International Mathematical Forum, Vol. 8, 2013, no. 10, 481 - 493
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Union Curves in W_2 Relative to a Congruence and Union Curvature of a Curve in W_2

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Abstract

In this paper, firstly we obtained differential equation of the union curves in W_2. Then, we showed that obtained equation is equivalent to a pair of differential equations of the second order. We denoted it by (η_1, η_2) and we named η_1 and η_2 as the components of the union curvature vector field. We defined union curves in W_2 relative a congruence by means of the union curvature vector field. Finally, we expressed union curvature of a curve in W_2 in terms of the components of the union curvature vector field.

Mathematics Subject Classification: 531325, 53A25

Keywords: Weyl space, union curvature vector field, union curve

1. Introduction

A manifold with a conformal metric g_{ij} and a symmetric connection ∇_k satisfying the compatibility condition

$$\nabla_k g_{ij} - 2T_k g_{ij} = 0 \tag{1.1}$$

is called a Weyl space, which will be denoted by $W_2(g_{ij}, T_k)$. The vector field T_k is named the complementary vector field. Under renormalization of the metric tensor in the form

$$\overset{\nu}{g}_{ij} = \lambda^2 g_{ij} \tag{1.2}$$

the complementary vector field T_k is transformed by the law

$$\overset{\nu}{T}_k = T_k + \partial_k \ln \lambda \tag{1.3}$$

where λ is a scalar function [1].

The coefficients Γ^i_{kl} of the symmetric connection ∇_k are given by

$$\Gamma^i_{kl} = \left\{ \begin{array}{c} \mathrm{i} \\ \mathrm{kl} \end{array} \right\} - g^{im} \left(g_{mk} T_l + g_{ml} T_k - g_{kl} T_m \right). \tag{1.4}$$

If, under transformation (1.2), the quantity A is changed according to the rule

$$\overset{\nu}{A} = \lambda^p A \tag{1.5}$$

then A is called satellite of g_{ij} with weight $\{p\}$.

The prolonged derivative and prolonged covariant derivative of A are, respectively, defined by ([2],[3])

$$\dot{\partial}_k A = \partial_k A - p T_k A \tag{1.6}$$

and

$$\dot{\nabla}_k A = \nabla_k A - p T_k A. \tag{1.7}$$

Let $W_2(g_{ij}, T_k)$ be two dimensional Weyl space and $W_3(g_{ab}, T_c)$ be three dimensional Weyl space $(i, j, k = 1, 2; a, b, c = 1, 2, 3)$. Let x^a and u^i be the coordinates of $W_3(g_{ab}, T_c)$ and $W_2(g_{ij}, T_k)$, respestively. The metrics of $W_2(g_{ij}, T_k)$ and $W_3(g_{ab}, T_c)$ are connected by the relations

$$g_{ij} = g_{ab} x^a_i x^b_j \tag{1.8}$$

where x^a_i is the covariant derivative of x^a with respect to u^i.

The prolonged covariant derivative of A with respect to u^k and x^c are $\dot{\nabla}_k A$ and $\dot{\nabla}_c A$, respectively. These are related by the conditions

$$\dot{\nabla}_k A = x^c_k \dot{\nabla}_c A. \tag{1.9}$$

Let the normal vector field n^a of $W_2(g_{ij}, T_k)$ be normalized by the condition $g_{ab}n^a n^b = 1$. The moving frame $\{x_i^a, n^a\}$ and its reciprocal $\{x_a^i, n_a\}$ are connected by the relations [1]

$$n^a n_a = 1\,,\ n_a x_i^a = 0\,,\ n^a x_a^i = 0\,,\ x_i^a x_a^j = \delta_i^j. \tag{1.10}$$

Since the weight of x_i^a is $\{0\}$, the prolonged covariant derivative of x_i^a, relative to u^k, is given by [1]

$$\overset{\bullet}{\nabla}_k x_i^a = \nabla_k x_i^a = w_{ik} n^a \tag{1.11}$$

where w_{ik} are the coefficients of the second fundamental form of $W_2(g_{ij,}T_k)$.

