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Rectilinear Crossing Number of a Zero Divisor Graph

M. Malathi, S. Sankeetha and J. Ravi Sankar

Department of Mathematics, Saradha Gangadharan College
Puducherry, India - 605 004
malathisundar@hotmail.com, ravisankar.maths@gmail.com

S. Meena

Department of Mathematics, Government Arts College
Chidambaram, India - 608 104
meenasaravanan14@gmail.com

Abstract

In this paper, we evaluate the rectilinear crossing number of $\Gamma(Z_n)$. We mainly focus on finding the rectilinear drawing of zero divisor complete graphs especially for $p = 7, 11$. Finally, we compare the rectilinear crossing number with the crossing number thereby forming an inequality by different conjectures.

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1 Introduction

Let **R** be a commutative ring with unity and let Z(R) be its set of zero divisors. The zero divisor graph of **R** denoted by $\Gamma(R)$ is a graph which is undirected with vertices $Z(R)^* = Z(R) - \{0\}$, the set of non-zero divisors of R, and for distinct $x, y \in Z(R)^*$, the vertices x and y are adjacent iff $xy = 0$. Throughout this paper, we consider the commutative ring by R and the zero divisor graph $\Gamma(R)$ by $\Gamma(Z_n)$.The idea of a zero divisor graph of a commutative ring was

introduced by I. Beck [2]. For notation and graph theory terminology are considered as in [1, 6, 7, 8].

The crossing number cr(G) of a graph G is the minimum number of edge crossings among the drawings of G in the plane such that the edges of G are Jordan arcs [3]. The rectilinear crossing number of a graph with minimum number of edge crossing drawn in a plane satisfies the following conditions: (i) edges are line segments. (ii) no three vertices are collinear. (iii) no three edges may intersect at a common vertex. The rectilinear crossing number of G is denoted by $\bar{cr}(G)$. It follows from the definition that, $cr(G) \leq \bar{cr}(G)$. Turan's brick factory problem asks the question of finding the crossing number of a complete bipartite graph. Zarankiewicz [9] proposed a solution to the problem in 1953. Guy [4] points out an error in Zarankiewicz proof that was discovered in 1965 by Kainen and Ringel. Recently, J. Ravi Sankar et. al [7] find out the crossing number of zero divisor graph and they gave a conjecture for $cr(\Gamma(Z_{pq})) = (p-1)(p-3)(q-1)(q-3)/16$ and $cr(\Gamma(Z_{p^2})) = (p-1)(p-3)^2(p-5)/64$. The rectilinear crossing for $k_{m,n}$ can be obtained by following its definition [3]. H.F.Jensen [5] gave an upper bound for rectilinear crossing number which availed itself by a formula, denoted by $j(n)$, for the exact number of edge crossings, in particular, $\bar{cr}(K_n) \leq j(n) = \frac{1}{432}\lfloor 7p^4 - 84p^3 + 354p^2 - 596p + 427 \rfloor$.

2 Crossing Number of a Zero Divisor Graph

In this section, we recall the crossing number of zero divisor graphs .

Lemma 2.1 [6] *A graph $\Gamma(Z_n)$ is connected if and only if n is a composite number.*

Theorem 2.2 [8] *In $\Gamma(Z_{2p})$, where p is any prime, then $cr(\Gamma(Z_{2p})) = 0$.*

Theorem 2.3 [8] *For any graph $\Gamma(Z_{3p})$, where p is any prime > 3, then $cr(\Gamma(Z_{3p})) = 0$.*

Theorem 2.4 [8] *For any graph $\Gamma(Z_{5p})$, where $p > 5$ is any prime, then $cr(\Gamma(Z_{5p})) = 2\left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{p-2}{2} \right\rfloor$.*

Theorem 2.5 [8] *For any prime $p \geq 3$, then $cr(\Gamma(Z_{4p})) = (p-1)(p-3)/4$.*

Theorem 2.6 [8] *If p and q are distinct prime numbers with $q > p$ then, $cr(\Gamma(Z_{pq})) = \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{p-2}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor \left\lfloor \frac{q-2}{2} \right\rfloor$.*

