Integral Representation and Certain Properties of M – Series Associated with Fractional Calculus

Kuldeep Singh Gehlot

Government Bangur P.G. College, Pali
 Pali-Marwar, Rajasthan, India-306401
 drksgehlol@rediffmail.com

Copyright © 2013 Kuldeep Singh Gehlot. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

This paper deals with the Integral Representations and Certain properties of M-Series. The main results are given in two sections namely 2 and 3. In section 2, we calculate four different Integral Representations of M-Series. In section 3, we calculate certain relation between M-Series and the Riemann- Liouville Fractional Integral and Derivatives are investigated. Also point out some particular cases.

Keywords: M-Series, Mittag-Leffler function, Generalized Hypergeometric Function, Fractional Calculus Operators.

Mathematics Subject Classification: 26A33, 33C20, 33E12.

1. Introduction

The M-series, introduced by [4], as

$${}_pM_q^\alpha [a_1, \dots, a_p; b_1, \dots, b_q; z] = {}_pM_q^\alpha [z] = \sum_{k=0}^{\infty} \frac{(a_1)_k, \dots, (a_p)_k}{(b_1)_k, \dots, (b_q)_k} \frac{z^k}{\Gamma(\alpha k + 1)}. \quad (1)$$

Here, $\alpha \in \mathbb{C}$, $\text{Re}(\alpha) > 0$ and $(a_j)_k, (b_j)_k$ are Pochhammer symbols. The series (1) is defined when none of the parameters b_j s, $j = 1, 2, \dots, q$, is a negative integer or zero. If any numerator parameter a_j is a negative integer or zero, then the series terminates to a polynomial in z . This series is convergent for all z if $p \leq q$, it is convergent if $p = q + 1$ and divergent, if $p > q + 1$. When $p = q + 1$ and $|z| = 1$, the series can converge in some cases. Let $\beta = \sum_{j=1}^p a_j - \sum_{j=1}^q b_j$. It can

be shown that when $p = q + 1$ the series is absolutely convergent for $|z| = 1$ if $\operatorname{Re}(\beta) < 0$, conditionally convergent for $z = -1$ if $0 \leq \operatorname{Re}(\beta) < 1$, and divergent for $|z| = 1$ if $1 \leq \operatorname{Re}(\beta)$.

Mittag- Leffler Function [1] is an entire function of order $\frac{1}{\alpha}$ defined by

$$E_{\alpha}[z] = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \quad (2)$$

when there is no upper (p) and lower (q) parameter in equation (1), we have

$${}_0M_{\alpha}^{\alpha}[z] = E_{\alpha}[z] = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \quad (3)$$

The Fractional Integral operators for $\vartheta > 0$ ([3], Definition 2.1, Page 33) are defined as,

$$(I_{0+}^{\vartheta}f)(x) = \frac{1}{\Gamma(\vartheta)} \int_0^x \frac{f(t)}{(x-t)^{1-\vartheta}} dt, (x > 0), \quad (4)$$

$$(I_{-}^{\vartheta}f)(x) = \frac{1}{\Gamma(\vartheta)} \int_x^{\infty} \frac{f(t)}{(t-x)^{1-\vartheta}} dt, (x > 0). \quad (5)$$

The Fractional Derivative for $\vartheta > 0$ ([3], Definition 2.2, Page 35) are defined as,

$$\begin{aligned} (D_{0+}^{\vartheta}f)(x) &= \left(\frac{d}{dx}\right)^{[\vartheta]+1} (I_{0+}^{1-\{\vartheta\}}f)(x) \\ &= \frac{1}{\Gamma(1-\{\vartheta\})} \left(\frac{d}{dx}\right)^{[\vartheta]+1} \int_0^x \frac{f(t)}{(x-t)^{\{\vartheta\}}} dt, (x > 0), \end{aligned} \quad (6)$$

$$\begin{aligned} (D_{-}^{\vartheta}f)(x) &= \left(\frac{d}{dx}\right)^{[\vartheta]+1} (I_{-}^{1-\{\vartheta\}}f)(x) \\ &= \frac{1}{\Gamma(1-\{\vartheta\})} \left(-\frac{d}{dx}\right)^{[\vartheta]+1} \int_x^{\infty} \frac{f(t)}{(t-x)^{\{\vartheta\}}} dt, (x > 0), \end{aligned} \quad (7)$$

where $[\vartheta]$ means the maximal integer not extending ϑ and $\{\vartheta\}$ is the fractional part of ϑ .

