$k$-* Paranormal Composition Operator

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Abstract

In this article, composition operators and weighted composition operator of $k$-* Paranormal operators, $(M,k)^*$ class of operators and their adjoints are characterized in $L^2$ spaces. We also discuss the relationship between $(M,k)^*$ class and $k$-* Paranormal operators.

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Introduction

Let $H$ be an infinite dimensional complex separable Hilbert space and $B(H)$ the algebra of all bounded linear operators defined on Hilbert space $H$. Let $T \in B(H)$. An operator $T$ is called Normal if $TT^* = T^*T$, quasi-normal if $T(T^*T) = (T^*T)T$, it is hyponormal if $T^*T \geq TT^*$, which is equivalent to the condition $\|T^*x\| \leq \|Tx\|$, for all $x$ in $H$. We say that an operator $T$ is
quasi-hyponormal if \( T^*T^2 \geq (T^*)^2 \), which is equivalent to the condition 
\[ \|T^2x\| \geq \|T^*Tx\|, \text{ for all } x \text{ in } H. \] 
We say that an operator \( T \) is \(*\)-Paranormal if \( \|T^*x\|^2 \leq \|T^2x\| \|x\|, \) for all unit vectors \( x \) in \( H \) or \( \|T^*x\|^2 \leq \|T^2x\|, \) for all unit vectors \( x \) in \( H \). An operator \( T \) is called \( k\)-* Paranormal operator if \( \|T^*x\|^k \leq \|T^kx\|, \) for all unit vectors \( x \) in \( H \) and \( k \geq 2 \) [11].

In general, \( k\)-* Paranormal \( \Rightarrow \) \(*\)-Paranormal, \( k \geq 2 \)
An operator \( T \) is of \((M,k)\)* class if \( T^{*k}T^k \geq (TT^*)^k \), for \( k \geq 1 \), which is equivalent to the condition \( \|T^kx\| \geq \|(TT^*)^{k/2}\|, \) for all \( x \) in \( H \) and \( k \geq 1 \) [8].

Let \((X, \Sigma, \lambda)\) be a sigma-finite measure space. The relation of being almost everywhere, denoted by a.e., is an equivalence relation in \( L^2(X, \Sigma, \lambda) \) and this equivalence relation splits \( L^2(X, \Sigma, \lambda) \) into equivalence classes. Let \( T \) be a measurable transformation from \( X \) into itself. \( L^2(X, \Sigma, \lambda) \) is denoted by \( L^2(\lambda) \).

The equation \( C_Tf = f \circ T, \ f \in L^2(\lambda) \) defined a composition transformation on \( L^2(\lambda) \). \( T \) induces a composition operator \( C_T \) on \( L^2(\lambda) \) if

(i) the measure \( \lambda \circ T^{-1} \) is absolutely continuous with respect to \( \lambda \).

(ii) the Radon-Nikodym derivative \( \frac{d(\lambda T^{-1})}{d\lambda} \) is essentially bounded.

Harrington and Whitley [4] have shown that if \( C_T \in B(L^2(\lambda)) \), then \( C_T^*C_Tf = f_Tf \) and \( C_TC_T^*f = (f_T \circ T)Pf \), for all \( f \) in \( L^2(\lambda) \), where \( P \) denotes the projection of \( L^2(\lambda) \) onto \( R(C_T) \). Thus it follows that \( C_T \) had dense range iff \( C_T^*C_T \)

is the operator of multiplication by \( f_T \circ T \), where \( f_T \) denotes \( \frac{d(\lambda T^{-1})}{d\lambda} \), the Radon Nikdoym Derivative. Every essentially bounded complex-valued measurable function \( f_T \) induces a bounded operator \( M_{f_T} \) on \( L^2(\lambda) \) which is defined by \( M_{f_T}f = f_Tf \), for every \( f \) in \( L^2(\lambda) \).

Further, \( C_T^*C_T = M_{f_T} \). Let us denote \( \frac{d(\lambda T^{-1})}{d\lambda} \) by \( h \) and \( \frac{d(\lambda T^k)}{d\lambda} \) by \( h_k \) where \( k \) is a positive integer greater than or equal to 1.

