On the Local Property of $|N, p_n, q_n|_k$-Summability of Factored Fourier Series

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Abstract

In this paper we have establish a theorem on the local property of $|N, p_n, q_n|_k$ summability of factored Fourier series.

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1 Introduction

Let $\sum a_n$ be a given infinite series with sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ and $\{q_n\}$ are sequences of positive real constants such that

$$r_n = \sum_{v=0}^n p_{n-v} q_v \to \infty \quad \text{as} \quad n \to \infty \quad (r_{-j} = p_{-j} = q_{-j} = 0, \ i \geq 1).$$

The sequence-to-sequence transformation

$$\sum_{v=0}^{\infty} p_{n-v} q_v.$$
defines the sequence \( \{ t_n \} \) of the \( (N, p_n, q_n) \)-means of the sequence of \( \{ s_n \} \) generated by the sequences of coefficients \( \{ p_n \} \) and \( \{ q_n \} \). The series \( \sum a_n \) is said to be summable \( \| N, p_n, q_n \|_k \) for \( k \geq 1 \), if

\[
\sum_{n=1}^{\infty} \left| \frac{r_{n-1}}{\Delta r_{n-1}} \right|^k \left| t_n - t_{n-1} \right|^k < \infty.
\]

When \( q_n = 1 \) for all \( n \), \( \| N, p_n, q_n \|_k \) -summability reduces to \( \| N, p_n \|_k \) -summability. A sequence \( \{ \lambda_n \} \) is said to be a convex if

\[
0 < \Delta^2 \lambda_n = \Delta \lambda_{n+1} - \Delta \lambda_n \quad \text{for all } n.
\]

Let \( f \) be a periodic function with period \( 2\pi \), integrable in the sense of Lebesgue over \( (-\pi, \pi) \). Without loss of generally we may assume that the constant term in the Fourier series of \( f \) is zero, so that

\[
\int_{-\pi}^{\pi} f(t) dt = 0
\]

and

\[
f(t) \approx \sum_{n=1}^{\infty} \left( a_n \cos nt + b_n \sin nt \right) = \sum_{n=1}^{\infty} A_n(t).
\]

## 2 Known Theorems

Dealing with the \( \| N, p_n \|_k \) -summability of an infinite series was proved the following theorem:

**Theorem 2.1** Let \( k \geq 1 \). Suppose \( \{ \lambda_n \} \) be a convex sequence such that \( \sum n^{-1} \lambda_n \) is convergent and \( \{ p_n \} \) be a sequence such that

\[
\Delta X_n = O \left( \frac{1}{n} \right)
\]

\[
\frac{p_{n+r}}{p_n} = O \left( \frac{p_{n+r-1} P_r}{P_{n+r-1} p_r} \right)
\]
Local property of \(|N, p_n, q_n|_k\)-summability

\[
\sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{k-1} p_{n-1} = O \left( \frac{p_r}{p_r} \right) \hspace{1cm} (8)
\]

\[
\sum_{n=1}^{\infty} \frac{X_n^{k-1} |\hat{\lambda}_n|^k}{n} < \infty \hspace{1cm} (9)
\]

\[
\sum_{n=1}^{\infty} \frac{X_n^{k-1} |\Delta \hat{\lambda}_n|^k}{n} < \infty \hspace{1cm} (10)
\]

where \(X_n = \frac{P_n}{np_n}\). Then the summability \(|N, p_n|_k\) of the series \(\sum_{n=1}^{\infty} A_n(t) \hat{\lambda}_n X_n\) at a point can be ensured by the local property [1].

3 Main Theorem

Since the behaviour of the Fourier series, as far as convergence is concerned, for a particular value of \(t\) depends on the behaviour of the function in the immediate neighbourhood of this point only, hence the truth of the Theorem 3.1 is related to the sequence to be a bounded.

**Theorem 3.1** : Let \(k \geq 1\) and let \(\{S_n\}\) the sequence partial sums of the series in (5). Suppose \(\{\hat{\lambda}_n\}\) be a convex sequence such that

\[
\sum_{n=0}^{\infty} \frac{|\Delta \hat{\lambda}_n|^k + |\hat{\lambda}_n|^k}{q_n^k} < \infty \hspace{1cm} (11)
\]

Also, \(\{p_n\}\) is a positive decreasing sequence and \(\{q_n\}\) is a positive increasing sequence such that

\[
\sum_{j=0}^{n-1} \left( \sum_{s=j}^{n} \frac{p_{n-s} q_s}{r_n} \right)^{\frac{k}{k-1}} = O \left( \frac{1}{n} \right), \hspace{1cm} (12)
\]

\[
\sum_{n=j+2}^{n+1} \frac{r_n}{n \Delta r_{n-1}} = O \left( \frac{1}{r_j} \right), \hspace{1cm} (13)
\]
where \( X_n = \frac{r_n}{p_n q_n} \). Then the summability \( |N, p_n, q_n|_k \) of the series 
\[
\sum_{n=1}^{\infty} A_n(t) \xi_n X_n
\]
at a point can be ensured by the local property.