On the other hand, it is easy to see that the prolonged covariant derivative of n^a is given by

$$\overset{\bullet}{\nabla}_k n^a = -w_{kl} g^{il} x_i^a. \tag{1.12}$$

By means of (1.10), the prolonged covariant derivative of x_a^j is found to be [8]

$$\overset{\bullet}{\nabla}_k \ x_a^j = \Omega_k^j \, n_a. \tag{1.13}$$

Let $\underset{r}{v}^i$ $(i, r = 1, 2)$ be the contravariant compenents of the vector field $\underset{r}{v}$ in $W_2(g_{ij}, T_k)$. Suppose that vector fields $\underset{r}{v}$ $(r = 1, 2)$ are normalized by the conditions $g_{ij}\underset{r}{v}^i \underset{r}{v}^j = 1$.

The reciprocal vector fields $\overset{r}{v}$ $(r = 1, 2)$ are defined by the relations [4]

$$\underset{r}{v}^i \overset{r}{v}_j = \delta_j^i\,,\ \underset{r}{v}^i \overset{s}{v}_i = \delta_r^s\,,\ (i, j, r, s = 1, 2). \tag{1.14}$$

The prolonged covariant derivatives of the vector field $\underset{r}{v}$ and its reciprocal $\overset{r}{v}$ are, respectively, given by [5]

$$\overset{\bullet}{\nabla}_k \ \underset{r}{v}^i = \overset{s}{\underset{r}{T}}_k \ \underset{s}{v}^i\,,\ \overset{\bullet}{\nabla}_k \ \overset{r}{v}_i = -\overset{r}{\underset{s}{T}}_k \ \overset{s}{v}_i. \tag{1.15}$$

Let $\underset{r}{v}^a$ and $\underset{r}{v}^i$ are the contravariant components of the vector field $\underset{r}{v}$ relative to $W_3(g_{ab}, T_c)$ and $W_2(g_{ij}, T_k)$,respectively. Denoting the components of $\overset{r}{v}$ relative to $W_3(g_{ab}, T_c)$ and $W_2(g_{ij}, T_k)$ by $\overset{r}{v}_a$ and $\overset{r}{v}_i$ we have [6]

$$\underset{r}{v}^a = x_i^a \ \underset{r}{v}^i,\ \overset{r}{v}_a = x_a^i \ \overset{r}{v}_i. \tag{1.16}$$

If $\underset{rr}{\kappa}$ is the normal curvature of $W_2(g_{ij}, T_k)$ in the direction of $\underset{r}{v}$, we have

$$\underset{rr}{\kappa} = w_{ij} \underset{r}{v}^i \underset{r}{v}^j. \tag{1.17}$$

Since the weight of w_{ij} is $\{1\}$ and that of $\underset{r}{v}^i$ is $\{-1\}$, $\underset{rr}{\kappa}$ is a satellite of g_{ij} with weight of $\{-1\}$.

The quantities

$$\overset{p}{\underset{r}{\wr}} = \overset{p}{\underset{r}{T}}_k \underset{r}{v}^k \qquad (r, p = 1, 2) \tag{1.18}$$

are called the geodesic curvatures of the lines of the net $\left(\underset{1}{v}, \underset{2}{v}\right)$ [5].

The vector fields

$$\underset{p}{c}^i = \overset{r}{\underset{p}{\wr}} \underset{r}{v}^i \qquad (i, r, p = 1, 2) \tag{1.19}$$

are called the geodesic vector fields of the net $\left(\underset{1}{v}, \underset{2}{v}\right)$ relative to $W_2(g_{ij}, T_k)$ [5].

If the components of the geodesic vector fields relative to $W_3(g_{ab}, T_c)$ are denoted by $\underset{r}{\overline{c}}^{\,a}$, then we have [6]

$$\underset{r}{v}^c \dot{\nabla}_c \underset{r}{v}^a = \underset{r}{\overline{c}}^{\,a} = \left(w_{ik} \underset{r}{v}^i \underset{r}{v}^k\right) n^a + \underset{r}{c}^i x_i^a. \tag{1.20}$$

Since the net $\left(\underset{1}{v}, \underset{2}{v}\right)$ is orthogonal, we have by [5]

$$\overset{r}{\underset{r}{T}}_k = 0\,, \ \overset{p}{\underset{r}{T}}_k + \overset{r}{\underset{p}{T}}_k = 0 \ \ (r \neq p). \tag{1.21}$$

The quantities

$$\epsilon_{ij} = e_{ij}\sqrt{g}\,, \ \epsilon^{ij} = e^{ij}\sqrt{g} \ \ (i, j = 1, 2) \tag{1.22}$$

are the components under a positive transformation of a skew-symmetric covariant and contravariant tensor of the second order, respectively, where $e_{11} = e_{22} = e^{11} = e^{22} = 0,\ e_{12} = e^{12} = 1,\ e_{21} = e^{21} = -1$ [7].

$$\delta_{abc} = \delta^{abc} = \begin{cases} 0 & \text{when two or three of the indices have the same values} \\ 1 & \text{when the respective indices have the values} \\ & 1,2,3\,;\ 2,3,1;\ \text{or } 3,1,2 \\ -1 & \text{when the respective indices have the values} \\ & 1,3,2\,;\ 3,2,1;\ \text{or } 2,1,3 \end{cases} \tag{1.23}$$

where $a, b, c = 1, 2, 3$ [7].

