On Density and Hypercyclicity

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Abstract

The aim of the paper ahead Birhoff, Maclane, Godefroy-Shapiro and
Kitai-Getner-Shapiro and the results of their theorems and hypercyclic operators on space $H(C)$. In Birhoffs theorem is shown that, if $b$ is non-zero, then the shift with the vector $b$ is an hypercyclic operator. Maclane in 1952 showed that the Differentiation operator on $H(C)$ is an hypercyclic operator. Bourdon and Shapiro also studied the behavior of composition operators on this space.

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1 Introduction

Let $X$ be a Frechet space and $T$ be a bounded linear operator on $X$. For each $x \in X$ put

\[ \text{Orb}(T, x) = \{ T^n(x) : n \geq 0 \} = \{ x, Tx, T^2x, T^3x, \ldots \} \]

The set $\text{Orb}(T, x)$ is called orbit of vector $x$ under the operator $T$ and the operator $T$ is called hypercyclic operator if there exist vector $x$ in $X$ such that the set $\text{Orb}(T, x)$ is dense in $X$, in this case the vector $x$ is called hypercyclic vector for the operator. If $X^*$ be the dual space of space $X$ and both operators $T : X \rightarrow X$ and $T^* : X^* \rightarrow X^*$ are hypercyclic, then the operator $T$ is called dual hypercyclic. For more information readers can see [1–5].

2 Preliminary Notes

Suppose that $H(C)$ be the space of all functions of one complex variable with the uniform convergence topology on compact subsets of $C$. Consider Banach space $E$, Frechet algebra by elements of dual space with uniform convergence topology over the balls of $E$. Space $H_{bc}(E)$ containing all bounded functions on compact subsets of $E$, the space $H_{bc}(E)$ includes all functions $f = \sum_{n=0}^{\infty} P_n$ in which

\[ P_n \in \text{Span}\{ \varphi^n : \varphi E^* \} , \quad n = 0, 1, 2, 3, \ldots \]

\[ \| P_n\|_1 = (\text{Sup}_{\|x\| < 1} |P_n|)_1 \rightarrow 0 , \quad n \rightarrow \infty \]

The operator $\phi : H(E) \rightarrow H(E)$ by definition $\phi(f) = Df$ is called differentiation operator. Let $\varphi \in H(C)$, then the operator $C_\varphi : H(C) \rightarrow H(C)$ by definition $C_\varphi(f) = f \circ \varphi$ on $H(C)$ is hypercyclic, if and only if the operator $\varphi$
is a shift with a non-zero vector $b \in C$. In other words, $\exists 0 \neq b \in C, \varphi(z) = z + b(\text{see}[1])$. Differentiation operator on $H(C)$ is a hypercyclic operator (see[6]). If $\phi(z) = \sum_{|\alpha| \geq 0} C_{\alpha} z^{\alpha}$ be non-constant entire function on $C$, Then the operator $\phi_{D} : H(C^{n}) \rightarrow H(C^{n})$ by definition $\phi_{D}(f) = \sum_{|\alpha| \geq 0} C_{\alpha} D^{\alpha} f$, $f \in H(C)$ is hypercyclic operator (see[10]). Also all continuous linear operator on $H(C^{n})$ substitute with translation, if and only if, be for one $\varphi \in H(C^{n})$ is of exponential form $T = \varphi' D$.

Theorem 2.1 (Hypercyclicity Criterion) Let $X$ be an $F$-space and $T : X \rightarrow X$ be a continuous linear operator and assume that $U, V$ are two dense subsets of $X$ and $\{n_{k}\}_{k=1}^{\infty}$ be a sequence of positive integers, and there are sequences $S_{n_{k}} : V \rightarrow X$ of mapping such that,

1. $T^{n_{k}} \rightarrow 0, k \rightarrow \infty$, Pointwise on $U$
2. $S_{n_{k}} \rightarrow 0, k \rightarrow \infty$, Pointwise on $V$
3. $T^{n_{k}} S_{n_{k}} = I_{V}$

then the operator $T$ is hypercyclic

3 Main Results

Theorem 3.1 If $E$ be a Banach Space then the collection $B = \{e^{\varphi} : \varphi \in E^{*}\}$ is an independently linear subset of $H_{bc}(E)$.

Theorem 3.2 Let $U$ be an open subset of $E^{*}$, then $S = \text{Span}\{e^{\varphi} : \varphi \in U\}$ is a dense subset of $H_{bc}(E)$.