**Proof:** Let \( \{ t_n \} \) denote the \((N, p_n, q_n)\)-mean of the series \( \sum_{n=1}^{\infty} A_n(t) \xi_n X_n \).

Then by definition and using Abel transformation, we have
\[
t_n - t_{n-1} = \sum_{v=0}^{n} \left( \frac{p_{n-v} - p_{n-v-1}}{r_n} \right) q_v \sum_{j=0}^{n} A_j \lambda_j X_j
\]
\[
= \sum_{j=0}^{n-1} A_j \lambda_j X_j \sum_{v=j}^{n} \left( \frac{p_{n-v} - p_{n-v-1}}{r_n} \right) q_v
\]
\[
= \sum_{j=0}^{n-1} \left\{ \lambda_j X_j \sum_{v=j}^{n} \left( \frac{p_{n-v} - p_{n-v-1}}{r_n} \right) q_v \right\} S_j + \lambda_n X_n S_n \left( \frac{p_0 - p_{n-1}}{r_n} \right) q_n
\]
\[
= \sum_{j=0}^{n-1} \left\{ \Delta \lambda_j X_j \sum_{v=j}^{n} \left( \frac{p_{n-v} - p_{n-v-1}}{r_n} \right) q_v \right\} S_j, \text{ with } p_0 = 0.
\]

\[
= \sum_{j=0}^{n-1} \left\{ \Delta \lambda_j X_j \sum_{v=j}^{n} \left( \frac{p_{n-v} - p_{n-v-1}}{r_n} \right) q_v \right\} S_j + \lambda_{j+1} X_{j+1} \left( \frac{p_{n-j} - p_{n-j-1}}{r_n} \right) q_j S_j
\]
\[
= \sum_{j=0}^{n-1} \left\{ X_j \Delta \lambda_j \sum_{v=j}^{n} \left( \frac{p_{n-v} - p_{n-v-1}}{r_n} \right) q_v \right\} S_j + \sum_{j=0}^{n-1} \lambda_{j+1} X_{j+1} \left( \frac{p_{n-j} - p_{n-j-1}}{r_n} \right) q_j S_j
\]
\[
+ \sum_{j=0}^{n-1} \left\{ \lambda_{j+1} \Delta X_j \sum_{v=j}^{n} \left( \frac{p_{n-v} - p_{n-v-1}}{r_n} \right) q_v \right\} S_j
\]
\[
= \sum_{j=0}^{n-1} X_j \Delta \lambda_j \sum_{v=j}^{n} \frac{p_{n-v} q_v S_j - \sum_{j=0}^{n-1} X_j \Delta \lambda_j \sum_{v=j}^{n} \frac{p_{n-v} - p_{n-v-1} q_v S_j}{r_n}}{r_n}
\]
\[
+ \sum_{j=0}^{n-1} \lambda_{j+1} X_{j+1} \frac{p_{n-j} q_j S_j - \sum_{j=0}^{n-1} \lambda_{j+1} X_{j+1} \frac{p_{n-j} - p_{n-j-1} q_j S_j}{r_n}}{r_n}
\]
\[
+ \sum_{j=0}^{n-1} \lambda_{j+1} \Delta X_j \sum_{v=j}^{n} \frac{p_{n-v} q_v S_j - \sum_{j=0}^{n-1} \lambda_{j+1} \Delta X_j \sum_{v=j}^{n} \frac{p_{n-v} - p_{n-v-1} q_v S_j}{r_n}}{r_n}
\]
\[
= t_{n,1} + t_{n,2} + t_{n,3} + t_{n,4} + t_{n,5} + t_{n,6} \text{ respectively.}
\]
Local property of \( |N, p_n, q_n|_k \) -summability

To complete proof of the theorem by Minkowski inequality, it is sufficient to show that

\[
\sum_{n=1}^{\infty} \left( \frac{r_{n-1}}{\Delta r_{n-1}} \right)^{k-1} |q_{n,i}|^k < \infty, \quad i = 1, 2, \ldots, 6.
\]