2. Preliminaries

Let us consider a congruence such that it occurs the lines. Let us denote that congruence by v. Let v^a be the contravariant components of v in the x's (with respect to W_3) and let v be normalized by the condition $g_{ab}v^a v^b = 1$. The vector field v with the components v^a, in general, not normal to W_2 and can be specified by

$$v^a = t^i x_i^a + rn^a \quad (a = 1, 2, 3\,;\, i = 1, 2) \tag{2.1}$$

where t^i and r are parameters [8].

Since $g_{ab}v^a v^b = 1$, with the help (1.8) and (1.10)

$$1 = g_{ab}v^a v^b = g_{ab}(t^i x_i^a + rn^a)(t^j x_j^b + rn^b) = g_{ij}t^i t^j + r^2 \tag{2.2}$$

is obtained.

From (2.2)

$$g_{ij}t^i t^j = 1 - r^2 \tag{2.3}$$

is found.

$$\cos\theta = g_{ab}v^a n^b = r \tag{2.4}$$

where θ is the angle between v^a and n^b , $g_{ab}v^a v^b = 1$ and $g_{ab}n^a n^b = 1$.

So

$$\sin\theta = \sqrt{1 - r^2} = \sqrt{g_{ij}t^i t^j}. \tag{2.5}$$

3. Osculating Plane of a Curve in W_2 and Differential Equation of The Union Curves in W_2

Definition 3.1 Let C be a curve in W_2. The osculating plane of the curve C relative to W_3 is defined in the form [8]

$$\delta_{abc}\,(\bar{x}^a - x^a)\,\underset{1}{v}^b \underset{1}{v}^d \overset{\bullet}{\nabla}_d\,\underset{1}{v}^c = 0 \ (a, b, c = 1, 2, 3) \tag{3.1}$$

by that determinantal equation where $\underset{1}{v}$ is the tangent vector field of the curve C.

Let us obtain differential equation of the union curves by means of equation (3.1): Remembering that, [6]

$$\underset{1}{v}^d \overset{\bullet}{\nabla}_d \underset{1}{v}^c = \underset{1}{\bar{c}}^c = \underset{1}{c}^i x_i^c + \underset{11}{\kappa} n^c \ (i = 1, 2) \tag{3.2}$$

where $\underset{1}{\bar{c}}^c$ and $\underset{1}{c}^i$ are the components of the geodesic curvature vector field relative to W_3 and W_2, respectively and $\underset{11}{\kappa}$ is the normal curvature of C in the direction of $\underset{1}{v}$.

If (3.2) is used in (3.1), then we have

$$\delta_{abc} \left(\bar{x}^a - x^a\right) \underset{1}{v}^b \left(\underset{1}{c}^i x_i^c + \underset{11}{\kappa} n^c\right) = 0 \tag{3.3}$$

If the osculating plane to C contains line of the congruence, the coordinates

$$\bar{x}^a = x^a + t v^a = x^a + t \left(t^m x_m^a + r n^a\right) \ (m = 1, 2) \tag{3.4}$$

must satisfy equation (3.3) for all t. Then, we get from (3.3) and (3.4)

$$\delta_{abc} \left(t^m x_m^a + r n^a\right) \left(\underset{i}{v}^j x_j^b\right) \left(\underset{1}{c}^i x_i^c + \underset{11}{\kappa} n^c\right) = 0. \tag{3.5}$$

From (3.5),

$$\begin{aligned} & t^m \delta_{abc} x_m^a x_j^b x_i^c \underset{1}{v}^j \underset{1}{c}^i + \underset{11}{\kappa} t^m \delta_{abc} x_m^a x_j^b n^c \underset{1}{v}^j + \\ & + r \delta_{abc} n^a x_j^b x_i^c \underset{1}{c}^i \underset{1}{v}^j + r \underset{11}{\kappa} \delta_{abc} n^a x_j^b n^c \underset{1}{v}^j = 0 \ (j = 1, 2) \end{aligned} \tag{3.6}$$

or

$$\underset{11}{\kappa} t^j \delta_{abc} x_m^a x_j^b n^c \underset{1}{v}^j + r \delta_{abc} n^a x_j^b x_i^c \underset{1}{c}^i \underset{1}{v}^j = 0 \tag{3.7}$$

where $\delta_{abc} x_m^a x_j^b x_i^c = 0$ and $\delta_{abc} n^a x_j^b n^c = 0$.

