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A Short Note on Universality of Some Quadratic Forms

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Abstract

In a paper by J. Deutsch [1], a quaternionic proof of the universality of seven quaternary quadratic forms was given. The proof relies on a construction very similar to that of Hurwitz quaternions, and its associated division algorithm. Of course, these results are evident, if one uses the Conway-Schneeberger Fifteen Theorem [2], as the author also mentioned, however it is interesting to give a direct proof for some specific quadratic forms based on simple argument. It is the purpose of this short note to prove five of the seven quadratic forms mentioned and proven by Deutsch, using the universality of the classical quadratic form associated to the celebrated Lagrange's Theorem of Four Squares and Euler's trick.

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1 Introduction

Lagrange's Theorem of Four Squares says that any positive integer can be expressed as a sum of four integer squares. A widely circulated proof is the one based on Euler [3]. We will say that a positive definite quadratic form is universal if it represents all positive integers. Accordingly the quadratic form x^2+y^2+

z^2+w^2 would be universal by Lagrange's Theorem of Four Squares. As a short hand, we will denote this quadratic form by $(1,1,1,1)$ and we will prove the universality of quadratic forms of $(1,1,1,4),(1,1,2,8),(1,1,2,2),(1,2,2,4)$, and $(1,2,4,8)$, by assuming the universality of $(1,1,1,1)$.

In the course of the following proofs, we will use a very simple case of Euler's trick: If x,y are integers of the same parity, then there exist integers z,w such that $x^2+y^2=2(z^2+w^2)$. This is because of the following identity

$$x^2+y^2=2\left(\left(\frac{x+y}{2}\right)^2+\left(\frac{x-y}{2}\right)^2\right).$$

2 Universality of Five Quaternary Quadratic Forms

The form (1,1,1,4) (i.e. $x^2+y^2+z^2+4w^2$, similarly for the others mentioned below):

Proof. Since it is evident that if m can be represented by a quadratic form, then $4^k m$ $(k\geq 1)$ can also be represented, we just need to consider two cases: n odd and $n=2m$, m odd.

If n is odd, we have by the case of $(1,1,1,1)$ that $n=x_0^2+x_1^2+x_2^2+x_3^2$, where $x_3=2y_3$ can be assumed to be even, without loss of generality. Therefore $n=x_0^2+x_1^2+x_2^2+4y_3^2$ is of the required type.

If $n=2m$ with m odd, then $2m=x_0^2+x_1^2+x_2^2+x_3^2$, where x_i's cannot all be odd (otherwise $4|2m$ gives a contradiction). We then conclude as in the above case.

□

The form (1,1,2,8):

Proof. As above, we may assume that n is odd, or $n=2m$, m odd. Assume first that n is odd. Writing $n=x_0^2+x_1^2+x_2^2+x_3^2$, we have by parity consideration that either exactly three odd x_i's, say x_0,x_1, and x_2 or exactly one odd, say x_0.

In the first case, we write $n=x_0^2+x_1^2+x_2^2+4y_3^2=x_0^2+2(y_1^2+y_2^2)+4y_3^2$ by Euler's trick (note that y_1 and y_2 must have different parity, since $x_1^2+x_2^2$ is congruent to 2 mod 4). Without loss of generality, assume $y_2=2z_2$ is even. Then

$$n=x_0^2+(2y_3)^2+2y_1^2+8z_2^2,$$

as required.

For the second case, write $n = x_0^2 + 4(y_1^2 + y_2^2 + y_3^2)$, where we may assume by the pigeonhole principle that y_1 and y_2 have the same parity without loss of generality. Therefore we may write $y_1^2 + y_2^2 = 2(z_1^2 + z_2^2)$. Now we have

$$n = x_0^2 + 4(2z_1^2 + 2z_2^2) + (2y_3)^2 = x_0^2 + (2y_3)^2 + 2(2z_1)^2 + 8z_2^2,$$

as required. This concludes the case when n is odd.

It remains to show the case when $n = 2m$, m odd.

But $n = 2m = 2(x_0^2 + x_1^2 + 2x_2^2 + 8x_3^2)$ (using the case when m is odd) shows that $x_0^2 + x_1^2$ must be odd. We may assume that x_0 is odd and $x_1 = 2y_1$. Then

$$n = 2m = 2(x_0^2 + 4y_1^2 + 2x_2^2 + 8x_3^2) = (2x_2)^2 + (4x_3)^2 + 2x_0^2 + 8y_1^2,$$

as required.

□

The form (1,1,2,2):

Proof. This follows directly from the case of $(1, 1, 2, 8)$:

$$n = x_0^2 + x_1^2 + 2x_2^2 + 8x_3^2 = x_0^2 + x_1^2 + 2x_2^2 + 2(2x_3)^2.$$

□

The form (1,2,2,4):

Proof. We deal with the case n odd and $n = 2m$, m odd separately. If n is odd, then by the case of (1,1,2,2), we have $n = x_0^2 + x_1^2 + 2x_2^2 + 2x_3^2$, where we may assume that $x_1 = 2y_1$ is even. Therefore $n = x_0^2 + 2x_2^2 + 2x_3^2 + 4y_1^2$, as required.

If $n = 2m$ with m odd, then $n = 2(x_0^2 + 2x_1^2 + 2x_2^2 + 4x_3^2)$ by what has just been established for odd m. Therefore

$$n = (2x_1)^2 + 2x_0^2 + 2(2x_3)^2 + 4x_2^2,$$

as required.

□

The form (1,2,4,8):

Proof. We deal with n odd and $n = 2m$ with m odd. If n is odd, then $n = x_0^2 + 2x_1^2 + 2x_2^2 + 4x_3^2$ by the case of (1,2,2,4). Now if either x_1 or x_2 is even, say $x_2 = 2y_2$ is even, then $n = x_0^2 + 2x_1^2 + 4x_3^2 + 8y_2^2$. Therefore we may assume that both x_1 and x_2 are odd. Then we may write $x_1^2 + x_2^2 = 2(y_1^2 + y_2^2)$, and hence $n = x_0^2 + 4(y_1^2 + y_2^2 + x_3^2)$, where we may assume that y_1 and y_2 have the same parity (by the pigeonhole principle), thus $y_1^2 + y_2^2 = 2(z_1^2 + z_2^2)$. It follows that

$$n = x_0^2 + 8(z_1^2 + z_2^2) + 4x_3^2 = x_0^2 + 2(2z_1)^2 + 4x_3^2 + 8z_2^2,$$

as required.

Now if $n = 2m, m$ odd, then we may write $n = 2(x_0^2 + 2x_1^2 + 4x_2^2 + 8x_3^2)$ by what has just been proven. Then we have

$$n = 2(x_0^2 + 2x_1^2 + 4x_2^2 + 8x_3^2) = (4x_3)^2 + 2x_0^2 + 4x_1^2 + 8x_2^2,$$

as required.

□

Concluding Remarks. In the same paper of J. Deutsch [1], the forms (1,1,3,3), (1,2,3,6) were also mentioned and proven. The form (1,2,5,10) was mentioned but left unproven, for which an elementary proof was given by P. Clark and others [4]. In an upcoming paper, we will give a quaternionic proof of an analogous Jacobi type formula for the number of representations of a positive integer in terms of the quadratic form (1,1,3,3), based on Lipschitz type quaternions. This would certainly imply the universality of the quadratic form (1,1,3,3).

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