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# Conditional Fourier-Feynman Transforms and Convolutions over Continuous Paths

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**Abstract**

In the present paper, we evaluate the analytic conditional Fourier-Feynman transforms and convolution products of bounded functions which are important in Feynman integration theories and quantum mechanics. We then investigate the inverse transforms of the functions with their relationships and finally that the conditional analytic Fourier-Feynman transforms of the conditional convolution products for the functions, can be expressed in terms of the products of the conditional Fourier-Feynman transform of each function.

**Mathematics Subject Classification:** 28C20

**Keywords:** analogue of Wiener measure, conditional analytic Feynman integral, conditional analytic Wiener integral, conditional Fourier-Feynman-transform, convolution, Wiener space

# 1 Introduction and preliminaries

Let $C_0[0,t]$ denote the Wiener space, that is, the space of real-valued continuous paths $x$ on the closed interval $[0,t]$ with $x(0)=0$. On the space $C_0[0,t]$, Chang and Skoug [2] introduced the concepts of conditional Fourier-Feynman transform and conditional convolution product and then, examined the effects that drift has on the conditional Fourier-Feynman transform, the conditional convolution product, and various relationships that occur between them. Moreover, on $C[0,t]$, the space of real-valued continuous paths on $[0,t]$, Kim [12] extended the relationships between the conditional convolution product and the $L_p(1\le p\le\infty)$-analytic conditional Fourier-Feynman transform of the functions in a Banach algebra which corresponds to the Cameron-Storvick's Banach algebra $\mathcal{S}$ [1]. The second author [3, 4, 5, 6, 7] also established the relationships between them for various functions on $C[0,t]$. In particular, he [6] derived an evaluation formula for the $L_p$-analytic conditional Fourier-Feynman transforms and convolution products of bounded functions with the conditioning functions $X_n$ and $X_{n+1}$ on $C[0,t]$ given by $X_n(x)=(x(t_0),x(t_1),\cdots,x(t_n))$ and $X_{n+1}(x)=(x(t_0),x(t_1),\cdots,x(t_n),x(t_{n+1}))$, where $n$ is a positive integer and $0=t_0<t_1<\cdots<t_n<t_{n+1}=t$ is a partition of $[0,t]$, and then, derived their relationships. Note that $X_n$ is independent of the present positions of paths in $C[0,t]$ while $X_{n+1}$ wholly depends on the present positions.

In this paper, when $n=0$, we further develop the relationships on the space $(C[0,t],w_\varphi)$, an analogue of Wiener space associated with a probability measure $\varphi$ on the Borel class of $\mathbb{R}$ [11, 13, 14]. In fact, using simple formulas for the conditional expectations given $X_0$ and $X_1$, we proceed to evaluate the $L_p$-analytic conditional Fourier-Feynman transforms and convolution products for the functions of the form

$$\int_{L_2[0,t]}\exp\{i(v,x)\}d\sigma(v)\int_{\mathbb{R}^r}\exp\Big\{i\sum_{j=1}^{r}z_j(v_j,x)\Big\}d\rho(z_1,\cdots,z_r) \tag{1}$$

for $w_\varphi$-a.e. $x\in C[0,t]$, where $\{v_1,\cdots,v_r\}$ is an orthonormal set in $L_2[0,t]$, $\sigma$ is a complex Borel measure of bounded variation on $L_2[0,t]$ and $\rho$ is a bounded complex Borel measure on $\mathbb{R}^r$. We then investigate the inverse transforms and the relationships between them of the functions given by (1). Finally, we show that the $L_p$-analytic conditional Fourier-Feynman transform of the conditional convolution product for the functions $\Psi_1$ and $\Psi_2$ given by (1), can be expressed by the formula

$$\begin{aligned}&T_q^{(p)}[[(\Psi_1*\Psi_2)_q|X_0](\cdot,\xi_0)|X_0](y,\zeta_0)\\ =\ &\Big[T_q^{(p)}[\Psi_1|X_0]\Big(\frac{1}{\sqrt{2}}y,0\Big)\Big]\Big[T_q^{(p)}[\Psi_2|X_0]\Big(\frac{1}{\sqrt{2}}y,0\Big)\Big]\end{aligned}$$

for a nonzero real $q$, $w_\varphi$-a.e. $y\in C[0,t]$ and $P_{X_0}$-a.e. $\xi_0,\zeta_0\in\mathbb{R}$. Thus the analytic conditional Fourier-Feynman transforms of the conditional convolution products for the functions, can be interpreted as the product of the analytic conditional Fourier-Feynman transform of each function. Note that they are independent of the initial positions of paths in $C[0,t]$ while they depend only on the present time $t$.