If we take indice i instead of indice j and indice j instead of indice m in the first term of the equation (3.7), we get

$$\underset{11}{\kappa} t^i \delta_{abc} x_j^a x_i^b n^c \underset{1}{v}^i + r \delta_{abc} n^a x_j^b x_i^c \underset{1}{c}^i \underset{1}{v}^j = 0. \tag{3.8}$$

Taking the indices b, c, a instead of the indices a, b, c, respectively in the first term of (3.8), we obtain

$$\underset{11}{\kappa} t^j \delta_{bca} x_j^b x_i^c n^a \underset{1}{v}^i + r \delta_{abc} n^a x_j^b x_i^c \underset{1}{c}^i \underset{1}{v}^j = 0. \tag{3.9}$$

Since $\delta_{bca} = \delta_{abc}$, from (3.9)

$$\underset{11}{\kappa} t^j \delta_{abc} n^a x^b_j x^c_i \underset{1}{v}^i + r\delta_{abc} n^a x^b_j x^c_i \underset{1}{c}^i \underset{1}{v}^j = 0 \tag{3.10}$$

or

$$\delta_{abc} n^a x^b_j x^c_i \left(\underset{11}{\kappa} t^j \underset{1}{v}^i + r \underset{1}{c}^i \underset{1}{v}^j \right) = 0 \tag{3.11}$$

is obtained.

Summing on j and i in the equation (3.11), we get

$$\delta_{abc} n^a x^b_1 x^c_2 \left(\underset{11}{\kappa} t^1 \underset{1}{v}^2 + r \underset{1}{v}^1 \underset{1}{c}^2 \right) +$$

$$+\delta_{abc} n^a x^b_2 x^c_1 \left(\underset{11}{\kappa} t^2 \underset{1}{v}^1 + r \underset{1}{v}^2 \underset{1}{c}^1 \right) = 0 \tag{3.12}$$

where $\delta_{abc} n^a x^b_1 x^c_1 = 0$ and $\delta_{abc} n^a x^b_2 x^c_2 = 0$.

If b and c be interchanged in the second term of (3.12), we find

$$\delta_{abc} n^a x^b_1 x^c_2 \left(\underset{11}{\kappa} t^1 \underset{1}{v}^2 + r \underset{1}{v}^1 \underset{1}{c}^2 \right) +$$

$$+\delta_{acb} n^a x^b_1 x^c_2 \left(\underset{11}{\kappa} t^2 \underset{1}{v}^1 + r \underset{1}{v}^2 \underset{1}{c}^1 \right) = 0 \tag{3.13}$$

or since $\delta_{acb} = -\delta_{abc}$

$$\delta_{abc} n^a x^b_1 x^c_2 \left(\underset{11}{\kappa} t^1 \underset{1}{v}^2 + r \underset{1}{v}^1 \underset{1}{c}^2 \right) -$$

$$-\delta_{abc} n^a x^b_1 x^c_2 \left(\underset{11}{\kappa} t^2 \underset{1}{v}^1 + r \underset{1}{v}^2 \underset{1}{c}^1 \right) = 0 \tag{3.14}$$

or

$$\delta_{abc} n^a x^b_1 x^c_2 \left\{ \left(\underset{11}{\kappa} t^1 \underset{1}{v}^2 + r \underset{1}{v}^1 \underset{1}{c}^2 \right) - \left(\underset{11}{\kappa} t^2 \underset{1}{v}^1 + r \underset{1}{v}^2 \underset{1}{c}^1 \right) \right\} = 0. \tag{3.15}$$