Theorem 2.7 [8] *If p is any prime, then $cr(\Gamma(Z_{p^2})) = \frac{1}{4}\left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{p-2}{2} \right\rfloor \left\lfloor \frac{p-3}{2} \right\rfloor \left\lfloor \frac{p-4}{2} \right\rfloor$.*

Theorem 2.8 [8] *For any graph $\Gamma(Z_{pqr})$ with $p = 2$, $q = 3$ and r is any prime then, $cr(\Gamma(Z_{pqr})) = (r-1)(2r-5)/2$.*

3 Rectilinear Crossing Number of $\Gamma(Z_n)$

In this section, we evaluate the rectilinear crossing number of a zero divisor graph in a commutative ring.

Theorem 3.1 *In $\Gamma(Z_{2p})$, where p is any prime, then $\bar{cr}(\Gamma(Z_{2p})) = 0$.*

Proof: *Using theorem (2.2), $\Gamma(Z_{2p})$ is a planar star graph and hence $\bar{cr}(\Gamma(Z_{2p})) = 0$.*

Theorem 3.2 *For any graph $\Gamma(Z_{3p})$, where p is any prime $p > 3$, then $\bar{cr}(\Gamma(Z_{3p})) = 0$.*

Proof: *Using theorem (2.3), $\Gamma(Z_{3p})$ is a planar complete bipartite graph and hence $\bar{cr}(\Gamma(Z_{3p})) = 0$.*

Theorem 3.3 *If p and q are distinct prime numbers with $q > p$ then, $\bar{cr}(\Gamma(Z_{pq})) = (p-1)(p-3)(q-1)(q-3)/16$.*

Proof: *The vertex set of $\Gamma(Z_{pq})$ is $\{p, 2p, .., p(q-1), q, 2q, .., (p-1)q\}$. The proof is by induction on p and q.*

Case(1)*: Let $p = 5$.*

Subcase(i)*: When $q = 7$, the vertex set of $\Gamma(Z_{5q})$ is $\{5, 10, ..., 5(q-1), q, 2q, 3q, 4q\}$. Clearly $|V(\Gamma(Z_{5q}))| = q + 3$. Let u and v be any two vertices in $\Gamma(Z_{5q})$ with maximum and minimum degree, respectively. Let $u = q$ and $v = 10$ then $5q$ must divide uv then u and v are adjacent.*

Let $u = q$ and $v = 2q$ then $5q$ does not divide uv. That is $uv = 2q^2 \neq 0$. Then the vertex set V can be partitioned into two parts X and Y, where $X = \{q, 2q, 3q, 4q\}$ and $Y = \{5, 10, ..., 5(q-1)\}$. Clearly any vertices u and v in X are non-adjacent and the same hold for Y. Let $u = q$ in X and $v = 10$ are in Y then $5q$ divides $uv = 10q$. Finally we note that, every vertex in X are adjacent to all the vertices in Y. Moreover $V(\Gamma(Z_{5q})) = X \cup Y$ and $X \cap Y = \phi$.

If $q = 7 > 5$, then $|X| = 4$ and $|Y| = q-1 = 6$. Let $X = \{v_1, v_2, v_3, v_4\}$ and $Y = \{u_1, u_2, ...u_{q-1}\}$. In a drawing D, we denote by $\bar{cr}_D(v_i, u_j)$ the number of crossings of lines, one terminating at v_i, the other at u_j and by $\bar{cr}_D(v_i)$ the number of crossings lines which terminate at v_i, $\bar{cr}_D(v_i) = \sum_{j=1}^{6} \bar{cr}_D(v_i, u_j)$. Clearly, $\bar{cr}_D(\Gamma(Z_{5q})) = \sum_{i=1}^{4} \sum_{j=1}^{6} \bar{cr}_D(v_i, u_j)$.