Let $\mathbb{C}$, $\mathbb{C}_+$ and $\mathbb{C}_+^{\sim}$ denote the sets of complex numbers, complex numbers with positive real parts and nonzero complex numbers with nonnegative real parts, respectively.

Let $C = C[0,t]$ be the space of all real-valued continuous functions on the closed interval $[0,t]$ and let $(C[0,t], \mathcal{B}(C[0,t]), w_\varphi)$ [11, 13, 14] be the analogue of Wiener space associated with a probability measure $\varphi$ on the Borel class of $\mathbb{R}$, where $\mathcal{B}(C[0,t])$ denotes the Borel class of $C[0,t]$.

Let $\{d_j : j = 1, 2, \cdots\}$ be a complete orthonormal subset of $L_2[0,t]$ such that each $d_j$ is of bounded variation. For $v$ in $L_2[0,t]$ and $x$ in $C[0,t]$, let $(v,x) = \lim_{n\to\infty} \sum_{j=1}^n \langle v, d_j\rangle \int_0^t d_j(s)dx(s)$ if the limit exists, where $\langle\cdot,\cdot\rangle$ denotes the inner product over $L_2[0,t]$. $(v,x)$ is called the Paley-Wiener-Zygmund integral of $v$ according to $x$. Note that by $\langle\cdot,\cdot\rangle_{\mathbb{R}^r}$, the dot product on the $r$-dimensional Euclidean space $\mathbb{R}^r$ is also denoted.

Let $F : C[0,t] \to \mathbb{C}$ be integrable and $X$ be a random vector on $C[0,t]$ assuming that the value space of $X$ is a normed space equipped with the Borel $\sigma$-algebra. Then, we have the conditional expectation $E[F|X]$ of $F$ given $X$ from a well known probability theory. Furthermore, there exists a $P_X$-integrable complex-valued function $\psi$ on the value space of $X$ such that $E[F|X](x) = (\psi \circ X)(x)$ for $w_\varphi$-a.e. $x \in C[0,t]$, where $P_X$ is the probability distribution of $X$. The function $\psi$ is called the conditional $w_\varphi$-integral of $F$ given $X$ and it is also denoted by $E[F|X]$.

Throughout this paper, let $X_0 : C[0,t] \to \mathbb{R}$ and $X_1 : C[0,t] \to \mathbb{R}^2$ be given by $X_0(x) = x(0)$ and $X_1(x) = (x(0), x(t))$. Let $F : C[0,t] \to \mathbb{C}$ be a function such that $F(\lambda^{-\frac12}\cdot)$ is integrable for $\lambda > 0$. Then we have the following formula from [9], $E[F(\lambda^{-\frac12}\cdot)|X_1(\lambda^{-\frac12}\cdot)](\xi_0,\xi_1) = E[F(\lambda^{-\frac12}(x - x(0) - \frac{\cdot}{t}(x(t) - x(0))) + \xi_0 + \frac{\cdot}{t}(\xi_1 - \xi_0))]$ for $P_{X_1^\lambda}$-a.e. $(\xi_0,\xi_1) \in \mathbb{R}^2$, where $P_{X_1^\lambda}$ is the probability distribution of $X_1(\lambda^{-\frac12}\cdot)$ on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$. For $y \in C[0,t]$ let

$$I_F^\lambda(y,\xi_0,\xi_1) = E\Big[F\Big(\lambda^{-\frac12}\Big(x - x(0) - \frac{\cdot}{t}(x(t) - x(0))\Big) + y + \xi_0 + \frac{\cdot}{t}(\xi_1 - \xi_0)\Big)\Big]$$