Using (11-13) conditions, we have

\[
\sum_{n=1}^{m+1} \left( \frac{r_{n-1}}{\Delta r_{n-1}} \right)^{k-1} |q_{n,1}|^k = \sum_{n=2}^{m+1} \left( \frac{r_{n-1}}{\Delta r_{n-1}} \right)^{k-1} \sum_{j=0}^{n-1} X_j \Delta \lambda_j \sum_{v=j}^{n} \frac{p_{n-v} q_v S_j}{r_n}
\]

\[
= O(1) \sum_{n=2}^{m+1} \left( \frac{r_{n-1}}{\Delta r_{n-1}} \right)^{k-1} \sum_{j=0}^{n-1} |X_j|^{k-1} \Delta \lambda_j \left[ \sum_{v=j}^{n} \left( \sum_{v=j}^{n} \frac{p_{n-v} q_v}{r_n} \right)^{k-1} \right]
\]

\[
= O(1) \sum_{n=2}^{m+1} \left( \frac{r_{n-1}}{\Delta r_{n-1}} \right)^{k-1} \sum_{j=0}^{n-1} |X_j|^{k-1} \Delta \lambda_j \left[ \sum_{v=j+2}^{n} \left( \sum_{v=j+2}^{n} \frac{p_{n-v} q_v}{r_n} \right)^{k-1} \right]
\]

\[
= O(1) \sum_{j=1}^{m+1} \frac{|X_j|^{k-1} \Delta \lambda_j}{r_j^{k-1}} = O(1) \sum_{j=1}^{m+1} |\Delta \lambda_j|^{k-1} = O(1) \text{ as } m \to \infty.
\]

Again, as above

\[
\sum_{n=2}^{m+1} \left( \frac{r_{n-1}}{\Delta r_{n-1}} \right)^{k-1} |q_{n,2}|^k = O(1) \text{ as } m \to \infty.
\]

Further, again using (11-13) conditions, we have

\[
\sum_{n=2}^{m+1} \left( \frac{r_{n-1}}{\Delta r_{n-1}} \right)^{k-1} |q_{n,3}|^k = \sum_{n=2}^{m+1} \left( \frac{r_{n-1}}{\Delta r_{n-1}} \right)^{k-1} \sum_{j=0}^{n-1} X_j \Delta \lambda_{j+1} \sum_{v=j+1}^{n} \frac{p_{n-v} q_v S_j}{r_n}
\]

\[
= O(1) \sum_{n=2}^{m+1} \left( \frac{r_{n-1}}{\Delta r_{n-1}} \right)^{k-1} \sum_{j=0}^{n-1} |X_{j+1}|^{k-1} \Delta \lambda_{j+1}^{k-1} \left[ \sum_{v=j+1}^{n} \left( \sum_{v=j+1}^{n} \frac{p_{n-v} q_v}{r_n} \right)^{k-1} \right]
\]
Similarly, as above
\[
\sum_{n=2}^{m+1} \frac{\Delta r_{n-1}}{r_n} |r_{n,4}|_k^k = O(1) \text{ as } m \to \infty.
\]

Now, using (11-13) conditions, \(\{q_n\}\) is an increasing sequence and \(\{r_n\}\) is a decreasing sequence
\[
\sum_{n=2}^{m+1} \frac{\sum_{j=0}^{n-1} \Delta X_j |\lambda_{j+1}|_k^k \left( \sum_{i=1}^{n} \frac{p_{n-i} q_i}{r_i} \right)_{j+1}^{k-1}}{n \Delta r_{n-1}} = O(1) \text{ as } m \to \infty.
\]

Finally, as above
\[
O(1) \left\{ \left| \sum_{j=1}^{m} \frac{\Delta X_j |\lambda_{j+1}|_k^k}{q_j r_j} + \sum_{j=1}^{m} \frac{\Delta r_j |\lambda_{j+1}|_k^k}{q_j r_j} \right|_k^k \right\} = O(1) \text{ as } m \to \infty.
\]
Local property of \( |N, p_n, q_n|_k \) -summability

\[
\sum_{n=2}^{m+1} \frac{r_{n-1}}{\Delta r_{n-1}} \left| q_{n,v} \right|^k = \sum_{n=2}^{m+1} \frac{r_{n-1}}{\Delta r_{n-1}} \left| \sum_{j=0}^{n-1} \lambda_{j+1} \Delta X_j \sum_{v=j}^n p_{n-v} r_{n-1} S_j \right|^k = O(1) \text{ as } m \to \infty.
\]

This completes the proof of the Theorem.

References


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