Then \( C_T^*C_T = M_h \) and \( C_T^*C_T^* = M_{h^2} \). In general, \( C_T^*C_T^k = M_{h_k} \) where \( M_{h_k} \)

is the multiplication operator on \( L^2(\lambda) \) induced by complex valued measurable function \( h_k \).

Let the set \( A \) be contained in the space \( X \) then the characteristic function of \( A \), written \( \chi_A \), is the function on \( A \) defined by:

\[
\chi_A(x) = 1, \text{ for } x \in A; \quad \chi_A(x) = 0, \text{ for } x \in (X - A)
\]

1 \( k\)-* Paranormal and \((M,k)\)* Composition Operators

In this section we characterize \( k\)-* Paranormal composition operator, \((M,k)\)* composition operator.
Theorem 1.1 ([3, Proposition 2.2]). For each positive integer \( k \geq 2 \), an operator \( T \in B(H) \) is \( k^* \)-Paranormal iff \( T^{*k}T^k - k\mu^{k-1}TT^* + (k-1)\mu kI \geq 0 \), for all \( \mu > 0 \).

Using this theorem, we characterize the \( k^* \)-Paranormal composition operators on \( L^2(\lambda) \) in the following.

Theorem 1.2. For each positive integer \( k \geq 2 \), \( C_T \in B(L^2(\lambda)) \) is \( k^* \)-Paranormal operator iff \( h_k - k\mu^{k-1}(h \circ T)P + (k-1)\mu k \geq 0 \), for all \( \mu > 0 \), where \( P \) denotes the Projection of \( L^2(\lambda) \) onto \( R(C_T) \).

Proof. For each positive integer \( k \geq 2 \)

\( C_T \) is \( k^* \)-Paranormal operator iff

\[
C_T^{*k}C_T^k - k\mu^{k-1}C_TC_T^* + (k-1)\mu kI \geq 0,
\]

for all \( \mu > 0 \).

Thus,

\[
\langle (C_T^{*k}C_T^k - k\mu^{k-1}C_TC_T^*) + (k-1)\mu kI \rangle(g), g \rangle \geq 0,
\]

for all \( g \) in \( L^2(\lambda) \) and for all \( \mu > 0 \).

\[
\Leftrightarrow \langle (C_T^{*k}C_T^k - k\mu^{k-1}C_TC_T^*) + (k-1)\mu kI \rangle(\chi_E), \chi_E \rangle \geq 0,
\]

for every characteristic function \( \chi_E \) of \( E \) in \( \Sigma \) such that \( \lambda(E) < \infty \).

Since, \( C_T^{*k}C_T^k = M_{h_k} \) and \( C_TC_T^* = M_{(h \circ T)p} \).

Therefore,

\[
\langle (M_{h_k} - k\mu^{k-1}M_{(h \circ T)p} + (k-1)\mu kI) \rangle(\chi_E), \chi_E \rangle \geq 0
\]

for every characteristic function \( \chi_E \) of \( E \) in \( \Sigma \) such that \( \lambda(E) < \infty \)

\[
\Leftrightarrow \int_E (M_{h_k} - k\mu^{k-1}M_{(h \circ T)p} + (k-1)\mu kI)(\chi_E)d\lambda \geq 0
\]

for every characteristic function \( \chi_E \) of \( E \) in \( \Sigma \) such that \( \lambda(E) < \infty \)

\[
\Leftrightarrow \int_E (h_k - k\mu^{k-1}(h \circ T)P + (k-1)\mu k) d\lambda \geq 0,
\]

for every \( E \) in \( \Sigma \) such that \( \lambda(E) < \infty \)

\[
\Leftrightarrow h_k - k\mu^{k-1}(h \circ T)P + (k-1)\mu k \geq 0, \text{ for every } \mu > 0
\]

Hence, \( C_T \) is \( k^* \)-Paranormal operator iff \( h_k - k\mu^{k-1}(h \circ T)P + (k-1)\mu k \geq 0 \) for every \( \mu > 0 \).  \( \square \)
Corollary 1.3. Let \( C_T \in B(L^2(\lambda)) \) with dense range. Then \( C_T \) is \( k^* \) Paranormal operator iff \( h^k \circ T \leq h_k \) a.e.