unless otherwise specified, where the expectation is taken over the variable $x$. Moreover, we have from [8], $E[F(\lambda^{-\frac12}\cdot)|X_0(\lambda^{-\frac12}\cdot)](\xi_0) = (\frac{\lambda}{2\pi t})^{\frac12}\int_{\mathbb{R}} I_F^\lambda(0,\xi_0,\xi_1) \cdot \exp\{-\frac{\lambda(\xi_1-\xi_0)^2}{2t}\}d\xi_1$ for $P_{X_0^\lambda}$-a.e. $\xi_1 \in \mathbb{R}$, where $P_{X_0^\lambda}$ is the probability distribution of $X_0(\lambda^{-\frac12}\cdot)$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. For $y \in C[0,t]$, let

$$K_F^\lambda(y,\xi_0) = \Big(\frac{\lambda}{2\pi t}\Big)^{\frac12}\int_{\mathbb{R}} I_F^\lambda(y,\xi_0,\xi_1)\exp\Big\{-\frac{\lambda(\xi_1-\xi_0)^2}{2t}\Big\}d\xi_1.$$

If $I_F^\lambda(0,\xi_0,\xi_1)$ has the analytic extension $J_\lambda^*(F)(\xi_0,\xi_1)$ on $\mathbb{C}_+$ as a function of $\lambda$, then it is called the conditional analytic Wiener $w_\varphi$-integral of $F$ given

$X_1$ with parameter $\lambda$ and denoted by $E^{anw_\lambda}[F|X_1](\xi_0,\xi_1) = J^*_\lambda(F)(\xi_0,\xi_1)$ for $(\xi_1,\xi_1) \in \mathbb{R}^2$. Moreover, if for a nonzero real $q$, $E^{anw_\lambda}[F|X_1](\xi_0,\xi_1)$ has the limit as $\lambda$ approaches to $-iq$ through $\mathbb{C}_+$, then it is called the conditional analytic Feynman $w_\varphi$-integral of $F$ given $X_1$ with parameter $q$ and denoted by $E^{anf_q}[F|X_1](\xi_0,\xi_1) = \lim_{\lambda\to -iq} E^{anw_\lambda}[F|X_1](\xi_0,\xi_1)$. Similarly, the definitions of $E^{anw_\lambda}[F|X_0](\xi_0)$ and $E^{anf_q}[F|X_0](\xi_0)$ are understood with $K_F^\lambda(0,\xi_0)$ if $X_1$ is replaced by $X_0$.

For a given extended real number $p$ with $1 < p \le \infty$, suppose that $p$ and $p'$ are related by $\frac{1}{p} + \frac{1}{p'} = 1$(possibly $p' = 1$ if $p = \infty$). Let $F_n$ and $F$ be measurable functions such that $\lim_{n\to\infty} \int_C |F_n(x) - F(x)|^{p'} dw_\varphi(x) = 0$. Then we write $\text{l.i.m.}_{n\to\infty}(w^{p'})(F_n) = F$ and call $F$ the limit in the mean of order $p'$. A similar definition is understood when $n$ is replaced by a continuously varying parameter.

Let $F$ and $G$ be defined on $C[0,t]$. For $\lambda \in \mathbb{C}_+$ and $w_\varphi$-a.e. $y \in C[0,t]$, let $T_\lambda[F|X_1](y,\xi_0,\xi_1) = E^{anw_\lambda}[F(y+\cdot)|X_1](\xi_0,\xi_1)$ for $P_{X_1}$-a.e. $(\xi_0,\xi_1) \in \mathbb{R}^2$ if it exists. For a nonzero real $q$ and $w_\varphi$-a.e. $y \in C[0,t]$, define the $L_1$-analytic conditional Fourier-Feynman transform $T_q^{(1)}[F|X_1]$ of $F$ given $X_1$ by the formula $T_q^{(1)}[F|X_1](y,\xi_0,\xi_1) = E^{anf_q}[F(y+\cdot)|X_1](\xi_0,\xi_1)$ for $P_{X_1}$-a.e. $(\xi_0,\xi_1) \in \mathbb{R}^2$ if it exists. For $1 < p \le \infty$, define the $L_p$-analytic conditional Fourier-Feynman transform $T_q^{(p)}[F|X_1]$ of $F$ given $X_1$ by the formula $T_q^{(p)}[F|X_1](\cdot,\xi_0,\xi_1) = \text{l.i.m.}_{\lambda\to -iq}(w^{p'})(T_\lambda[F|X_1](\cdot,\xi_0,\xi_1))$ for $P_{X_1}$-a.e. $(\xi_0,\xi_1) \in \mathbb{R}^2$, where $\lambda$ approaches to $-iq$ through $\mathbb{C}_+$. We also define the conditional convolution product $[(F*G)_\lambda|X_1]$ of $F$ and $G$ given $X_1$ by the formula, for $w_\varphi$-a.e. $y \in C[0,t]$

