Combinatorial Approach to M/M/1 Queues Using Hypergeometric Functions

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Abstract

In this paper we use a well known reflection principle for lattice path counting to apply for the analysis of the M/M/1 queuing system. The joint distribution of i arrivals and j departures over a time interval of length t is obtained, when there are k customers in the beginning of the system. The derivation uses the lattice path approach between two points, in two dimensional x - y plane with certain restrictions. Finally a known result of queue length is verified from Satty (1961).

Keywords: M/M/1 queues, Combinatorial Method, Lattice Path, Idle period, Bessel's function, Hypergeometric Functions

1. INTRODUCTION

Mohanty and Panny (1990) obtained a transient solution by first discretizing the continuous time model and then representing the same by a lattice path. Sen, Jain and Gupta (1993) used the same approach for solving the M/M/1 model.

In this paper we use the lattice path approach for solving M/M/1 queuing model and obtain an expression for the probability ($P_{i,j;k}(t)$) of i arrivals, j departures in the time interval of length t , under the assumption that the number of customers in the system at time $t=0$ is k . We then use this result to verify a known result of Satty (1961). For some related work based on lattice path combinatorics and applications,

one may refer to Mohanty (1979), Csaki, Mohanty and Saran (1990), Saran and Rani (1994) and Saran (1994, 1996).

Combinatorial Approach

For determining the transient solution, we first discretize the model by dividing the time interval $(0, t)$ into sub intervals(slots) of length h . In each subinterval, one of the following three mutually exclusive events occurs, viz.

1. Arrival of a customer
2. Departure of a customer
3. Neither arrival nor departure of a customer.

The events in different time slots are independent and each can be represented on graph by taking a unit horizontal step for an arrival, a unit vertical step for each departure and stay at the position if neither arrival nor departure of an event occurs. In other words, we can say that if at any point, the system is at state (u, v) , it takes one of the positions $(u+1, v)$ or $(u, v+1)$ or (u, v) in the next subinterval of length h .

Let the arrivals occur according to Poisson process with rate λ and service times, independent of arrivals, are iid exponentially distributed with mean $1/\mu$.

During busy period, in a time slot of length h

Probability of an arrival is: $\lambda h + o(h)$

Probability of a departure is: $\mu h + o(h)$ and

Probability of neither arrival nor departure is: $1 - (\lambda + \mu) h + o(h)$.

Whenever the system is empty, server sits idle, till a first customer arrives. Server starts providing service immediately after the first arrival.

During idle period

Probability of an arrival is: $\lambda h + o(h)$

Probability of a departure is: $o(h)$ and

Probability of neither arrival nor departure is: $1 - \lambda h + o(h)$.

2. COMPUTATION OF $P_{i,j,k}(t)$

Let the system become empty r times in the period of duration t .

Case (i) Let $r = 0$.

Fig. 1

In this case the server remains busy in $(0,t)$ and the corresponding lattice path from $A(k,0)$ to $B(k+i,j)$, as shown in Fig. 1, does not touch the line $y=x$.

Therefore the number of paths from $(k,0)$ to $(k+i,j)$ not touching the line $y=x$ is

given as $\binom{i+j}{j} - \binom{i+j}{i+k}$ and hence

$$\begin{aligned}
 P_{i,j,k}(t) &= \lim_{h \rightarrow 0} \left(\frac{t}{h} \right) \left[\binom{i+j}{j} - \binom{i+j}{i+k} \right] (\lambda h)^i (\mu h)^j (1 - (\lambda + \mu)h)^{\frac{t}{h} - (i+j)} \\
 &= \left[\binom{i+j}{j} - \binom{i+j}{i+k} \right] \cdot \frac{(\lambda t)^i (\mu t)^j}{(i+j)!} e^{-(\lambda + \mu)t} \quad (1)
 \end{aligned}$$

