing, a contradiction to [9, Theorem 10.5].

Now we make clear that, if the rank space is ℓ_q with $q > 1$, then, for each $p \geq 1$, there are bounded operators failing to be p-compact.

Proposition 3.5. Let $p \geq 1$ and $q > 1$. If $\mathcal{L}(Y, \ell_q) = \mathcal{K}_p(Y, \ell_q)$ then Y is finite dimensional.

Proof. Since $\ell_q \in \mathcal{C}_p$, then $\mathcal{L}(Y,\ell_q) = \Pi_p^d(Y,\ell_q)$. According to [15, Theorem 1.3], $\mathcal{L}(\ell_{q'},Y^*) = \Pi_p(\ell_{q'},Y^*)$. This implies that Y^* must be finite dimensional ([15, p. 22]). □

Remark 3.5. The proof of Lemma 3.1 essentially works because $\mathcal{L}(\ell_2,X) = \Pi_1^d(\ell_2,X)$ if X is a $\mathcal{L}_\infty$-space. If $q > 1$, the above result reveals that $\mathcal{L}(\ell_2,\ell_q) \neq \mathcal{K}_p(\ell_2,\ell_q) = \Pi_p^d(\ell_2,\ell_q)$. Thus, the procedure used to prove Lemma 3.1 and Proposition 3.3 is not useful to obtain characterizations of p-compact sets in ℓ_q $(q > 1)$.

References

[1] R. Aharoni and P. Saphar, On the reflexivity of the space $\Pi_p(E,F)$ of p-absolutely summing operators, $1 \leq p < \infty$, *Bull. London Math. Soc.*, **25** (1993), 362 - 368.

[2] R.M. Aron, M. Maestre and P. Rueda, p-Compact holomorphic mappings, *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math.*, **104** (2010), 353 - 364.

[3] J.L. Arregui and O. Blasco, (p,q)-Summing sequences, *J. Math. Anal. Appl.*, **274** (2002), 812 - 827.

[4] S. Aywa and J.H. Fourie, On summing multipliers and applications, *J. Math. Anal. Appl.*, **253** (2001), 166 - 186.

[5] J. Bourgain and O. Reinov, On the approximation properties for the space H^∞, *Math. Nachr.*, **122** (1985), 19 - 27.

[6] Y.S. Choi and J.M. Kim, The dual space of $(\mathcal{L}(X,Y),\tau_p)$ and the p-approximation property, *J. Funct. Anal.*, **259** (2010), 2437 - 2454.

[7] J.S. Cohen, Absolutely p-summing, p-nuclear operators and their conjugates, *Math. Ann.*, **201** (1973), 177 - 200.

[8] J.M. Delgado, C. Piñeiro and E. Serrano, Operators whose adjoints are quasi p-nuclear, *Studia Math*, **197** (2010), 291 - 304.

[9] J. Diestel, H. Jarchow and A. Tonge, *Absolutely summing operators*, Cambridge Univ. Press, Cambridge, 1995.

[10] J. Diestel and J.J. Uhl, *Vector measures*, Math. Surveys Monogr. 15, Amer. Math. Soc., Providence, RI, 1977.

[11] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Memoirs Amer. Math. Soc., 16, 1955.

[12] R. Khalil, On some Banach space sequences, *Bull. Austral. Math. Soc.*, **25** (1982), 231 - 241.

[13] S. Kwapien, On a theorem of L. Schwartz and its applications to absolutely summing operators, *Studia Math.*, **38** (1970), 193 - 201.

[14] S. Lassalle and P. Turco, On p-compact mappings and the p-approximation property, *J. Math. Anal. Appl.*, **389** 2012, 1204 - 1221.

[15] J.S. Morrell and J.R. Retherford, p-Trivial Banach spaces, *Studia Math.*, **43** (1972), 1 - 25.

[16] A. Persson, On some properties of p-nuclear and p-integral operators, *Studia Math.*, **33** (1969), 213 - 222.

[17] A. Persson and A. Pietsch, p-Nukleare und p-integrale Abbildungen in Banachräumen, *Studia Math.*, **33** (1969), 19 - 62.

[18] A. Pietsch, *Operator Ideals*, North–Holland Publishing Company, Amsterdam-New York-Oxford, 1980.

[19] C. Piñeiro and J.M. Delgado, p-Convergent sequences and Banach spaces in which p-compact sets are q-compact, *Proc. Amer. Math. Soc.*, **139** (2011), 957 - 967.

[20] O.I. Reinov, A survey of some results in connection with Grothendieck approximation property, *Math. Nachr.*, **119** (1984), 257 - 264.

[21] D.P. Sinha and A.K. Karn, Compact operators whose adjoints factor through subspaces of ℓ_p, *Studia Math.*, **150** (2002), 17 - 33.

[22] N. Tomczak-Jaegermann, *Banach-Mazur Distances and Finite-Dimensional Operator Ideals*, Pitman Monographs and Surveys in Pure and Applied Mathematics, New York, 1989.

Received: November 11, 2013