Since $\delta_{abc} n^a x^b_1 x^c_2 \neq 0$, we get from (3.15)

$$\left(\underset{11}{\kappa} t^1 \underset{1}{v}^2 + r \underset{1}{v}^1 \underset{1}{c}^2 \right) - \left(\underset{11}{\kappa} t^2 \underset{1}{v}^1 + r \underset{1}{v}^2 \underset{1}{c}^1 \right) = 0 \tag{3.16}$$

or

$$e_{12} \left(\underset{11}{\kappa} t^1 \underset{1}{v}^2 + r \underset{1}{v}^1 \underset{1}{c}^2 \right) + e_{21} \left(\underset{11}{\kappa} t^2 \underset{1}{v}^1 + r \underset{1}{v}^2 \underset{1}{c}^1 \right) = 0 \tag{3.17}$$

where $e_{11} = e_{22} = e^{11} = e^{22} = 0$, $e_{12} = e^{12} = 1$, $e_{21} = e^{21} = -1$ or

$$e_{ji} \left(\underset{11}{\kappa} t^j \underset{1}{v}^i + r \underset{1}{v}^j \underset{1}{c}^i \right) = 0. \tag{3.18}$$

Under the condition that the osculating plane of C contains line of the congruence, equation (3.18) is obtained. Equation (3.18) is the differential equation of the union curves in W_2.

Corollary 3.1 If $r = 0$, then we get from (3.18)

$$\underset{11}{\kappa} e_{ji} t^j \underset{1}{v}^i = 0 \tag{3.19}$$

or

$$\underset{11}{\kappa} \left(t^1 \underset{1}{v}^2 - t^2 \underset{1}{v}^1 \right) = 0. \tag{3.20}$$

That is, the only union curves are those in the directions given by $t^1 \underset{1}{v}^2 - t^2 \underset{1}{v}^1$ and by $\underset{11}{\kappa} = w_{ik} \underset{1}{v}^i \underset{1}{v}^k = 0$, the asymptotic directions.

Corollary 3.2 Let r be different zero. If we take $t^j/r = l^j$ in (3.18), then the equation of the union curves in (3.18) takes the form

$$e_{ji} \left(\underset{11}{\kappa} \frac{t^j}{r} \underset{1}{v}^i + \underset{1}{v}^j \underset{1}{c}^i \right) = 0. \tag{3.21}$$

or

$$e_{ji} \left(\underset{11}{\kappa} l^j \underset{1}{v}^i + \underset{1}{v}^j \underset{1}{c}^i \right) = 0 \tag{3.22}$$

or

$$\underset{11}{\kappa} e_{ji} l^j \underset{1}{v}^i + e_{ji} \underset{1}{v}^j \underset{1}{c}^i = 0. \tag{3.23}$$

If i and j be interchanged in the first term of (3.23)

$$\underset{11}{\kappa} e_{ij} l^i \underset{1}{v}^j + e_{ji} \underset{1}{v}^j \underset{1}{c}^i = 0 \tag{3.24}$$

or

$$e_{ji} \underset{1}{v}^j \left(\underset{1}{c}^i - \underset{11}{\kappa} l^i \right) = 0 \tag{3.25}$$

is found where $e_{ij} = -e_{ji}$.

From equation (3.25)

Corollary 3.3 If the components $\underset{1}{c}^i$ of the geodesic vector fields are zero, then the union curves, geodesic curves and asymptotic curves coincide in W_2.

Corollary 3.4 If the congruence is normal to W_2 ($l^i = 0$), then the union curves are geodesic curves in W_2.

4. Union Curvature of a Curve in W_2

In this section, firstly, we will show that the equation (3.25) is equivalent to a pair of differential equations of the second order. Later, we will define union curvature vector field and union curve of the curve C in W_2.

Geodesic curvature vector field with the components $\underset{1}{c}^i$ is orthogonal to the tangent vector field $\underset{1}{v}$ of C.
Hence,

$$g_{ij}\underset{1}{c}^i\underset{1}{v}^j = g_{ij}\overset{p}{\underset{1}{\wr}}\,\underset{p}{v}^i\underset{1}{v}^j = \overset{2}{\underset{1}{\wr}}g_{ij}\underset{2}{v}^i\underset{1}{v}^j = 0 \ (p=1,2) \tag{4.1}$$

where $\overset{1}{\underset{1}{\wr}} = 0$ from (1.18) and (1.21), $g_{ij}\underset{2}{v}^i\underset{1}{v}^j = 0$.