2. Integral representation of M-series

Theorem 2.1 For $a_j > 0, b_n - a_j > 0, \forall j = 1 \text{ to } p \text{ and } n = 1 \text{ to } q$ and convergent condition of M-Series are satisfies, then

$${}_pM_q^{\alpha}[z] = \prod_{j=1}^p \prod_{n=1}^q \frac{1}{B(a_j, b_n - a_j)} \int_0^1 (1-t)^{b_n - a_j - 1} t^{a_j - 1} {}_0M_{\alpha}^{\alpha}[tz] dt. \quad (8)$$

Proof : we know $(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$, is Pochhammer symbol, there for

$$\frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} = \frac{\prod_{j=1}^p \frac{\Gamma(a_j + k)}{\Gamma(a_j)}}{\prod_{n=1}^q \frac{\Gamma(b_n + k)}{\Gamma(b_n)}} = \prod_{j=1}^p \prod_{n=1}^q \frac{\Gamma(b_n)}{\Gamma(a_j)} \frac{\Gamma(a_j + k) \Gamma(b_n - a_j)}{\Gamma(b_n + k) \Gamma(b_n - a_j)},$$

$$\frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} = \prod_{j=1}^p \prod_{n=1}^q \frac{\Gamma(b_n)}{\Gamma(a_j) \Gamma(b_n - a_j)} \frac{\Gamma(a_j + k) \Gamma(b_n - a_j)}{\Gamma(a_j + k + b_n - a_j)},$$

we know $\int_0^1 (1-t)^{p-1} t^{q-1} dt = \frac{\Gamma(p) \Gamma(q)}{\Gamma(p+q)}$, we have

$$\frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} = \prod_{j=1}^p \prod_{n=1}^q \frac{\Gamma(b_n)}{\Gamma(a_j) \Gamma(b_n - a_j)} \int_0^1 (1-t)^{b_n - a_j - 1} t^{a_j + k - 1} dt, \quad (9)$$

from definition of M-series, equation (1),

$${}_p M_q^\alpha [z] = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{z^k}{\Gamma(\alpha k + 1)},$$

using equation (9), we have

$${}_p M_q^\alpha [z] = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \prod_{j=1}^p \prod_{n=1}^q \frac{1}{B(a_j, b_n - a_j)} \int_0^1 (1-t)^{b_n - a_j - 1} t^{a_j + k - 1} dt,$$

$${}_p M_q^\alpha [z] = \prod_{j=1}^p \prod_{n=1}^q \frac{1}{B(a_j, b_n - a_j)} \int_0^1 (1-t)^{b_n - a_j - 1} t^{a_j - 1} \sum_{k=0}^{\infty} \frac{(tz)^k}{\Gamma(\alpha k + 1)} dt,$$

$${}_p M_q^\alpha [z] = \prod_{j=1}^p \prod_{n=1}^q \frac{1}{B(a_j, b_n - a_j)} \int_0^1 (1-t)^{b_n - a_j - 1} t^{a_j - 1} {}_0 M_0^\alpha [tz] dt,$$

Hence.

Corollary 2.1.1 For $\alpha = 1, a_j > 0, b_n - a_j > 0, \forall j = 1$ to p and $n = 1$ to q and convergent condition of M-Series are satisfies, then equation (8) reduces in the form

$${}_p F_q [z] = \prod_{j=1}^p \prod_{n=1}^q \frac{1}{B(a_j, b_n - a_j)} \int_0^1 (1-t)^{b_n - a_j - 1} t^{a_j - 1} e^{tz} dt. \quad (10)$$

Remark : The formula (10) is a known relation, the integral representation of ${}_p F_q [z]$ function ([2], Theorem 28, page 85).