Since, the rectilinear crossing number of $\Gamma(Z_{5q})$ is either the sum of all rectilinear crossing number of the vertices in X or the sum of all rectilinear crossing number of the vertices in Y. Now, we consider the vertex set X. The proof is based on having vertices of X placed horizontally and Y is placed randomly so that no three vertices are collinear. The first thing is to place $\left(\frac{q-1}{2}\right)$ vertices on one side of the horizontal line and $\left(\frac{q}{2}\right)$ vertices on the other side. Then we connect all the vertices on the horizontal line with all the vertices placed randomly by means of line segment such that no three edges cross at a common point.

On computing the rectilinear crossing number of $\Gamma(Z_{5q})$, *we get* $\bar{cr}(7) = 0$, $\bar{cr}(14) = 0$, $\bar{cr}(21) = 6$, $\bar{cr}(28) = 6$.

$$\bar{cr}(\Gamma(Z_{5q})) = \sum_{i=1}^{4} \bar{cr}(v_i) = 12 = 2 \times 1 \times 3 \times 2$$
$$= \left(\tfrac{5-1}{2}\right)\left(\tfrac{5-3}{2}\right)\left(\tfrac{7-1}{2}\right)\left(\tfrac{7-3}{2}\right) = \left(\tfrac{p-1}{2}\right)\left(\tfrac{p-3}{2}\right)\left(\tfrac{q-1}{2}\right)\left(\tfrac{q-3}{2}\right)$$

Subcase(ii): *When* $q = 11$.

The vertex set of $\Gamma(Z_{5q})$ *is* $\{5, 10, ..., 5(q-1), q, 2q, 3q, 4q\}$. *Similarly as in subcase(i), we get* $\bar{cr}(11) = 0$, $\bar{cr}(22) = 0$, $\bar{cr}(33) = 20$, $\bar{cr}(44) = 20$.

$$\bar{cr}(\Gamma(Z_{5q})) = \sum_{i=1}^{4} \bar{cr}(v_i) = 40 = 2 \times 1 \times 5 \times 4$$
$$= \left(\tfrac{5-1}{2}\right)\left(\tfrac{5-3}{2}\right)\left(\tfrac{11-1}{2}\right)\left(\tfrac{11-3}{2}\right) = \left(\tfrac{q-1}{2}\right)\left(\tfrac{q-3}{2}\right)\left(\tfrac{q-1}{2}\right)\left(\tfrac{q-3}{2}\right)$$

Case(2): *Let* $p = 7$.

Subcase(i): *When* $q = 11$, *the vertex set of* $\Gamma(Z_{7q})$ *is* $\{7, 14, ..., 7(q-1), q, 2q, 3q, 4q, 5q, 6q\}$. *Let* $u = 7$ *and* $v = q$ *in* $V(\Gamma(Z_{7q}))$ *then* $7q$ *must divide* uv, *which implies that* u *and* v *are adjacent vertices in* $\Gamma(Z_{7q})$. *Let* $u = 7$ *and* $v = 14$ *then* $7q$ *does not divide* uv. *Using theorem* (2.4) *partition the vertex set of* $\Gamma(Z_{7q})$ *into two parts* X *and* Y. *Clearly no two vertices in* X *are adjacent and the same hold for* Y. *Next, we calculate either the sum of the rectilinear crossing number of the vertices in* X *or the sum of the rectilinear crossing number of the vertices in* Y.

Now place the six vertices $\{q, 2q, 3q, 4q, 5q, 6q\}$ *in* X *are placed horizontally and* $(q-1)$ *vertices* $\{7, 14, ..., 7(q-1)\}$ *in* Y *are placed randomly in such a way that no three vertices are collinear. Then, we connect all the vertices placed horizontally with all the vertices that are randomly placed. Clearly,* $\bar{cr}(q) = 0$, $\bar{cr}(2q) = 0$, $\bar{cr}(3q) = 20$, $\bar{cr}(4q) = 20$, $\bar{cr}(5q) = 40$ *and* $\bar{cr}(6q) = 40$. *Then,*