Proof. Since, \( C_T \in B(L^2(\lambda)) \) has dense range. Therefore

\[
C_TC^*_Tf = M(h \circ T)f = (h \circ T)f
\]

Now,

\[
C_T \text{ is } k^* \text{ Paranormal} \\
\iff h_k - k\mu^{k-1}(h \circ T) + (k - 1)\mu^k \geq 0, \quad \text{for every } \mu > 0 \\
\iff (h \circ T)^k \leq h_k \text{ a.e.} \\
\iff h^k \circ T \leq h_k \text{ a.e.}
\]

\[ \square \]

Example 1.4. Let \( X = \mathbb{N} \) and let \( \lambda \) be a counting measure on \( X \). Define \( T : \mathbb{N} \to \mathbb{N} \) by

\[
T(1) = T(2) = 1; \quad T(3) = 2 \\
T(4n + m - 1) = n + 2, \quad \text{for } m = 1, 2, 3, 4 \text{ and } n \in \mathbb{N}.
\]

Then, for each \( k \geq 3 \)

\[
(h^k \circ T)(n) \leq h_k(n) \text{ a.e., for every } n \in \mathbb{N}.
\]

Hence, \( T \) is \( k^* \) Paranormal operator.

Theorem 1.5. For each positive integer \( k \geq 2 \), \( C^*_T \) is \( k^* \) Paranormal operator iff

\[
h_k \circ T^kP_k - k\mu^{k-1}h + (k - 1)\mu^k \geq 0 \quad \text{a.e., for every } \mu > 0
\]

where \( P_k \) is the projection of \( L^2(\lambda) \) onto \( \overline{R(C^*_T)} \).

Proof. \( C^*_T \) is \( k^* \) Paranormal operator iff

\[
C^*_TC^*_k - k\mu^{k-1}C^*_TC_T + (k - 1)\mu^kI \geq 0, \quad \text{for every } \mu > 0.
\]

Thus

\[
\langle (C^*_TC^*_k - k\mu^{k-1}C^*_TC_T + (k - 1)\mu^kI)(\chi_E), \chi_E \rangle \geq 0,
\]

for every characteristic function \( \chi_E \) of \( E \) in \( \Sigma \) such that \( \lambda(E) < \infty \) and for every \( \mu > 0 \)

\[
\iff h_k \circ T^kP_k - k\mu^{k-1}h + (k - 1)\mu^k \geq 0 \text{ a.e., for every } \mu > 0.
\]

\[ \square \]
Corollary 1.6. Let $C^*_T \in B(L^2(\lambda))$ with dense range. Then, $C^*_T$ is $k^*$ Paranormal operator iff $h^k \leq h_k \circ T^k$.

Proof. Since $C^*_T \in B(L^2(\lambda))$ has dense range therefore $C^*_T$ is $k^*$ Paranormal iff $h_k \circ T^k - k\mu^{k-1}h + (k-1)\mu^k \geq 0$ a.e., for all $\mu > 0$ iff $h^k \leq h_k \circ T^k$ a.e. \[
\]

Theorem 1.7 ([7]). If $C_T$ is a composition operator on $L^2(\lambda)$, then $C_T$ is of class $(M, k)^*$ if and only if

$$
\|h_k^\frac{1}{2} \chi_E\| \geq \|(h \circ T)^{k/2}P(\chi_E)\|, \text{ for all } \chi_E \in L^2(\lambda)
$$

where $P$ is the projection onto $\overline{R(C_T)}$.

Fahri and Muhib [3] has characterized $(M, k)^*$ class of operators as follows:

Theorem 1.8. $T \in (M, k)^*$ iff $T^kT^k + 2\mu(TT^*)(TT^*)^k + \mu^2TT^kT^k \geq 0$, for all $\mu > 0$.

Theorem 1.9. For each positive integer $k \geq 1$, a composition operator on $L^2(\lambda)$ is of class $(M, k)^*$ iff $h_k + 2\mu(h \circ T)P + \mu^2h_k$ a.e., for all $\mu > 0$.

Proof.