Multiplying (4.1) by $\underset{1}{v}^1$ and (3.25) by $g_{2m}\underset{1}{v}^m$ $(m=1,2)$ and then subtracting of them, we find

$$\begin{aligned} &g_{11}\underset{1}{c}^1\underset{1}{v}^1\underset{1}{v}^1 + g_{12}\underset{1}{c}^1\underset{1}{v}^2\underset{1}{v}^1 + g_{21}\underset{1}{c}^2\underset{1}{v}^1\underset{1}{v}^1 + g_{22}\underset{1}{c}^2\underset{1}{v}^2\underset{1}{v}^1 - \\ &-g_{11}\underset{1}{v}^1\underset{1}{v}^1\underset{1}{c}^2 + \underset{11}{\kappa}g_{21}\underset{1}{v}^1\underset{1}{v}^1 l^2 - g_{22}\underset{1}{v}^2\underset{1}{v}^1\underset{1}{c}^2 + \underset{11}{\kappa}g_{22}\underset{1}{v}^2\underset{1}{v}^1 l^2 + \\ &+g_{21}\underset{1}{v}^1\underset{1}{v}^2\underset{1}{c}^1 - \underset{11}{\kappa}g_{21}\underset{1}{v}^1\underset{1}{v}^2 l^2 + g_{22}\underset{1}{v}^2\underset{1}{v}^2\underset{1}{c}^1 - \underset{11}{\kappa}g_{22}\underset{1}{v}^2\underset{1}{v}^2 l^1 = 0 \end{aligned} \tag{4.2}$$

or

$$\begin{aligned} &\underset{1}{c}^1\left(g_{11}\underset{1}{v}^1\underset{1}{v}^1 + g_{12}\underset{1}{v}^1\underset{1}{v}^2 + g_{21}\underset{1}{v}^2\underset{1}{v}^1 + g_{22}\underset{1}{v}^2\underset{1}{v}^2\right) - \\ &-\underset{11}{\kappa}\left(-g_{21}\underset{1}{v}^1 l^2\underset{1}{v}^1 - g_{22}\underset{1}{v}^2 l^2\underset{1}{v}^1 + g_{21}\underset{1}{v}^1 l^1\underset{1}{v}^2 + g_{22}\underset{1}{v}^2 l^1\underset{1}{v}^2\right) = 0 \end{aligned} \tag{4.3}$$

or

$$\underset{1}{c}^1 g_{ij}\underset{1}{v}^i\underset{1}{v}^j - \underset{11}{\kappa}\left(-g_{2m}\underset{1}{v}^m l^2\underset{1}{v}^1 + g_{2m}\underset{1}{v}^m l^1\underset{1}{v}^2\right) = 0 \tag{4.4}$$

or

$$\underset{1}{c}^1 - \underset{11}{\kappa}g_{2m}\underset{1}{v}^m\left(e_{21}l^2\underset{1}{v}^1 + e_{12}l^1\underset{1}{v}^2\right) = 0 \tag{4.5}$$

where $g_{ij}\underset{1}{v}^i\underset{1}{v}^j = 1$
or

$$\underset{1}{c}^1 - \underset{11}{\kappa}g_{2m}\underset{1}{v}^m e_{ij}l^i\underset{1}{v}^j = 0. \tag{4.6}$$

Let us denote that equation by η^1:

$$\eta^1 = \underset{1}{c}^1 - \underset{11}{\kappa}g_{2m}\underset{1}{v}^m e_{ij}l^i\underset{1}{v}^j = 0. \tag{4.7}$$

Similarly, multiplying (4.1) by $\underset{1}{v}^2$ and (3.25) by $g_{1m}\underset{1}{v}^m$ and then summing of them, we find

$$\begin{aligned} & g_{11}\underset{1}{c}^1\underset{1}{v}^1\underset{1}{v}^2 + g_{12}\underset{1}{c}^1\underset{1}{v}^2\underset{1}{v}^2 + g_{21}\underset{1}{c}^2\underset{1}{v}^1\underset{1}{v}^2 + g_{22}\underset{1}{c}^2\underset{1}{v}^2\underset{1}{v}^2 + \\ & + g_{11}\underset{1}{v}^1\underset{1}{v}^1\underset{1}{c}^2 - \underset{11}{\kappa} g_{11}\underset{1}{v}^1\underset{1}{v}^1 l^2 + g_{12}\underset{1}{v}^2\underset{1}{v}^1\underset{1}{c}^2 - \underset{11}{\kappa} g_{12}\underset{1}{v}^2\underset{1}{v}^1 l^2 - \\ & - g_{11}\underset{1}{v}^1\underset{1}{v}^2\underset{1}{c}^1 + \underset{11}{\kappa} g_{11}\underset{1}{v}^1\underset{1}{v}^2 l^2 - g_{12}\underset{1}{v}^2\underset{1}{v}^2\underset{1}{c}^1 + \underset{11}{\kappa} g_{12}\underset{1}{v}^2\underset{1}{v}^2 l^1 = 0 \end{aligned} \tag{4.8}$$