$$\bar{cr}(\Gamma(Z_{7q})) = \sum_{i=1}^{6} \sum_{j=1}^{q-1} \bar{cr}_D(v_i, u_j), \text{ where, } v_i \in X \text{ and } u_j \in Y.$$
$$\bar{cr}(\Gamma(Z_{7q})) = \text{Sum of the rectilinear crossing number in } X \text{ or } Y$$
$$= \bar{cr}(11) + \bar{cr}(22) + \bar{cr}(33) + \bar{cr}(44) + \bar{cr}(55) + \bar{cr}(66) = 120$$
$$= 3 \times 2 \times 5 \times 4 = \left(\tfrac{7-1}{2}\right)\left(\tfrac{7-3}{2}\right)\left(\tfrac{11-1}{2}\right)\left(\tfrac{11-3}{2}\right) = \left(\tfrac{q-1}{2}\right)\left(\tfrac{q-3}{2}\right)\left(\tfrac{q-1}{2}\right)\left(\tfrac{q-3}{2}\right).$$

Subcase(ii): *When* $q = 13$.

The vertex set of $\Gamma(Z_{7q})$ *is* $\{7, 14, ..., 7(q-1), q, 2q, 3q, 4q, 5q, 6q\}$. *Now proceeding as in subcase(i), we get the rectilinear crossing as,*

$$\bar{cr}(\Gamma(Z_{5q})) = 180 = 3 \times 2 \times 6 \times 5 = \left(\tfrac{7-1}{2}\right)\left(\tfrac{7-3}{2}\right)\left(\tfrac{13-1}{2}\right)\left(\tfrac{13-3}{2}\right)$$
$$\bar{cr}(\Gamma(Z_{pq})) = \left(\tfrac{p-1}{2}\right)\left(\tfrac{p-3}{2}\right)\left(\tfrac{q-1}{2}\right)\left(\tfrac{q-3}{2}\right) = (p-1)(p-3)(q-1)(q-3)/16.$$

Theorem 3.4 *For any graph* $\Gamma(Z_{pqr})$, *where* $p = 2$, $q = 3$ *and* $r > 3$ *then* $\bar{cr}(\Gamma(Z_{pqr})) = (r-1)(r-2)$.

Proof: *The vertex set of* $\Gamma(Z_{pqr})$ *is* $\{2, .., 2(qr-1), 3, .., 3(pr-1), r, 2r, .., 5r\}$. *Let* P *be the set of all pendant vertices in* $\Gamma(Z_{pqr})$. *Clearly* P *is a multiple of 2 other than multiple of 3. Let* $u = 2$ *and* $v = \frac{pqr}{2}$, *then* uv *is divided by* pqr *which implies that* u *and* v *are adjacent. Let* $u = 3$ *and* $v = 2r$, *then* pqr *divides* uv *i.e,* u *and* v *are adjacent. Let* S *be the set containing the common multiples of 2 and 3. Let* $u = 18$ *and* $v = r$. *Then* pqr *divides* uv. *Clearly any*

vertices $v \in S$ are adjacent to multiple of r. Let Y be the set which contains multiple of 3 other than multiple of 2. Then let $u = 3$ and $v = 2r$. Clearly pqr divides uv i.e u and v are adjacent. Similarly let $u = 3$ and $v = 4r$, then u and v are adjacent which implies $2r$ and $4r$ are adjacent to multiples of 3. Now place the multiples of r horizontally and placing the remaining vertices other than the pendant vertices randomly in such a way that no three vertices are collinear, no edges crosses itself and the edges are line segments.

***Case(i)**: Let $r = 5$.*

The vetex set $V(\Gamma(Z_{30}))$ can be partitioned into two parts X and Y where $X = \{5, 10, 15, 20, 25\}$ and $Y = \{3, 6, 9, 12, 18, 21, 24, 27\}$ then $\bar{cr}(10) = 0$, $\bar{cr}(20) = 0$, $\bar{cr}(25) = 2$,$\bar{cr}(15) = 6$, and $\bar{cr}(5) = 4$.

$$\bar{cr}(\Gamma(Z_{30})) = \sum_{i=1}^{5} \sum_{j=1}^{8} \bar{cr}_D(v_i, u_j), \text{ where, } v_i \in X \text{ and } u_j \in Y.$$