Lemma 2.1.1 For $a_j > 0, b_n - a_j > 0, \forall j = 1$ to p and $n = 1$ to q and convergent condition of M-Series are satisfies, then

$${}_pM_q^\alpha [z] = \frac{1}{B(a_1, b_1 - a_1)} \int_0^1 (1-t)^{b_1-a_1-1} t^{a_1-1} {}_{p-1}M_{q-1}^\alpha [tz] dt, \quad (11)$$

Proof : we know $(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$, is Pochhammer symbol.

$$\begin{aligned} \frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} &= \frac{\prod_{j=2}^p (a_j)_k (a_1)_k}{\prod_{n=2}^q (b_n)_k (b_1)_k} = \frac{\prod_{j=2}^p (a_j)_k \frac{\Gamma(a_1+k)}{\Gamma(a_1)}}{\prod_{n=2}^q (b_n)_k \frac{\Gamma(b_1+k)}{\Gamma(b_1)}}, \\ \frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} &= \frac{\prod_{j=2}^p (a_j)_k \Gamma(b_1) \Gamma(a_1+k) \Gamma(b_1-a_1)}{\prod_{n=2}^q (b_n)_k \Gamma(a_1) \Gamma(b_1+k) \Gamma(b_1-a_1)}, \\ \frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} &= \frac{\prod_{j=2}^p (a_j)_k \Gamma(b_1)}{\prod_{n=2}^q (b_n)_k \Gamma(a_1) \Gamma(b_1-a_1)} \frac{\Gamma(a_1+k) \Gamma(b_1-a_1)}{\Gamma(b_1-a_1+a_1+k)}, \end{aligned}$$

we know $\int_0^1 (1-t)^{p-1} t^{q-1} dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$, then we get

$$\frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} = \frac{\prod_{j=2}^p (a_j)_k}{\prod_{n=2}^q (b_n)_k} \frac{1}{B(a_1, b_1 - a_1)} \int_0^1 (1-t)^{b_1-a_1-1} t^{a_1+k-1} dt, \quad (12)$$

from definition of M-series,

$${}_pM_q^\alpha [z] = \sum_{k=0}^{\infty} \frac{(a_1)_k, \dots, (a_p)_k}{(b_1)_k, \dots, (b_q)_k} \frac{z^k}{\Gamma(\alpha k + 1)},$$

using equation (12), we have

$$\begin{aligned} {}_pM_q^\alpha [z] &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \frac{\prod_{j=2}^p (a_j)_k}{\prod_{n=2}^q (b_n)_k B(a_1, b_1 - a_1)} \int_0^1 (1-t)^{b_1-a_1-1} t^{a_1+k-1} dt \\ {}_pM_q^\alpha [z] &= \frac{1}{B(a_1, b_1 - a_1)} \int_0^1 (1-t)^{b_1-a_1-1} t^{a_1-1} \sum_{k=0}^{\infty} \frac{\prod_{j=2}^p (a_j)_k}{\prod_{n=2}^q (b_n)_k} \frac{(tz)^k}{\Gamma(\alpha k + 1)} dt, \\ {}_pM_q^\alpha [z] &= \frac{1}{B(a_1, b_1 - a_1)} \int_0^1 (1-t)^{b_1-a_1-1} t^{a_1-1} {}_{p-1}M_{q-1}^\alpha [tz] dt. \end{aligned}$$

Hence.

Theorem 2.2 For $\beta > 0$, and convergent conditions of M-series are satisfies, then

$${}_pM_q^\alpha [z] = \frac{z^{-\beta-1}}{\Gamma(\beta+1)} \int_0^\infty e^{-\frac{t}{z}} t^\beta {}_pM_{q+1}^\alpha [a_1, \dots, a_p; b_1, \dots, b_q, (\beta+1); t] dt. \quad (13)$$

Proof : Consider

$$A \equiv \int_0^\infty e^{-\frac{t}{z}} t^\beta {}_pM_{q+1}^\alpha [a_1, \dots, a_p; b_1, \dots, b_q, (\beta+1); t] dt,$$

using (1), we have

$$A \equiv \int_0^{\infty} e^{-\frac{t}{z}} t^{\beta} \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} \frac{t^k}{(\beta+1)_k \Gamma(\alpha k + 1)} dt,$$

put $\frac{t}{z} = u$, we obtain

$$A \equiv \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} \frac{z^{k+\beta+1}}{(\beta+1)_k \Gamma(\alpha k + 1)} \int_0^{\infty} e^{-u} u^{k+\beta} dt,$$