In order to find an expression for queue length at time t , we substitute $k+i-j = p$ or $i = p + j - k$ and sum over j (from 0 to ∞), we get the same

$$\begin{aligned}
&= \left(\sqrt{\frac{\lambda}{\mu}} \right)^{p-k} \sum_{j=0}^{\infty} \left[\frac{(\sqrt{\lambda\mu} \cdot t)^{p+2j-k}}{j!(p+j-k)!} - \frac{(\sqrt{\lambda\mu} \cdot t)^{p+2j-k}}{(p+j)!(j-k)!} \right] e^{-(\lambda+\mu)t} \\
&= \left(\sqrt{\frac{\lambda}{\mu}} \right)^{n-k} \left[I_{p-k}(2\sqrt{\lambda\mu} \cdot t) - I_{p+k}(2\sqrt{\lambda\mu} \cdot t) \right] e^{-(\lambda+\mu)t} \quad (2)
\end{aligned}$$

where $I_n(x) = \sum_{j=0}^{\infty} \frac{x^{n+2j}}{j!(n+j)!}$ is modified Bessel's function of first kind.

Case (ii) Let $r \neq 0$.

Fig. 2

In this case, the server sits idle r times in duration of time interval $(0, t)$ and he has to wait r times for single server. Let T_q denote q^{th} waiting time for single server, $q=1, 2, \dots, r$. In the corresponding lattice path (see Fig. 2) from $A(k, 0)$ to $B(k+i, j)$, we

show this by r contact points with the line $y=x$ and after each contact point the server sits idle for time $T_q, q=1,2,\dots,r$ for single horizontal step (single arrival). The above Fig. 2 demonstrates the arrival departure process for single server, who sits idle three times ($r=3$).

The number of paths from $(k,0)$ to $(k+i, j)$ which touches the line $y=x$, r times is =

$$\binom{i+j-r}{k+i-1} - \binom{i+j-r}{i+k}.$$

In this case

$$\begin{aligned} P_{i,j,k}(t) &= \lim_{h \rightarrow 0} \sum_{r=0}^{j-(k-1)} \sum_{T_1+T_2+\dots+T_r \leq t} \binom{i+j-r}{k+i-1} - \binom{i+j-r}{i+k} \\ &\quad \times (\lambda h)^{i-r} (\mu h)^j (1 - (\lambda + \mu)h)^{\frac{t-T}{h} - (i+j-r)} \cdot \prod_{q=1}^r (\lambda h)(1 - \lambda h)^{\frac{T_q}{h} - 1} \\ &= \sum_{r=1}^{j-(k-1)} \int_{T_1+T_2+\dots+T_r \leq t} \left[\binom{i+j-r}{k+i-1} - \binom{i+j-r}{i+k} \right] \\ &\quad \times \frac{(\lambda(t-T))^{i-r} (\mu(t-T))^j}{(i+j-r)!} \cdot e^{-(\lambda+\mu)(t-T)} \cdot e^{-\lambda T} \lambda^r dT_1 dT_2 \dots dT_r \end{aligned}$$

where,

$$\begin{aligned} e^{-\lambda T - (\lambda+\mu)(t-T)} &= e^{-\lambda t - \mu(t-T)} \\ &= e^{-\lambda t} \sum_{\beta=0}^{\infty} \frac{(-\mu(t-T))^{\beta}}{\beta!} \end{aligned}$$

and, therefore,

$$\begin{aligned} P_{i,j,k}(t) &= \sum_{r=1}^{j-(k-1)} \left[\binom{i+j-r}{k+i-1} - \binom{i+j-r}{i+k} \right] \sum_{\beta=0}^{\infty} \frac{\lambda^i \mu^j (-\mu)^{\beta}}{(i+j-r)! \beta!} e^{-\lambda t} \int_{T_1+T_2+\dots+T_r \leq t} (t-T)^{i+j-r+\beta} dT_1 dT_2 \dots dT_r \\ &= \sum_{r=1}^{j-(k-1)} \left[\binom{i+j-r}{k+i-1} - \binom{i+j-r}{i+k} \right] \sum_{\beta=0}^{\infty} \frac{\lambda^i \mu^j (-\mu)^{\beta}}{(i+j-r)! \beta!} e^{-\lambda t} \left[\frac{(i+j-r+\beta)!}{(i+j-r)!} \cdot t^{i+j-\beta} \right] \\ &= \sum_{r=1}^{j-(k-1)} \left[\binom{i+j-r}{k+i-1} - \binom{i+j-r}{i+k} \right] \frac{(\lambda t)^i (\mu t)^j}{(i+j-r)!} \sum_{\beta=0}^{\infty} \left[\frac{(i+j-r+\beta)!}{(i+j-r)!} \frac{(-\mu t)^{\beta}}{\beta!} \right] e^{-\lambda t} \\ &= \sum_{r=1}^{j-(k-1)} \left[\binom{i+j-r}{k+i-1} - \binom{i+j-r}{i+k} \right] \frac{(\lambda t)^i (\mu t)^j}{(i+j-r)!} \cdot {}_1F_1(i+j-r+1; i+j+1; -\mu t) e^{-\lambda t} \end{aligned}$$