$C_T$ is of class $(M, k)^*$

$\Leftrightarrow C^*TC^*_T + 2\mu(C_TC^*_T)^k + \mu^2C^*TC^*_T \geq 0$ a.e., for all $\mu > 0$

$\Leftrightarrow \langle C^*TC^*_T + 2\mu(C_TC^*_T)^k + \mu^2(C^*TC^*_T)(\chi_E), \chi_E \rangle \geq 0,$

for every $\chi_E \in L^2(\lambda)$ and for every $\mu > 0$.

$\Leftrightarrow \langle (M_{h_k} + 2\mu M_{(h \circ T)h_k} + \mu^2M_{h_k})(\chi_E), \chi_E \rangle \geq 0,$

for every $\chi_E \in L^2 \in L^2(\lambda)$ and for every $\mu > 0$.

$\Leftrightarrow \int_E (h_k + 2\mu(h \circ T)^kp + \mu^2h_k) d\lambda \geq 0$ a.e., for every $\mu > 0$

and for every $E \in \Sigma$ with $\lambda(E) < \infty$

$\Leftrightarrow h_k + 2\mu(h \circ T)^kp + \mu^2h_k \geq 0$ a.e., for every $\mu > 0$

$\Leftrightarrow h_k + 2\mu(h^k \circ T)p + \mu^2h_k \geq 0$ a.e., for every $\mu > 0$

\[
\]

Corollary 1.10. If $C_T \in B(L^2(\lambda))$ and has dense range. Then $C_T$ is of class $(M, k)^*$ iff $(h_k \circ T) \leq h_k$ a.e.

Proof. $C_T \in B(L^2(\lambda))$ with dense range is of class $(M, k)^*$ iff $h_k + 2\mu(h^k \circ T) + \mu^2h_k \geq 0$ a.e., for every $\mu > 0$ iff $h^k \circ T \leq h_k$ a.e. (using the elementary property of real quadratic form). \[
\]
Theorem 1.11. A composition operator $C_T \in B(L^2(\lambda))$ with dense range is $k$-$^*$ Paranormal operator iff it is of class $(M, k)^*$, for all $k \geq 2$.

Proof. [7, Theorem 2.1] If $C_T \in (M, k)^*$, $k \geq 2$, then $C_T$ is $k$-$^*$ Paranormal operator.

Conversely, let $C_T \in B(L^2(\lambda))$ be a composition operation with dense range which is $k$-$^*$ Paranormal operator.

Then, by Theorem 1.2

\[
\begin{align*}
  h_k - k\mu^{k-1}(h \circ T) + (k-1)\mu^k &\geq 0 \quad \text{a.e., for all } \mu > 0 \\
  (h \circ T)^k &\leq h_k \quad \text{a.e.} \\
  h_k - (h \circ T)^k &\leq 0 \quad \text{a.e.} \\
  \int_E (h_k - (h \circ T)^k) d\lambda &\geq 0 \quad \text{for every } E \in \Sigma \text{ with } \lambda(E) < \infty \\
  \int_E (Mh_k - M^{k}_{h\circ T})(\chi_E) d\lambda &\geq 0 \quad \text{a.e., for every } E \in \Sigma \text{ with } \lambda(E) < \infty \\
  \langle (C_T^* C_T^k - (C_T C_T^*)^k)(\chi_E), \chi_E \rangle &\geq 0, \quad \text{for every } \chi_E \in L^2(\lambda) \\
  C_T \text{ of class } (M, k)^* \quad \text{(by definition of } (M, k)^*)
\end{align*}
\]

\[\square\]

2 Weighted Composition Operators

In this section we characterize the weighted $k$-$^*$ Paranormal composition operators and weighted $(M, k)^*$ class of composition operators.

A weighted composition operator $W$ induced by $T$ is defined as $Wf = w(f \circ T)$, is a complex-valued $\Sigma$ measurable function. When $w = 1$, we say that $W$ is a composition operator.