using definition of gamma function

$$A \equiv \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} \frac{z^{k+\beta+1} \Gamma(k + \beta + 1)}{(\beta+1)_k \Gamma(\alpha k + 1)},$$

$$A \equiv z^{\beta+1} \Gamma(\beta + 1) \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} \frac{z^k \Gamma(k + \beta + 1)}{(\beta+1)_k \Gamma(\beta + 1) \Gamma(\alpha k + 1)},$$

$$A \equiv z^{\beta+1} \Gamma(\beta + 1) \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} \frac{z^k (\beta + 1)_k}{(\beta+1)_k \Gamma(\alpha k + 1)},$$

$$A \equiv z^{\beta+1} \Gamma(\beta + 1) \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} \frac{z^k}{\Gamma(\alpha k + 1)},$$

$$A \equiv z^{\beta+1} \Gamma(\beta + 1) {}_p M_q [z].$$

Hence.

Lemma 2.2.1 For there is no upper (p) and lower (q) parameter in equation (13), and $\alpha > 0, \beta > 0$ then, there holds a Integral Representation formula for Mittag-Leffler function

$$E_{\alpha}[z] = \frac{1}{z^{\beta+1} \Gamma(\beta + 1)} \int_0^{\infty} e^{-\frac{t}{z}} t^{\beta} {}_0 M_1 [-; (\beta + 1); t] dt. \quad (14)$$

Theorem 2.3 For $\beta, \gamma \in \mathbb{C}, \beta > \gamma > 0$ and convergent conditions of M-series are satisfies, then

$${}_p M_{q+1} [a_1, \dots, a_p, \gamma; b_1, \dots, b_q, \beta; z]$$

$$= \frac{1}{\gamma \cdot B(\gamma, \beta - \gamma)} \int_0^1 \left(1 - t^{\frac{1}{\gamma}}\right)^{\beta - \gamma - 1} {}_p M_q \left[zt^{\frac{1}{\gamma}}\right] dt. \quad (15)$$

Proof: consider the integral,

$$A \equiv \int_0^1 \left(1 - t^{\frac{1}{\gamma}}\right)^{\beta-\gamma-1} {}^\alpha_p M_q \left[zt^{\frac{1}{\gamma}}\right] dt,$$

using the definition of M-series, equation (1), we have

$$A \equiv \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} \frac{1}{\Gamma(\alpha k + 1)} \int_0^1 \left(1 - t^{\frac{1}{\gamma}}\right)^{\beta-\gamma-1} \left(zt^{\frac{1}{\gamma}}\right)^k dt,$$

put $t^{\frac{1}{\gamma}} = u$, we have

$$A \equiv \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} \frac{z^k \gamma}{\Gamma(\alpha k + 1)} \int_0^1 (1 - u)^{\beta-\gamma-1} (u)^{k+\gamma-1} du,$$

using definition of Beta Function,

$$A \equiv \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} \frac{z^k \gamma}{\Gamma(\alpha k + 1)} \frac{\Gamma(\beta - \gamma) \Gamma(\gamma + k)}{\Gamma(\beta + k)},$$

$$A \equiv \gamma \cdot \Gamma(\beta - \gamma) \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} \frac{z^k}{\Gamma(\alpha k + 1)} \frac{(\gamma)_k \Gamma(\gamma)}{(\beta)_k \Gamma(\beta)},$$

$$A \equiv \frac{\gamma \cdot \Gamma(\beta - \gamma) \Gamma(\gamma)}{\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} \frac{z^k}{\Gamma(\alpha k + 1)} \frac{(\gamma)_k}{(\beta)_k},$$

$$A \equiv \gamma \cdot B(\gamma, \beta - \gamma) {}_{p+1}M_{q+1}[a_1, \dots, a_p, \gamma; b_1, \dots, b_q, \beta; z].$$

Hence.