Proof. Let $\varphi_{0} \in E^{*}$ and $\Lambda : H_{bc}(E) \rightarrow H_{bc}(E)$ by $\Lambda(\psi) = e^{\varphi_{0}} \cdot \psi$. Suppose $\psi_{1}, \psi_{2} \in H_{bc}(E)$ and $\Lambda(\psi_{1}) = \Lambda(\psi_{2})$, so $e^{\varphi_{0}} \cdot \psi_{1} = e^{\varphi_{0}} \cdot \psi_{2}$. Since $e^{\varphi_{0}} \neq 0$, then $\psi_{1} = \psi_{2}$, that is the operator $\Lambda$ is one-one operator. Since constant operator and identity operator are continuous, then the operator $\Lambda$ is continuous. Now since $\Lambda(\psi)^{-1} = e^{-\varphi_{0}} \cdot \psi$ is continuous operator, then the operator $\Lambda$ is a homeomorphism and

$$\text{Span}\{e^{\varphi_{0} + \varphi} : \varphi \in U\} = H_{bc}(E) \Leftrightarrow \text{Span}\{e^{\varphi} : \varphi \in U\} = H_{bc}(E)$$

If $\lambda_{0} \in U$ then take $U_{0} = \{\varphi - \lambda_{0} : \varphi \in U\}$, then $0 = \lambda_{0} - \lambda_{0} \in U_{0}$. So without lost of generality we can suppose $0 \in U$. If $U$ be a non-empty open subset of $E^{*}$, such that the norm of all element in $U$ are not zero, then theorem is trivial. So assume that $\varphi_{0} \in U, \|\varphi_{0}\| \neq 0$ and define $U_{0} = \{\frac{1}{\|\varphi_{0}\|} \varphi : \varphi \in U\}$. Now we have

$$\frac{1}{\|\varphi_{0}\|} \varphi_{0} = \frac{1}{\|\varphi_{0}\|} \cdot \|\varphi_{0}\| = 1 \quad , \quad \frac{1}{\|\varphi_{0}\|} \varphi_{0} \in U_{0}.$$
So we have an open non-empty subset of $E^*$ contain an element of norm 1. Now take $\delta > 0$ such that, $U = \{ \varphi \in E^* : \| \varphi \| < \delta \}$. Specially, for $0 \in U$ we have $1 \in U$. Now we just to proof that, 

$$\varphi^n \in S, \quad \forall n \geq 0, \quad \forall \varphi \in U$$

For this, suppose that $\varphi^n \in U$ for $\varphi^n \in U$ and $n \leq k - 1$. In this way we have

$$\psi_t = \frac{e^{t\varphi} - 1 - t\varphi - \frac{(t\varphi)^2}{2!} - \ldots - \frac{(t\varphi)^k}{k!}}{t^k}$$

Since $t\varphi \in U$, assume that $x \in E$ be given, then

$$|(\psi_t - \frac{\varphi^k}{k!})(x)| = \left| \frac{1}{tk}(e^{t\varphi} - 1 - t\varphi - \frac{(t\varphi)^2}{2!} - \ldots - \frac{(t\varphi)^k}{k!})(x) \right|$$

$$\leq t \sum_{n \geq k+1} t^{n-k-1} \frac{|\varphi(x)|^n}{n!} \leq t e^{\delta\|x\|}$$

Then in the space $H_{bc}(E)$ we have

$$\psi_t \to \frac{\varphi^k}{k!}, \quad t \to \infty$$

So $\frac{\varphi^k}{k!} \in \overline{S}$, and by this the proof is complete.

**Theorem 3.3** Let $T : X \to X$ be a hypercyclic operator and $U : X \to Y$ be a one by one operator with the dense range, then $UTU^{-1} : Y \to Y$ is hypercyclic.

**Proof.** Take hypercyclic vector $x \in X$, so we have to show that $U(x) \in Y$ is a hypercyclic vector for $UTU^{-1}$. Since $U(x) \in Y$ is a hypercyclic vector for $UTU^{-1}$ then

$$Orb(UTU^{-1}, U(x)) = \{(UTU^{-1})^n(U(x)) : n = 1, 2, 3, \ldots\}$$

$$= \{UT^nU^{-1}(U(x)) : n = 1, 2, 3, \ldots\}$$

$$= \{UT^n(U^{-1}(U(x))) : n = 1, 2, 3, \ldots\}$$

$$= \{UT^n(x) : n = 1, 2, 3, \ldots\}$$

$$= U(\{T^n(x) : n = 1, 2, 3, \ldots\})$$

$$= U(Orb(T, x))$$

so

$$Orb(UTU^{-1}, U(x)) = \overline{U(Orb(T, x))} = Y.$$
Since \( T(A) \subseteq \overline{T(A)} \) then \( U(X) = U(\overline{Orb(T,x)}) \subseteq U(\overline{Orb(T,x)}) \subseteq Y \) so \( Y = U(X) \subseteq U(Orb(T,x)) \subseteq Y \), now we have \( \overline{U(Orb(T,x))} = Y \), in other hand \( (UTU^{-1}, U(x)) = Y \). This concluded that the vector \( U(x) \in Y \) is a hypercyclic operator for \( UTU^{-1} \).

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**References**


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