Using the result ${}_1F_1(\alpha; \gamma; x) = e^x {}_1F_1(\gamma - \alpha; \gamma; -x)$, we get

$${}_1F_1(i + j - r + 1; i + j + 1; -\mu t) = e^{-\mu t} {}_1F_1(r; i + j + 1; \mu t)$$

and, therefore,

$$P_{i,j;k}(t) = e^{-(\lambda+\mu)t} \sum_{r=1}^{j-(k-1)} \left[\binom{i+j-r}{k+i-1} - \binom{i+j-r}{k+i} \right] \frac{(\lambda t)^i (\mu t)^j}{(i+j-r)!} {}_1F_1(r; i+j+1; \mu t).$$

Now using the result ${}_1F_1(r; i+j+1; \mu t) = \frac{1}{B(r, i+j-r+1)} \int_0^1 (1-\theta)^{i+j-r} \theta^{r-1} e^{\mu t \theta} d\theta$

we get

$$\begin{aligned} P_{i,j;k}(t) &= e^{-(\lambda+\mu)t} \sum_{r=1}^{j-(k-1)} \left[\frac{1}{(k+i-1)!(j-(k+r)+1)!} - \frac{1}{(k+i)!(j-k-r)!} \right] \frac{(\lambda t)^i (\mu t)^j}{(r-1)!} \\ &\quad \times \int_0^1 (1-\theta)^{i+j-r} \theta^{r-1} e^{\mu t \theta} d\theta. \\ &= e^{-(\lambda+\mu)t} (\lambda t)^i (\mu t)^j \sum_{r=1}^{j-(k-1)} \left[\frac{1}{(k+i-1)!(j-(k+r)+1)!(r-1)!} - \frac{1}{(k+i)!(j-k-r)!(r-1)!} \right] \\ &\quad \times \int_0^1 \left(\frac{\theta}{1-\theta} \right)^{r-1} (1-\theta)^{i+j-1} e^{\mu t \theta} d\theta. \\ &= e^{-(\lambda+\mu)t} (\lambda t)^i (\mu t)^j \int_0^1 \left\{ \frac{1}{(k+i-1)!(j-k)!} \sum_{r=1}^{j-(k-1)} \binom{j-k}{r-1} \left(\frac{\theta}{1-\theta} \right)^{r-1} - \frac{1}{(k+i)!(j-k-1)!} \sum_{r=1}^{j-(k-1)} \binom{j-k-1}{r-1} \left(\frac{\theta}{1-\theta} \right)^{r-1} \right\} \\ &\quad \times (1-\theta)^{i+j-1} e^{\mu t \theta} d\theta \\ &= e^{-(\lambda+\mu)t} (\lambda t)^i (\mu t)^j \int_0^1 \left\{ \frac{1}{(k+i-1)!(j-k)!} \left(1 + \frac{\theta}{1-\theta} \right)^{j-k} - \frac{1}{(k+i)!(j-k-1)!} \left(1 + \frac{\theta}{1-\theta} \right)^{j-(k+1)} \right\} \\ &\quad \times (1-\theta)^{i+j-1} e^{\mu t \theta} d\theta \\ &= e^{-(\lambda+\mu)t} (\lambda t)^i (\mu t)^j \int_0^1 \left\{ \frac{(1-\theta)^{i+k-1}}{(k+i-1)!(j-k)!} - \frac{(1-\theta)^{i+k}}{(k+i)!(j-k-1)!} \right\} e^{\mu t \theta} d\theta \end{aligned}$$