Theorem 2.4 For $\rho \in \mathbb{C}$, $\text{Re}(\rho) > 0$ and convergent conditions of M-series are satisfies, then

$${}_{p+1}M_q \left[a_1, \dots, a_p, \rho; b_1, \dots, b_q; \frac{a}{z} \right] = \frac{z^\rho}{\Gamma(\rho)} \int_0^\infty t^{\rho-1} e^{-tz} {}^\alpha_p M_q [at] dt. \quad (16)$$

Proof: Consider the integral,

$$B \equiv \int_0^\infty t^{\rho-1} e^{-tz} {}^\alpha_p M_q [at] dt,$$

using the definition of M-series, equation (1), we have

$$B \equiv \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} \frac{a^k}{\Gamma(\alpha k + 1)} \int_0^\infty e^{-tz} t^{\rho+k-1} dt,$$

using definition of Gamma Function, we have

$$B \equiv \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} \frac{a^k}{\Gamma(\alpha k + 1)} \frac{\Gamma(\rho + k)}{z^{\rho+k}},$$

$$B \equiv \frac{\Gamma(\rho)}{z^\rho} \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} \frac{(\rho)_k}{\Gamma(\alpha k + 1)} \left(\frac{a}{z}\right)^k,$$

$$B \equiv \frac{\Gamma(\rho)}{z^\rho} {}_{p+1}M_q \left[a_1, \dots, a_p, \rho; b_1, \dots, b_q; \frac{a}{z} \right].$$

Hence.

3. Certain properties of M-series associated with Fractional Calculus

Theorem 3.1 Let $\alpha > 0, \vartheta > 0, \gamma > 0$ and $a \in R$. Let I_{0+}^ϑ be the left $-$ sided operator of Riemann- Liouville fractional integral (given by equation (4)). Then there holds a formula,

$$\left(I_{0+}^\vartheta \left[t^{\gamma-1} {}_pM_q \left[at \right] \right] \right) (x) = \frac{x^{\gamma+\vartheta-1} \Gamma(\gamma)}{\Gamma(\gamma + \vartheta)} \times {}_{p+1}M_{q+1} [a_1, \dots, a_p, \gamma; b_1, \dots, b_q, \gamma + \vartheta; ax]. \quad (17)$$

Proof: By virtue of (1) and (4), we have

$$A \equiv \left(I_{0+}^\vartheta \left[t^{\gamma-1} {}_pM_q \left[at \right] \right] \right) (x),$$

$$A \equiv \frac{1}{\Gamma(\vartheta)} \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} \frac{a^k}{\Gamma(\alpha k + 1)} \int_0^x \frac{t^{\gamma+k-1}}{(x-t)^{1-\vartheta}} dt,$$

put $t = xu$, we have

$$A \equiv \frac{1}{\Gamma(\vartheta)} \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} \frac{a^k x^{k+\gamma+\vartheta-1}}{\Gamma(\alpha k + 1)} \int_0^1 u^{k+\gamma-1} (1-u)^{\vartheta-1} du,$$

using the definition of Beta function,

$$A \equiv \frac{x^{\gamma+\vartheta-1}}{\Gamma(\vartheta)} \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} \frac{(ax)^k}{\Gamma(\alpha k + 1)} \frac{\Gamma(\gamma + k) \Gamma(\vartheta)}{\Gamma(\gamma + \vartheta + k)},$$

$$A \equiv \frac{x^{\gamma+\vartheta-1} \Gamma(\gamma)}{\Gamma(\gamma + \vartheta)} \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_k (\gamma)_k}{\prod_{n=1}^q (b_n)_k (\gamma + \vartheta)_k} \frac{(ax)^k}{\Gamma(\alpha k + 1)},$$

$$A \equiv \frac{x^{\gamma+\vartheta-1} \Gamma(\gamma)}{\Gamma(\gamma + \vartheta)} {}_{p+1}M_{q+1} [a_1, \dots, a_p, \gamma; b_1, \dots, b_q, \gamma + \vartheta; ax].$$

Hence.