Then, $\bar{cr}(\Gamma(Z_{30}))$ =sum of all crossing in X or Y

$$\begin{aligned}\bar{cr}(\Gamma(Z_{30})) &= \bar{cr}(5) + \bar{cr}(10) + \bar{cr}(15) + \bar{cr}(20) + \bar{cr}(25) = 12\\ &= (5-1)(5-2) = (r-1)(r-2).\end{aligned}$$

***Case(ii)**: Let $r = 7$.*

The vertex set $V(\Gamma(Z_{42}))$ can be partitioned into two parts X and Y with $X = \{7, .., 35\}$ and $Y = \{3, 6, 9, 12, 15, 18, 24, 27, 30, 33, 36, 39\}$ then $\bar{cr}(7) = 0$, $\bar{cr}(35) = 0$, $\bar{cr}(21) = 12$,$\bar{cr}(14) = 6$, and $\bar{cr}(28) = 12$.

$$\bar{cr}(\Gamma(Z_{42})) = \sum_{i=1}^{5} \sum_{j=1}^{12} \bar{cr}_D(v_i, u_j), \text{ where, } v_i \in X \text{ and } u_j \in Y.$$

Then, $\bar{cr}(\Gamma(Z_{42}))$ =sum of all crossing in X or Y

$$\begin{aligned}&=\bar{cr}(7) + \bar{cr}(14) + \bar{cr}(21) + \bar{cr}(28) + \bar{cr}(35)\\ &= 30 = (7-1)(7-2) = (r-1)(r-2).\end{aligned}$$

Theorem 3.5 *For any prime $p \geq 3$, then $\bar{cr}(\Gamma(Z_{4p})) = (p-1)(p-3)/4$.*

In the next section, we introduce the progress on relationship between crossing number and rectilinear crossing number of zero divisor graph.

4 Relationship between $cr(\Gamma(Z_n))$ and $\bar{cr}(\Gamma(Z_n))$

Our present interest is that of finding $\bar{cr}(\Gamma(Z_{p^2}))$ and comparing the same with $cr(\Gamma(Z_{p^2}))$. Before that let us briefly discuss the rectilinear drawings of the complete graphs $\Gamma(Z_{p^2})$ especially for $p = 7, 11$. Using theorem (2.8), $\Gamma(Z_{p^2})$ is a complete graph.

Theorem 4.1 *A rectilinear drawing of $\Gamma(Z_{p^2})$, $p = 7$, comprising of nested non-concentric triangles has three crossings.*

Proof:

To prove this, first we discuss k_5 principle, which says that for a drawing of k_n with triangular convex hull A and $(n-3)$ vertices contained within it is B, then the drawing has exactly $\binom{n-3}{2}$ $AB \times BA$ edge crossings. Therefore by

using k_5 principle the drawing of k_n has $\binom{3}{2}$ edge crossings. That is rectilinear drawing of $\Gamma(Z_{p^2})$, where $p = 7$ has exactly three $AB \times BA$ edge crossings.

Theorem 4.2 $\bar{cr}(\Gamma(Z_{p^2})) = 62$ *where* $p = 11$.

Proof: *The vertex set of $\Gamma(Z_{p^2})$ is $\{p, 2p, ..p(p-1)\}$. The drawing of $\Gamma(Z_{p^2})$ comprises non-concentric triangles in which there are three triangular convex hull A,B,C containing one in another and the tenth vertex is placed in such a way that the inner most hull forms a convex quadrilateral. Obviously, if the inner most hull C is a convex quadrilateral, then it is not concentric with the outer triangle B. Also if the tenth vertex is placed such that the second hull forms convex quadrilateral, even then the triangles seems to be non-concentric. The rectilinear drawing of $\Gamma(Z_{p^2})$ where $p = 11$ cannot have fewer than 60 crossings since $cr(\Gamma(Z_{p^2})) = 60$, and naturally $\bar{cr}(\Gamma(Z_{p^2})) > 60$. Therefore, the first two hulls of an optimal rectilinear drawing of $\Gamma(Z_{p^2})$ will consist of two nested triangles A, B containing a convex quadrilateral C . The rectilinear crossing number of $\Gamma(Z_{p^2})$ can be found as illustrated below.*

First we will place the 4 vertices $\{p, 2p, 3p, 4p\}$ such that they form a convex quadrilateral. Clearly, $\bar{cr}(k_4) = 1$. Place the next three vertices $\{5p, 6p, 7p\}$ such a way that they are not concentric with the inner convex hull C. The rectilinear crossing of the outer triangle B with the inner quadrilateral C is 10. Then the total $\bar{cr}(k_7) = 10 + 1 = 11$.