$$\begin{aligned}&[(F*G)_\lambda|X_1](y,\xi_0,\xi_1)\\ =&\begin{cases} E^{anw_\lambda}\Big[F\Big(\dfrac{y+\cdot}{\sqrt{2}}\Big)G\Big(\dfrac{y-\cdot}{\sqrt{2}}\Big)\Big|X_1\Big](\xi_0,\xi_1), & \lambda \in \mathbb{C}_+;\\ E^{anf_q}\Big[F\Big(\dfrac{y+\cdot}{\sqrt{2}}\Big)G\Big(\dfrac{y-\cdot}{\sqrt{2}}\Big)\Big|X_1\Big](\xi_0,\xi_1), & \lambda = -iq; \quad q \in \mathbb{R}-\{0\}\end{cases}\end{aligned}$$

if they exist for $P_{X_1}$-a.e. $(\xi_0,\xi_1) \in \mathbb{R}^2$. If $\lambda = -iq$, we replace $[(F*G)_\lambda|X_1]$ by $[(F*G)_q|X_1]$. Similar definitions and notations are understood with $\xi_0 \in \mathbb{R}$ if $X_1$ is replaced by $X_0$.

## 2 The position-dependent transforms

Throughout this paper, let $\{v_l : l = 1,\cdots,r\}$ be an orthonormal subset of $L_2[0,t]$ such that $\mathcal{G} \equiv \{v_l - \frac{1}{t}\int_0^t v_l(s)ds : l = 1,\cdots,r\}$ is an independent set. Let $\{e_l : l = 1,\cdots,r\}$ be the orthonormal set obtained from $\mathcal{G}$ by the Gram-Schmidt orthonormalization process. Now, for $l = 1,\cdots,r$, let $v_l - \frac{1}{t}\int_0^t v_l(s)ds = \sum_{j=1}^r \beta_{lj}e_j$ be the linear combinations of the $e_j$s and let $A = [\beta_{lj}]_{r\times r}$ be the coefficient matrix of the combinations. Define the linear transformation $T_A : \mathbb{R}^r \to \mathbb{R}^r$ given by $T_A\vec{z} = \vec{z}A$, where $\vec{z}$ is any row-vector in $\mathbb{R}^r$. For

$v \in L_2[0,t]$ let $c_l(v) = \langle v, e_l\rangle - \frac{1}{t}\int_0^t v(s)ds \int_0^t e(s)ds$ $(l = 1, \cdots, r)$, let $(\vec{v}, x) = ((v_1, x), \cdots, (v_r, x))$ for $x \in C[0,t]$ and let $\vec{V}_t = \frac{1}{t}(\int_0^t v_1(s)ds, \cdots, \int_0^t v_r(s)ds)$. Furthermore, for $\vec{z} \in \mathbb{R}^r$ and $\xi_0, \xi_1 \in \mathbb{R}$, let

$$\begin{aligned} & H_1(x, \xi_0, \xi_1, v, \vec{z}) \\ = \;& \exp\Big\{i\Big[(v,x) + \langle(\vec{v},x),\vec{z}\rangle_{\mathbb{R}^r} + (\xi_1 - \xi_0)\Big[\langle\vec{V}_t, \vec{z}\rangle_{\mathbb{R}^r} + \frac{1}{t}\int_0^t v(s)ds\Big]\Big]\Big\} \end{aligned} \tag{2}$$

and for $\lambda \in \mathbb{C}_+^\sim$ let

$$H_2(\lambda, v, \vec{z}) = \exp\Big\{\frac{-1}{2\lambda}\Big[\Big\|v - \frac{1}{t}\int_0^t v(s)ds\Big\|_2^2 - 2\langle\vec{c}(v), T_A\vec{z}\rangle_{\mathbb{R}^r} + \|T_A\vec{z}\|_{\mathbb{R}^r}^2\Big]\Big\}, \tag{3}$$

where $\vec{c}(v) = (c_1(v), \cdots, c_r(v))$. Note that by the Bessel's inequality,

$$|H_2(\lambda, v, \vec{z})| \le 1 \tag{4}$$

Using the same method as used in the proof of Theorem 2.6 in [10], we can prove the following lemma.