or

$$\begin{aligned} & \underset{1}{c}^2 \left(g_{21}\underset{1}{v}^2\underset{1}{v}^1 + g_{22}\underset{1}{v}^2\underset{1}{v}^2 + g_{11}\underset{1}{v}^1\underset{1}{v}^1 + g_{12}\underset{1}{v}^1\underset{1}{v}^2 \right) - \\ & - \underset{11}{\kappa} \left[\left(g_{11}\underset{1}{v}^1 + g_{12}\underset{1}{v}^2 \right) \underset{1}{v}^1 l^2 - \left(g_{11}\underset{1}{v}^1 + g_{12}\underset{1}{v}^2 \right) \underset{1}{v}^2 l^2 \right] = 0 \end{aligned} \tag{4.9}$$

or

$$\underset{1}{c}^2 g_{ij}\underset{1}{v}^i\underset{1}{v}^j - \underset{11}{\kappa} g_{1m}\underset{1}{v}^m \left(\underset{1}{v}^1 l^2 - \underset{1}{v}^2 l^1 \right) = 0 \tag{4.10}$$

or

$$\underset{1}{c}^2 - \underset{11}{\kappa} g_{1m}\underset{1}{v}^m \left(e_{12}\underset{1}{v}^1 l^2 + e_{21}\underset{1}{v}^2 l^1 \right) = 0 \tag{4.11}$$

where $g_{ij}\underset{1}{v}^i\underset{1}{v}^j = 1$
or

$$\underset{1}{c}^2 - \underset{11}{\kappa} g_{1m}\underset{1}{v}^m \left(-e_{21} l^2 \underset{1}{v}^1 - e_{12} l^1 \underset{1}{v}^2 \right) = 0 \tag{4.12}$$

where $e_{ij} = -e_{ji}$
or

$$\underset{1}{c}^2 + \underset{11}{\kappa} g_{1m}\underset{1}{v}^m \left(e_{21} l^2 \underset{1}{v}^1 + e_{12} l^1 \underset{1}{v}^2 \right) = 0 \tag{4.13}$$

or

$$\underset{1}{c}^2 + \underset{11}{\kappa} g_{1m}\underset{1}{v}^m e_{ij} l^i \underset{1}{v}^j = 0. \tag{4.14}$$

Let us denote that equation by η^2:

$$\eta^2 = \underset{1}{c}^2 + \underset{11}{\kappa} g_{1m}\underset{1}{v}^m e_{ij} l^i \underset{1}{v}^j = 0. \tag{4.15}$$

Definition 4.1 The vector field with the components η^i $(i = 1, 2)$ can be called union curvature vector field of the curve C in W_2.

From the equations (4.7) and (4.15)

Corollary 4.1 The union curvature vector field is a zero vector field at each point of a union curve.
Hence

Definition 4.2 The union curve in W_2 relative to a congruence is defined as a curve whose union curvature vector field is a zero vector field [9].

We know that

$$\sin\theta = \frac{\epsilon_{ij}\underset{1}{v}^i\underset{2}{v}^j}{\sqrt{g_{ij}\underset{1}{v}^i\underset{1}{v}^j}\sqrt{g_{kl}\underset{2}{v}^k\underset{2}{v}^l}} \tag{4.16}$$

where θ is the angle between the vector fields $\underset{1}{v}$ and $\underset{2}{v}$, $\epsilon_{ij} = \sqrt{g}\,e_{ij}$ [7].

Since $g_{ij}\underset{1}{v}^i\underset{1}{v}^j = 1$ and $g_{kl}\underset{2}{v}^k\underset{2}{v}^l = 1$, from (4.16)

$$\sin\theta = \epsilon_{ij}\underset{1}{v}^i\underset{2}{v}^j \tag{4.17}$$

is obtained.