Lemma 3.1.1 For there is no upper (p) and lower (q) parameter in equation (17), and $\alpha > 0, \vartheta > 0, \gamma > 0$ and $a \in R$. Let I_{0+}^{ϑ} be the left ϑ -sided operator of Riemann- Liouville fractional integral (given by equation (4)). Then there holds a formula,

$$(I_{0+}^{\vartheta} [t^{\gamma-1} E_{\alpha} [at]])(x) = \frac{x^{\gamma+\vartheta-1} \Gamma(\gamma)}{\Gamma(\gamma + \vartheta)} {}_1M_1 [\gamma; \gamma + \vartheta; ax]. \quad (18)$$

Lemma 3.1.2 For $\alpha = 1, \vartheta > 0, \gamma > 0$ and $a \in R$. Let I_{0+}^{ϑ} be the left ϑ -sided operator of Riemann- Liouville fractional integral (given by equation (4)). Then equation (17) will reduce in the form,

$$\begin{aligned} \left(I_{0+}^{\vartheta} \left[t^{\gamma-1} {}_pF_q [at] \right] \right) (x) \\ = \frac{x^{\gamma+\vartheta-1} \Gamma(\gamma)}{\Gamma(\gamma + \vartheta)} {}_{p+1}F_{q+1} [a_1, \dots, a_p, \gamma; b_1, \dots, b_q, \gamma + \vartheta; ax]. \end{aligned} \quad (19)$$

Theorem 3.2 Let $\alpha > 0, \vartheta > 0, \gamma > 0$ and $a \in R$. Let I_{0-}^{ϑ} be the right ϑ -sided operator of Riemann- Liouville fractional integral (given by equation (5)). Then there holds a formula,

$$\begin{aligned} \left(I_{-}^{\vartheta} \left[t^{-\gamma-\vartheta} {}_pM_q [at^{-1}] \right] \right) (x) \\ = \frac{x^{-\gamma} \Gamma(\gamma)}{\Gamma(\gamma + \vartheta)} {}_{p+1}M_{q+1} [a_1, \dots, a_p, \gamma; b_1, \dots, b_q, \gamma + \vartheta; ax^{-1}]. \end{aligned} \quad (20)$$

Proof: By virtue of (1) and (5), we have

$$\begin{aligned} A &\equiv \left(I_{-}^{\vartheta} \left[t^{-\gamma-\vartheta} {}_pM_q [at^{-1}] \right] \right) (x), \\ A &\equiv \frac{1}{\Gamma(\vartheta)} \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} \frac{a^k}{\Gamma(\alpha k + 1)} \int_x^{\infty} t^{-\gamma-\vartheta-k} (t-x)^{\vartheta-1} dt, \end{aligned}$$

put $t = \frac{x}{u}$, we have

$$A \equiv \frac{x^{-\gamma}}{\Gamma(\vartheta)} \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} \frac{(ax^{-1})^k}{\Gamma(\alpha k + 1)} \int_0^1 u^{\gamma+k-1} (1-u)^{\vartheta-1} du,$$

using the definition of Bata function,

$$\begin{aligned}
A &\equiv \frac{x^{-\gamma}}{\Gamma(\vartheta)} \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} \frac{(ax^{-1})^k}{\Gamma(\alpha k + 1)} \frac{\Gamma(\gamma + k)\Gamma(\vartheta)}{\Gamma(\gamma + \vartheta + k)}, \\
A &\equiv \frac{x^{-\gamma}\Gamma(\gamma)}{\Gamma(\gamma + \vartheta)} \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_k (\gamma)_k}{\prod_{n=1}^q (b_n)_k (\gamma + \vartheta)_k} \frac{(ax^{-1})^k}{\Gamma(\alpha k + 1)}, \\
A &\equiv \frac{x^{-\gamma}\Gamma(\gamma)}{\Gamma(\gamma + \vartheta)} {}_{p+1}M_{q+1}^{\alpha}[a_1, \dots, a_p, \gamma; b_1, \dots, b_q, \gamma + \vartheta; ax^{-1}].
\end{aligned}$$

Hence.

Lemma 3.2.1 For there is no upper (p) and lower (q) parameter in equation (20), and $\alpha > 0, \vartheta > 0, \gamma > 0$ and $a \in R$. Let I_{0-}^{ϑ} be the right –sided operator of Riemann- Liouville fractional integral (given by equation (5)). Then there holds a formula,

$$\left(I_{-}^{\vartheta} \left[t^{-\gamma-\vartheta} E_{\alpha} [at^{-1}] \right] \right) (x) = \frac{x^{-\gamma}\Gamma(\gamma)}{\Gamma(\gamma + \vartheta)} {}_1M_1^{\alpha} [\gamma; \gamma + \vartheta; ax^{-1}]. \quad (21)$$