$$\begin{aligned}
&= e^{-(\lambda+\mu)t} (\lambda t)^i (\mu t)^j \sum_{q=0}^{\infty} \left\{ \frac{(\mu t)^q}{(k+i-1)!(j-k)! q!} \int_0^1 (1-\theta)^{i+k-1} \theta^q d\theta - \frac{(\mu t)^q}{(k+i)!(j-(k+1))! q!} \int_0^1 (1-\theta)^{i+k} \theta^q d\theta \right\} \\
&= e^{-(\lambda+\mu)t} (\lambda t)^i (\mu t)^j \sum_{q=0}^{\infty} \left\{ \frac{(\mu t)^q}{(k+i-1)!(j-k)! q!} B(q+1, i+k) - \frac{(\mu t)^q}{(k+i)!(j-(k+1))! q!} B(q+1, i+k+1) \right\} \\
&= e^{-(\lambda+\mu)t} (\lambda t)^i (\mu t)^j \sum_{q=0}^{\infty} \left\{ \frac{1}{(k+i+q)!(j-k)!} - \frac{1}{(k+i+q+1)!(j-(k+1))!} \right\} (\mu t)^q \\
&= e^{-(\lambda+\mu)t} (\lambda t)^i (\mu t)^j \sum_{q=0}^{\infty} \left\{ \frac{2k+i-j+q+1}{(k+i+q+1)!(j-k)!} \right\} (\mu t)^q \tag{3}
\end{aligned}$$

which is the joint distribution for i arrivals and j departures in time interval of length t .

3. COMPUTATION OF AN EXPRESSION FOR QUEUE LENGTH

To find an expression for queue length putting $k+i-j=p$ (or $i=p+j-k$) and summing over j (from 0 to ∞), we get the same

$$\begin{aligned}
&= e^{-(\lambda+\mu)t} \sum_{j=0}^{\infty} \left\{ (\lambda t)^{p+j-k} (\mu t)^j \sum_{q=0}^{\infty} \left\{ \frac{2k+(p+k-k)-j+q+1}{(k+(p+j-k)+q+1)!(j-k)!} \right\} (\mu t)^q \right\} \\
&= e^{-(\lambda+\mu)t} \sum_{q=0}^{\infty} \sum_{j=0}^{\infty} \left\{ \frac{(\lambda t)^{p+j-k} (\mu t)^{j+q}}{(j-k)!(q+j+p)!} - \frac{(\lambda t)^{p+j-k} (\mu t)^{j+q}}{(j-k-1)!(q+j+p+1)!} \right\} \\
&= e^{-(\lambda+\mu)t} \left\{ \sum_{q=0}^{\infty} \left(\sqrt{\frac{\lambda}{\mu}} \right)^{p+q-k} \left\{ \sum_{j=0}^{\infty} \frac{(\sqrt{\lambda\mu} \cdot t)^{2j+q+p-k}}{(j-k)!(q+j+p)!} - \sum_{j=0}^{\infty} \frac{(\sqrt{\lambda\mu} \cdot t)^{2j+q+p-k}}{(j-k-1)!(q+j+p+1)!} \right\} \right\}
\end{aligned}$$

It may be noted that within the inner curly brackets, the limits in the first summation are $j \geq k$, and those in the second summation are $j \geq k+1$. Therefore, putting $j-k=s$ in the first summation and $j-k-1=s$ in the second summation and summing over s (from 0 to ∞), we get the queue length