A necessary and sufficient condition that vector field $\underset{2}{v}$ makes a right angle with vector field $\underset{1}{v}$ is that [7]

$$\epsilon_{ij}\underset{1}{v}^i\underset{2}{v}^j = 1. \tag{4.18}$$

From here

$$\begin{aligned} \epsilon_{12}\underset{1}{v}^1\underset{2}{v}^2 + \epsilon_{21}\underset{1}{v}^2\underset{2}{v}^1 &= 1 \\ \epsilon_{12}\underset{1}{v}^1\overset{2}{\underset{1}{\wr}}\underset{2}{v}^2 + \epsilon_{21}\underset{1}{v}^2\overset{2}{\underset{1}{\wr}}\underset{2}{v}^1 &= \overset{2}{\underset{1}{\wr}} \\ \epsilon_{12}\underset{1}{v}^1\underset{1}{c}^2 + \epsilon_{21}\underset{1}{v}^2\underset{1}{c}^1 &= \overset{2}{\underset{1}{\wr}} \end{aligned} \tag{4.19}$$

where $\overset{2}{\underset{1}{\wr}} = \overset{2}{\underset{1}{T}}_k\underset{1}{v}^k$ is the geodesic curvature of C which has tangent vector field $\underset{1}{v}$. Let us denote it by K_g :

$$K_g = \epsilon_{ij}\underset{1}{v}^i\underset{1}{c}^j. \tag{4.20}$$

Using (4.7) and (4.15) in (4.20), we get

$$\begin{aligned} K_g = \epsilon_{ij}\underset{1}{v}^i\underset{1}{c}^j &= \epsilon_{12}\underset{1}{v}^1\underset{1}{c}^2 + \epsilon_{21}\underset{1}{v}^2\underset{1}{c}^1 \\ &= \epsilon_{12}\underset{1}{v}^1\left[\eta^2 - \underset{11}{\kappa}\left(g_{11}\underset{1}{v}^1 + g_{12}\underset{1}{v}^2\right)\left(l^1\underset{1}{v}^2 - l^2\underset{1}{v}^1\right)\right] + \\ &\quad + \epsilon_{21}\underset{1}{v}^2\left[\eta^1 + \underset{11}{\kappa}\left(g_{21}\underset{1}{v}^1 + g_{22}\underset{1}{v}^2\right)\left(l^1\underset{1}{v}^2 - l^2\underset{1}{v}^1\right)\right] \\ &= \epsilon_{12}\underset{1}{v}^1\eta^2 + \epsilon_{21}\underset{1}{v}^2\eta^1 + \underset{11}{\kappa}\epsilon_{21}\left(l^1\underset{1}{v}^2 - l^2\underset{1}{v}^1\right)\left(g_{11}\underset{1}{v}^1\underset{1}{v}^1 + \right. \\ &\quad \left. + g_{12}\underset{1}{v}^1\underset{1}{v}^2 + g_{21}\underset{1}{v}^2\underset{1}{v}^1 + g_{22}\underset{1}{v}^2\underset{1}{v}^2\right) \end{aligned}$$

$$= \epsilon_{ij} \underset{1}{v}^i \eta^j + \underset{11}{\kappa} \left(\epsilon_{21} \underset{1}{v}^2 l^1 + \epsilon_{12} \underset{1}{v}^1 l^2 \right) g_{ij} \underset{1}{v}^i \underset{1}{v}^j$$

$$= \epsilon_{ij} \underset{1}{v}^i \eta^j + \underset{11}{\kappa} \epsilon_{ij} \underset{1}{v}^i \underset{1}{v}^j \tag{4.21}$$

where $\epsilon_{ij} = \sqrt{g} e_{ij} = \sqrt{g}(-e_{ji}) = -\sqrt{g} e_{ji} = -\epsilon_{ji}$ and $g_{ij} \underset{1}{v}^i \underset{1}{v}^j = 1$
or

$$K_g - \underset{11}{\kappa} \epsilon_{ij} \underset{1}{v}^i l^j = \epsilon_{ij} \underset{1}{v}^i \eta^j \tag{4.22}$$

or

$$K_u = K_g - \underset{11}{\kappa} \epsilon_{ij} \underset{1}{v}^i l^j = \epsilon_{ij} \underset{1}{v}^i \eta^j \tag{4.23}$$

where K_u is the union curvature of C in W_2.

Therefore, the union curvature of the curve C in W_2 is found in terms of the components of the union curvature vector field.

Corollary 4.2 If C is a union curve in W_2, since $K_u = 0$, K_g is obtained as

$$K_g = \underset{11}{\kappa} \epsilon_{ij} \underset{1}{v}^i l^j. \tag{4.24}$$

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Received: November, 2012