Lemma 3.2.2 For $\alpha = 1, \vartheta > 0, \gamma > 0$ and $a \in R$. Let I_{0-}^{ϑ} be the right –sided operator of Riemann- Liouville fractional integral (given by equation (5)). Then equation (20) will reduce in the form,

$$\begin{aligned}
&\left(I_{-}^{\vartheta} \left[t^{-\gamma-\vartheta} {}_pF_q [at^{-1}] \right] \right) (x) \\
&= \frac{x^{-\gamma}\Gamma(\gamma)}{\Gamma(\gamma + \vartheta)} {}_{p+1}F_{q+1} [a_1, \dots, a_p, \gamma; b_1, \dots, b_q, \gamma + \vartheta; ax^{-1}]. \quad (22)
\end{aligned}$$

Theorem 3.3 Let $\alpha > 0, \gamma > \vartheta > 0$ and $a \in R$. Let D_{0+}^{ϑ} be the left –sided operator of Riemann- Liouville fractional derivative (given by equation (6)). Then there holds a formula,

$$\begin{aligned}
&\left(D_{0+}^{\vartheta} \left[t^{\gamma-1} {}_pM_q^{\alpha} [at] \right] \right) (x) \\
&= \frac{x^{\gamma-\vartheta-1}\Gamma(\gamma)}{\Gamma(\gamma - \vartheta)} {}_{p+1}M_{q+1}^{\alpha} [a_1, \dots, a_p, \gamma; b_1, \dots, b_q, \gamma - \vartheta; ax]. \quad (23)
\end{aligned}$$

Proof: By virtue of (1) and (6), we have

$$A \equiv \left(D_{0+}^{\vartheta} \left[t^{\gamma-1} {}_pM_q^{\alpha} [at] \right] \right) (x),$$

$$A \equiv \frac{1}{\Gamma(r-\vartheta)} \left(\frac{d}{dx}\right)^r \int_0^x t^{\gamma-1} (x-t)^{r-\vartheta-1} \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} \frac{(at)^k}{\Gamma(\alpha k+1)} dt,$$

where $r = [\alpha] + 1$,

$$A \equiv \frac{1}{\Gamma(r-\vartheta)} \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} \frac{a^k}{\Gamma(\alpha k+1)} \left(\frac{d}{dx}\right)^r \int_0^x t^{k+\gamma-1} (x-t)^{r-\vartheta-1} dt,$$

put $t = xu$ and using Beta function, we have

$$A \equiv \frac{1}{\Gamma(r-\vartheta)} \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} \frac{a^k}{\Gamma(\alpha k+1)} \left(\frac{d}{dx}\right)^r x^{r+\gamma+k-\vartheta-1} B(r-\vartheta, \gamma+k),$$

$$A \equiv \frac{x^{\gamma-\vartheta-1} \Gamma(\gamma)}{\Gamma(\gamma-\vartheta)} \frac{\alpha}{p+1} M_{q+1} [a_1, \dots, a_p, \gamma; b_1, \dots, b_q, \gamma-\vartheta; ax].$$

Hence.

Lemma 3.3.1 For there is no upper (p) and lower (q) parameter in equation (23), and $\alpha > 0, \gamma > \vartheta > 0$ and $a \in R$. Let D_{0+}^{ϑ} be the left -sided operator of Riemann- Liouville fractional integral (given by equation (6)). Then there holds a formula,

$$(D_{0+}^{\vartheta} [t^{\gamma-1} E_{\alpha}[at]])(x) = \frac{x^{\gamma-\vartheta-1} \Gamma(\gamma)}{\Gamma(\gamma-\vartheta)} \frac{\alpha}{1} M_1 [\gamma; \gamma-\vartheta; ax]. \quad (24)$$

Lemma 3.3.2 For $\alpha = 1, \gamma > \vartheta > 0$ and $a \in R$. Let D_{0+}^{ϑ} be the left -sided operator of Riemann- Liouville fractional integral (given by equation (6)). Then equation (23) will reduce in the form,

$$\begin{aligned} & \left(D_{0+}^{\vartheta} \left[t^{\gamma-1} {}_p F_q [at] \right] \right) (x) \\ &= \frac{x^{\gamma-\vartheta-1} \Gamma(\gamma)}{\Gamma(\gamma-\vartheta)} {}_{p+1} F_{q+1} [a_1, \dots, a_p, \gamma; b_1, \dots, b_q, \gamma-\vartheta; ax]. \end{aligned} \quad (25)$$