$$\begin{aligned}
&= e^{-(\lambda+\mu)t} \left\{ \sum_{q=0}^{\infty} \left(\sqrt{\frac{\lambda}{\mu}} \right)^{p+q-k} \left\{ \sum_{s=0}^{\infty} \frac{(\sqrt{\lambda\mu} \cdot t)^{2s+q+p+k}}{(s)!(q+p+k+s)!} - \sum_{s=0}^{\infty} \frac{(\sqrt{\lambda\mu} \cdot t)^{2s+q+p+k+2}}{(s)!(q+p+k+s+2)!} \right\} \right\} \\
&= e^{-(\lambda+\mu)t} \left\{ \sum_{q=0}^{\infty} \left(\sqrt{\frac{\lambda}{\mu}} \right)^{p+q-k} \left\{ I_{q+p+k}(2\sqrt{\lambda\mu} \cdot t) - I_{q+p+k+2}(2\sqrt{\lambda\mu} \cdot t) \right\} \right\} \\
&= e^{-(\lambda+\mu)t} \left(\sqrt{\frac{\lambda}{\mu}} \right)^{2p+1} \left\{ \sum_{q=0}^{\infty} \left(\sqrt{\frac{\lambda}{\mu}} \right)^{q-p-k-1} \left\{ I_{q+p+k}(2\sqrt{\lambda\mu} \cdot t) - I_{q+p+k+2}(2\sqrt{\lambda\mu} \cdot t) \right\} \right\} \\
&= e^{-(\lambda+\mu)t} \left(\sqrt{\frac{\lambda}{\mu}} \right)^{2p+1} \left\{ \sum_{r=p+k+1}^{\infty} \left(\sqrt{\frac{\mu}{\lambda}} \right)^r \left\{ I_{r-1}(2\sqrt{\lambda\mu} \cdot t) - I_{r+1}(2\sqrt{\lambda\mu} \cdot t) \right\} \right\} \quad (4)
\end{aligned}$$

Since events represented in Case (i) and Case (ii) are mutually exclusive, therefore, the joint distribution of i arrivals and j departures in time interval of length t is given by the sum of equations (1) and (3) and the expression for queue length at time t is given by the addition of (2) and (4), i.e.,

$$\begin{aligned}
&= e^{-(\lambda+\mu)t} \left\{ \left(\sqrt{\frac{\lambda}{\mu}} \right)^{p-k} \left\{ I_{p-k}(2\sqrt{\lambda\mu} \cdot t) - I_{p+k}(2\sqrt{\lambda\mu} \cdot t) \right\} + \left(\sqrt{\frac{\lambda}{\mu}} \right)^{2p+1} \sum_{r=p+k+1}^{\infty} \left\{ I_{r-1}(2\sqrt{\lambda\mu} \cdot t) - I_{r+1}(2\sqrt{\lambda\mu} \cdot t) \right\} \right\} \\
&= e^{-(\lambda+\mu)t} \left\{ \left(\sqrt{\frac{\lambda}{\mu}} \right)^{p-k} \left\{ I_{p-k}(2\sqrt{\lambda\mu} \cdot t) - I_{p+k}(2\sqrt{\lambda\mu} \cdot t) \right\} + \left(\sqrt{\frac{\lambda}{\mu}} \right)^{p-k} \left\{ I_{p+k}(2\sqrt{\lambda\mu} \cdot t) - I_{p+k+2}(2\sqrt{\lambda\mu} \cdot t) \right\} \right. \\
&\quad \left. + \left(\sqrt{\frac{\lambda}{\mu}} \right)^{2p+1} \sum_{r=p+k+2}^{\infty} \left\{ I_{r-1}(2\sqrt{\lambda\mu} \cdot t) - I_{r+1}(2\sqrt{\lambda\mu} \cdot t) \right\} \right\} \\
&= e^{-(\lambda+\mu)t} \left\{ \left(\sqrt{\frac{\lambda}{\mu}} \right)^{p-k} \left\{ I_{p-k}(2\sqrt{\lambda\mu} \cdot t) - I_{p+k+2}(2\sqrt{\lambda\mu} \cdot t) \right\} + \left(\sqrt{\frac{\lambda}{\mu}} \right)^{2p+1} \sum_{r=p+k+2}^{\infty} \left\{ I_{r-1}(2\sqrt{\lambda\mu} \cdot t) - I_{r+1}(2\sqrt{\lambda\mu} \cdot t) \right\} \right\} \quad (5)
\end{aligned}$$