Now place the next three vertices $\{8p, 9p, 10p\}$ in such a way that they are non concentric with the previous hull. Then the rectilinear crossing of the outer 3 vertices with the inner quadrilateral C and its succeding hull B is 51. Therefore the total number of crossing is $\bar{cr}(\Gamma(Z_{p^2})) = 11 + 51 = 62$.

Theorem 4.3 *If p is any prime, $cr(\Gamma(Z_{p^2})) \leq \bar{cr}(\Gamma(Z_{p^2})) \leq j(p-1)$.*

Proof: *The vertex set of $\Gamma(Z_{p^2})$ is $p, 2p, 3p.....2(p-1)$. Let $u = 2p$, $v = 3p$. Then p^2 must divide uv. Clearly any two vertices in $V(\Gamma(Z_{p^2}))$ are adjacent vertices and $|V(\Gamma(Z_{p^2}))| = p - 1$. Since, p is any prime number then (p-1) is even, then the following cases arises.*

Case(i): *$\Gamma(Z_{p^2})$ is a planar graph when $p = 2, 3, 5$. Then, $\bar{cr}(\Gamma(Z_{p^2})) = 0$ and $cr(\Gamma(Z_{p^2})) = 0$. Clearly, $\bar{cr}(\Gamma(Z_{p^2})) = cr(\Gamma(Z_{p^2})) = 0$.*

Case(ii): *When $p = 7$, $cr(\Gamma(Z_7)) = \frac{(7-1)(7-3)^2(7-5)}{64} = 3$. By theorem (4.1), $\bar{cr}(\Gamma(Z_7)) = 3$ and $j(p-1) = \frac{1}{432}\lfloor 7(6)^4 - 84(6)^3 + 354(6)^2 - 596(6) + 427 \rfloor = \left\lfloor \frac{1692}{432} \right\rfloor = 3$. This implies that $\bar{cr}(\Gamma(Z_{p^2})) = cr(\Gamma(Z_{p^2})) = j(p-1) = 3$.*

Case(iii): *When $p = 11$, $cr(\Gamma(Z_{11})) = \frac{(11-1)(11-3)^2(11-5)}{64} = 60$. By theorem (4.2), $\bar{cr}(\Gamma(Z_{11})) = 62$. Then,*

$j(p-1) = \frac{1}{432}\lfloor 7(10)^4 - 84(10)^3 + 354(10)^2 - 596(10) + 427 \rfloor = \left\lfloor \frac{27388}{432} \right\rfloor = 63$. This implies that, $cr(\Gamma(Z_{p^2})) < \bar{cr}(\Gamma(Z_{p^2})) < j(p-1)$. Using all the above cases, $cr(\Gamma(Z_{p^2})) \leq \bar{cr}(\Gamma(Z_{p^2})) \leq j(p-1)$.

Theorem 4.4 *For any graph* $\Gamma(Z_{pqr})$*, where* $p = 2$*,* $q = 3$ *and* $r > 3$ *then* $\bar{cr}(\Gamma(Z_{pqr})) = cr(\Gamma(Z_{pqr})) + (r-1)/2$.

Proof: *Using theorem*(3.4), $\bar{cr}(\Gamma(Z_{pqr})) = (r-1)(r-2)$

$$= ((r-1)(2r-5))/2 + (r-1)/2$$

Since, $cr(\Gamma(Z_{pqr})) = (r-1)(2r-5)/2$.

Hence, $\bar{cr}(\Gamma(Z_{pqr})) = cr(\Gamma(Z_{pqr})) + (r-1)/2$.

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