Let $w_k$ denote $w(w \circ T)(w \circ T^2) \circ (w \circ T^{k-1})$. Then,

\[W^k f = w_k(f \circ T)^k.\]

To examine the weighted composition operators effectively, Alan Lambert [1] associated conditional expectation operator $E$ with $T$ as $E(\cdot \mid T^{-1}\Sigma) = E(\cdot)$. $E(f)$ is defined for each non-negative measurable function $f \in L^p(\lambda)$, $p \geq 1$ and is uniquely determined by conditions:

(i) $E(f)$ is $T^{-1}\Sigma$ measurable

(ii) If $B$ is any $T^{-1}\Sigma$ measurable set for which $\int_B f d\lambda$ converges,

\[\int_B f d\lambda = \int_B E(f) d\lambda\]
An operator of $L^p(\lambda), E$ is the projection onto the closure of range of $T$ and $E$ is the identity on $L^p(\lambda), p \geq 1$ iff $T^{-1}\Sigma = \Sigma$. Detailed discussion of $E$ is found in [5], [6], [10]. The following properties due to Cambell and Jamison [5] is well known.

**Proposition 2.1.** For $w \geq 0$

(i) $W^*Wf = h[E(w^2)] \circ T^{-1}f$

(ii) $WW^*f = w(h \circ T)E(wf)$

Since $W^kf = w_k(f \circ T^k)$ and $W^{*k}f = h_kE(w_kf) \circ T^k$, we have

$$W^{*k}W^k f = h_kE(w_k^2) \circ T^{-k}f, \quad \text{for every } f \in L^p(\lambda), (p \geq 1).$$

Now we give a characterization of $k^*$ Paranormal weighted composition operators and $(M, k)^*$ class of weighted composition operators.

**Theorem 2.1.** For each positive integer $k \geq 2$. Let $W \in B(L^2(\lambda))$. Then, $W$ is $k^*$ Paranormal operator iff

$$h_kE(w_k^2) \circ T^{-k} - k\mu^{k-1}w(h \circ T)E(w) + (k - 1)\mu^k \geq 0 \quad \text{a.e., for all } \mu > 0.$$  

**Proof.** $W$ is $k^*$ Paranormal

\[ \Leftrightarrow W^{*k}W^k - k\mu^{k-1}WW^* + (k - 1)\mu^k I \geq 0 \text{ a.e.; for every } \mu > 0 \]

\[ \Leftrightarrow \int_E (h_kE(w_k^2) \circ T^{-k} - k\mu^{k-1}w(h \circ T)E(w) + (k - 1)\mu^k I) \mu \geq 0 \]

a.e., for every $\mu > 0$ and for every $E \in \Sigma$ with $\lambda(E) < \infty$

i.e.

$$h_kE(w_k^2) \circ T^{-k} - k\mu^{k-1}w(h \circ T)E(w) + (k - 1)\mu^k \geq 0 \text{ a.e., for every } \mu > 0.$$ 

\[ \Box \]

**Corollary 2.2.** Let $T^{-1}\Sigma = \Sigma$ Then, $W$ is $k^*$ Paranormal operator iff

$$h_kw_k^2 \circ T^{-k} - k\mu^{k-1}w(h \circ T)w + (k - 1)\mu^k \geq 0 \quad \text{a.e., for every } \mu > 0.$$ 

**Theorem 2.3.** Let $W \in B(L^2(\lambda))$ and $k \geq 1$. Then, $W$ is of class $(M, k)^*$ operators iff

$$h_kE(w_k^2) \circ T^k + 2\mu(w(h \circ T)E(W))^k + \mu^2h_kE(w_k^2) \geq 0 \text{ a.e., for every } \mu > 0.$$
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Proof. $W$ is of class $(M, k)^*$

\[ \iff W^* w^k + 2\mu(WW^*)^k + \mu^2 W^*kW^k \geq 0 \quad \text{a.e., for every } \mu > 0 \]

\[ \iff \int_E [h_k E(w_k^2) \circ T^{-k} + 2\mu(w(h \circ T)E(w)) + \mu^2 h_k E(w_k^2)]d\lambda \geq 0, \]

for every $\mu > 0$ and for every $E \in \Sigma$ with $\lambda(E) < \infty$

\[ \iff h_k E(w_k^2) \circ T^{-k} + 2\mu(w(h \circ T)E(w))^k + \mu^2 h_k E(w_k^2) \geq 0 \quad \text{a.e., for every } \mu > 0. \]

\[ \square \]

Corollary 2.4. Let $T^{-1}\Sigma = \Sigma$. Then, $W$ is of class $(M, k)^*$ operators iff $(w(h \circ T)w)^k \leq h_k(w_k^2)$.

References


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