Now rewriting

$$\left(\sqrt{\frac{\lambda}{\mu}} \right)^{2p+1} \left\{ \sum_{r=p+k+2}^{\infty} \left(\sqrt{\frac{\mu}{\lambda}} \right)^r \left\{ I_{r-1}(2\sqrt{\lambda\mu} \cdot t) - I_{r+1}(2\sqrt{\lambda\mu} \cdot t) \right\} \right\}$$

$$\begin{aligned}
&= \left(\sqrt{\frac{\lambda}{\mu}}\right)^{2p+1} \left\{ \left(\sqrt{\frac{\mu}{\lambda}}\right)^{p+k+2} I_{p+k+1}(2\sqrt{\lambda\mu}, t) + \sqrt{\frac{\mu}{\lambda}} \sum_{r=p+k+2}^{\infty} \left(\sqrt{\frac{\mu}{\lambda}}\right)^r I_r(2\sqrt{\lambda\mu}, t) + \left(\sqrt{\frac{\mu}{\lambda}}\right)^{p+k+1} I_{p+k+2}(2\sqrt{\lambda\mu}, t) \right. \\
&\quad \left. - \sqrt{\frac{\lambda}{\mu}} \sum_{r=p+k+2}^{\infty} \left(\sqrt{\frac{\mu}{\lambda}}\right)^r I_r(2\sqrt{\lambda\mu}, t) \right\} \\
&= \left(\sqrt{\frac{\lambda}{\mu}}\right)^{p-k-2} I_{p+k+1}(2\sqrt{\lambda\mu}, t) + \left(\sqrt{\frac{\lambda}{\mu}}\right)^{p-k} I_{p+k+2}(2\sqrt{\lambda\mu}, t) + \left(\frac{\lambda}{\mu}\right)^p \left(1 - \frac{\lambda}{\mu}\right) \sum_{r=p+k+2}^{\infty} \left(\sqrt{\frac{\mu}{\lambda}}\right)^r I_r(2\sqrt{\lambda\mu}, t)
\end{aligned}$$

and putting it in (5), we get an expression for queue length as :

$$\begin{aligned}
&= e^{-(\lambda+\mu)} \left\{ \left(\sqrt{\frac{\lambda}{\mu}}\right)^{p-k} I_{p-k}(2\sqrt{\lambda\mu}, t) + \left(\sqrt{\frac{\lambda}{\mu}}\right)^{p-k-1} I_{p+k+1}(2\sqrt{\lambda\mu}, t) \right. \\
&\quad \left. + \left(\frac{\lambda}{\mu}\right)^p \left(1 - \frac{\lambda}{\mu}\right) \sum_{r=p+k+2}^{\infty} \left(\sqrt{\frac{\mu}{\lambda}}\right)^r I_r(2\sqrt{\lambda\mu}, t) \right\}
\end{aligned}$$

which is same as equation (4.24) given in Satty (1961).

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REFERENCES

- [1] D.G. Champernowne, An elementary method of solution of the queueing problem with a single server and constant parameter, J. Roy. Statist. Soc., Ser. B, 18 (1956), 125-128.
- [2] E. Csaki, S.G. Mohanty and J. Saran, On random walks in a plane, Ars Combinatoria, 29 (1990), 309-318.
- [3] S.G. Mohanty, Lattice Path Counting and Applications, Academic Press, 1979.

- [4] S.G. Mohanty, and W. Panny, A discrete time analogue of the M/M/1 queue and the transient solution: A geometric approach, *Sankhya, Ser. A*, 52 (1990), 364-370.
- [5] J. Saran, A note on the compositions of a positive integer, *Ars Combinatoria*, 38 (1994), 213-224.
- [6] J. Saran, Some distributions related to a random walk in a plane, *Ars Combinatoria*, 44 (1996), 119-127.
- [7] J. Saran and S. Rani, Some joint distributions concerning random walk in a plane, *Ars Combinatoria*, 38 (1994), 33-45.
- [8] T. L.Satty, *Elements of Queueing Theory with Applications*, McGraw Hill, New York, 1961.
- [9] K. Sen, J.L. Jain and J. M. Gupta, Lattice path approach to transient solution of M/M/1 with (0,k) control policy, *J. Statist. Plann. Infer.*, 34, No. 2 (1993), 259-267.

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