Exact Formulae for the Perfect Power Counting Function and the n -th Perfect Power

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Abstract

There exist in the literature various exact formulae for the prime counting function $\pi(x)$ and the n -th prime p_n . These formulae use the floor function. In this note we present exact formulae for the perfect power counting function $N(x)$ and the n -th perfect power P_n . These formulae also use the floor function.

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1 Introduction

A natural number of the form m^n where m is a positive integer and $n \geq 2$ is called a perfect power. The first few terms of the integer sequence of perfect powers are

$$1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128, \dots$$

and they form sequence A001597 in Sloane's Encyclopedia.

Let $N(x)$ be the number of perfect powers that do not exceed x . In [1] are obtained asymptotic formulae for $N(x)$. For example,

$$N(x) = \sqrt{x} + \sqrt[3]{x} + \sqrt[5]{x} - \sqrt[6]{x} + \sqrt[7]{x} - \sqrt[10]{x} + \sqrt[11]{x} + o\left(\sqrt[11]{x}\right).$$

In this note we obtain an exact formula for $N(x)$ using the floor function.

Let P_n be the n -th perfect power. That is, $P_1 = 1$, $P_2 = 4$, $P_3 = 8$, $\dots$. In [2] are obtained asymptotic formulae for P_n . For example,

$$P_n = n^2 - 2n^{\frac{5}{3}} - 2n^{\frac{7}{5}} + \frac{13}{3}n^{\frac{4}{3}} - 2n^{\frac{9}{7}} + 2n^{\frac{6}{5}} - 2n^{\frac{13}{11}} + o\left(n^{\frac{13}{11}}\right).$$

In this note we obtain exact formulae for P_n using the floor function.

In this note (as usual) $\lfloor \cdot \rfloor$ denotes the floor function.

2 An exact formula for $N(k)$

Lemma 2.1 *The number $h \geq 2$ is a perfect power if and only if at least one of the numbers*

$$h^{1/2}, h^{1/3}, \dots, h^{1/h}$$

is a positive integer.

Proof. If $h^{1/j} = a$, where a is a positive integer and $j \in \{2, \dots, h\}$, then $h = a^j$ is a perfect power.

On the other hand, we have

$$\begin{aligned} h &< 2^h < 3^h < 4^h < 5^h < \dots \\ h &< 2^{h+1} < 3^{h+1} < 4^{h+1} < 5^{h+1} < \dots \\ h &< 2^{h+2} < 3^{h+2} < 4^{h+2} < 5^{h+2} < \dots \\ &\vdots \end{aligned}$$

Since if $h \geq 3$ then $h < 2^h$. Consequently if $h \geq 3$ is a perfect power then

$$h = b^c,$$

where b is a positive integer and $c \in \{2, \dots, h-1\}$. Therefore

$$h^{1/c} = b.$$

The lemma is proved.

Theorem 2.2 *We have the following formula*

$$N(k) = 1 + \sum_{h=2}^k \left(1 - \prod_{j=2}^h \left(1 - \left\lfloor \frac{h^{1/j}}{h^{1/j}} \right\rfloor \right) \right) \quad (k \geq 1) \quad (1)$$

Proof. Let us consider the function

$$1 - \left\lfloor \frac{\lfloor x \rfloor}{x} \right\rfloor.$$

If $x > 0$ is an integer then

$$1 - \left\lfloor \frac{\lfloor x \rfloor}{x} \right\rfloor = 0.$$

On the other hand, If $x > 0$ is not an integer then

$$1 - \left\lfloor \frac{\lfloor x \rfloor}{x} \right\rfloor = 1.$$

Therefore (Lemma 2.1) if h is a perfect power then

$$1 - \prod_{j=2}^h \left(1 - \left\lfloor \frac{\lfloor h^{1/j} \rfloor}{h^{1/j}} \right\rfloor \right) = 1.$$

On the other hand (Lemma 2.1) if h is not a perfect power then

$$1 - \prod_{j=2}^h \left(1 - \left\lfloor \frac{\lfloor h^{1/j} \rfloor}{h^{1/j}} \right\rfloor \right) = 0.$$

Consequently, formula (1) is proved.

3 Formulae for P_n in terms of $N(k)$.

We need the following lemma.

Lemma 3.1 *We have the following inequality*

$$P_n \leq n^2 \quad (n \geq 1).$$

Proof. We have $P_1 = 1^2$, $P_2 = 2^2$, $P_3 = 8 < 3^2$. Consequently if $n \geq 3$ then $P_n < n^2$. Since the squares are perfect powers. The lemma is proved.

We have the following general theorem.

Theorem 3.2 *Suppose that the function $f(x)$ has the following properties.*

(i) $f(x)$ is continuous, nonnegative and strictly increasing in the interval $[0, \infty]$.

(ii) $f(1) = 1$

(iii) $\lim_{x \rightarrow \infty} f(x) = A \leq 2$

Then the following formula holds,

$$P_n = 1 + \sum_{k=1}^{n^2} \left\lfloor f \left(\frac{n}{1 + N(k)} \right) \right\rfloor \quad (n \geq 1). \quad (2)$$

Proof. Note that in the interval $(0, 1)$ we have $0 < f(x) < 1$ and in the interval $[1, \infty)$ we have $1 \leq f(x) < 2$. Besides, we have (Lemma 3.1) $P_n \leq n^2$ ($n \geq 1$). Clearly formula (2) is true if $n = 1$. Since $N(1) = 1$. Suppose that $n \geq 2$.

If $k = P_n, P_n + 1, P_n + 2, \dots, n^2$ we have (since $N(P_n) = n$)

$$0 < \frac{n}{1 + N(k)} < 1.$$

Therefore

$$\left\lfloor f\left(\frac{n}{1 + N(k)}\right) \right\rfloor = 0.$$

On the other hand, if $k = 1, 2, \dots, P_n - 1$ then $N(k)$ take the values $1, 2, \dots, n - 1$ and therefore

$$1 \leq \frac{n}{1 + N(k)}.$$

Consequently

$$\left\lfloor f\left(\frac{n}{1 + N(k)}\right) \right\rfloor = 1.$$

The theorem is proved.

Example 3.3 A simple family of functions that satisfy conditions (i), (ii) and (iii) is

$$f(x) = \frac{kx}{(k-1)x + 1} \quad (k \geq 2).$$

For example, if $k = 2$ equation (2) becomes

$$P_n = 1 + \sum_{k=1}^{n^2} \left\lfloor \frac{2\left(\frac{n}{1+N(k)}\right)}{\left(\frac{n}{1+N(k)}\right) + 1} \right\rfloor = 1 + \sum_{k=1}^{n^2} \left\lfloor \frac{2n}{n + 1 + N(k)} \right\rfloor \quad (n \geq 1). \quad (3)$$

Substituting equation (1) into (3) we obtain a formula for P_n .

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