Theorem 3.4 Let $\gamma > \vartheta > 0$, with $\gamma - \vartheta + \{\vartheta\} > 1$ and $a \in R$. Let D_-^{ϑ} be the right -sided operator of Riemann- Liouville fractional derivative (given by equation (7)). Then there holds a formula,

$$\left(D_-^{\vartheta} \left[t^{\vartheta-\gamma} {}_p M_q [at^{-1}] \right] \right) (x)$$

$$= \frac{x^{-\gamma} \Gamma(\gamma)}{\Gamma(\gamma - \vartheta)} {}^{p+1}M_{q+1}^{\alpha} [a_1, \dots, a_p, \gamma; b_1, \dots, b_q, \gamma - \vartheta; ax^{-1}]. \quad (26)$$

Proof: From (1) and (7) it follows that,

$$A \equiv \left(D_-^{\vartheta} \left[t^{\vartheta-\gamma} {}^pM_q^{\alpha} [at^{-1}] \right] \right) (x),$$

$$A \equiv \frac{1}{\Gamma(r - \vartheta)} \left(-\frac{d}{dx} \right)^r \int_x^{\infty} \frac{t^{\vartheta-\gamma}}{(t-x)^{1+\vartheta-r}} \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_k}{\prod_{n=1}^q (b_n)_k} \frac{(at^{-1})^k}{\Gamma(\alpha k + 1)} dt,$$

where $r = [\alpha] + 1$, and put $t = \frac{x}{u}$ and use Beta function, we have

$$A \equiv \frac{x^{-\gamma} \Gamma(\gamma)}{\Gamma(\gamma - \vartheta)} {}^{p+1}M_{q+1}^{\alpha} [a_1, \dots, a_p, \gamma; b_1, \dots, b_q, \gamma - \vartheta; ax^{-1}].$$

Hence.

Lemma 3.4.1 For there is no upper (p) and lower (q) parameter in equation (26), and $\alpha > 0, \gamma > \vartheta > 0$ and $a \in R$. Let D_{0-}^{ϑ} be the rightt -sided operator of Riemann- Liouville fractional integral (given by equation (7)). Then there holds a formula,

$$\left(D_-^{\vartheta} \left[t^{\vartheta-\gamma} E_{\alpha} [at^{-1}] \right] \right) (x) = \frac{x^{-\gamma} \Gamma(\gamma)}{\Gamma(\gamma - \vartheta)} {}_1M_1^{\alpha} [\gamma; \gamma - \vartheta; ax^{-1}]. \quad (27)$$

Lemma 3.4.2 For $\alpha = 1, \gamma > \vartheta > 0$ and $a \in R$. Let D_{0-}^{ϑ} be the rightt -sided operator of Riemann- Liouville fractional integral (given by equation (7)). Then equation (26) will reduce in the form,

$$\left(D_-^{\vartheta} \left[t^{\vartheta-\gamma} {}^pF_q [at^{-1}] \right] \right) (x)$$

$$= \frac{x^{-\gamma} \Gamma(\gamma)}{\Gamma(\gamma - \vartheta)} {}^{p+1}F_{q+1} [a_1, \dots, a_p, \gamma; b_1, \dots, b_q, \gamma - \vartheta; ax^{-1}]. \quad (28)$$

References

- [1] Mittag- Leililer , G., Sur la nouvelle fonction $E_{\alpha}[x]$, C.R. Acad. Sic. Paris. (Ser. II), (1903), 137, 554-558.
- [2] Rainville, E.D., Special Function, The Macmillan Company, New York, 1963.

- [3] Samko, S.G., Kilbas, A.A., Marichev, O.I., Fractional Integral and Derivatives. Theory and Applications. Gordon and Breach, Yverdon et al. (1993).
- [4] Sharma, M. Fractional Integral and Fractional Differentiation of the M-Series, Fractional Calculus and Applied Analysis, Vol. 11, No. 2 (2008), 187-191.

Received: October, 2012