**Lemma 2.1** *For $x \in C[0,t]$, $\lambda > 0$, $v \in L_2[0,t]$ and $\vec{z} \in \mathbb{R}^r$, let*

$$\begin{aligned} H_3(\lambda, v, \vec{z}, x) \;=\; & \exp\Big\{i\lambda^{-\frac{1}{2}}\Big[\Big(v, x - x(0) - \frac{\cdot}{t}(x(t) - x(0))\Big) \\ & + \Big\langle\Big(\vec{v}, x - x(0) - \frac{\cdot}{t}(x(t) - x(0))\Big), \vec{z}\Big\rangle_{\mathbb{R}^r}\Big]\Big\}. \end{aligned} \tag{5}$$

*Then $\int_C H_3(\lambda, v, \vec{z}, x)dw_\varphi(x) = H_2(\lambda, v, \vec{z})$, where $H_2$ is given by (3).*

Let $\hat{\mathrm{M}}(\mathbb{R}^r)$ be the space of all functions $\phi$ on $\mathbb{R}^r$ defined by

$$\phi(\vec{u}) = \int_{\mathbb{R}^r} \exp\{i\langle\vec{u}, \vec{z}\rangle_{\mathbb{R}^r}\}d\rho(\vec{z}), \tag{6}$$

where $\rho$ is in the space $\mathcal{M}(\mathbb{R}^r)$ of complex Borel measure of bounded variation over $\mathbb{R}^r$. Let $\mathcal{M}(L_2[0,t])$ be the class of all $\mathbb{C}$-valued Borel measures of bounded variation over $L_2[0,t]$ and let $\mathcal{S}_{w_\varphi}$ be the space of all functions $F$ which for $\sigma \in \mathcal{M}(L_2[0,t])$ have the form

$$F(x) = \int_{L_2[0,t]} \exp\{i(v,x)\}d\sigma(v) \tag{7}$$

for $w_\varphi$-a.e. $x \in C[0,t]$. Note that $\mathcal{S}_{w_\varphi}$ is a Banach algebra which is equivalent to $\mathcal{M}(L_2[0,t])$ with the norm $\|F\| = \|\sigma\|$, the total variation of $\sigma$ [11].

**Theorem 2.2** *Let $1 \le p \le \infty$. For $w_\varphi$-a.e. $x \in C[0,t]$, let $\Psi(x) = F(x)\phi(\vec{v}, x)$, where $\phi \in \hat{\mathrm{M}}(\mathbb{R}^r)$ and $F \in \mathcal{S}_{w_\varphi}$ are given by (6) and (7), respectively. Then for a nonzero real $q$, $w_\varphi$-a.e. $y \in C[0,t]$ and $P_{X_1}$-a.e. $(\xi_0, \xi_1) \in \mathbb{R}^2$,*

$$T_q^{(p)}[\Psi|X_1](y,\xi_0,\xi_1) = \int_{\mathbb{R}^r}\int_{L_2[0,t]} H_1(y,\xi_0,\xi_1,v,\vec{z})H_2(-iq,v,\vec{z})d\sigma(v)d\rho(\vec{z}), \quad (8)$$

*where $H_1$ and $H_2$ are given by (2) and (3), respectively.*

*Proof.* For $\lambda > 0$, $y \in C[0,t]$ and $\xi_0, \xi_1 \in \mathbb{R}$, we have by Lemma 2.1

$$\begin{aligned} I_\Psi^\lambda(y,\xi_0,\xi_1) &= \int_{\mathbb{R}^r}\int_{L_2[0,t]} H_1(y,\xi_0,\xi_1,v,\vec{z}) \int_C H_3(\lambda,v,\vec{z},x)dw_\varphi(x)d\sigma(v)d\rho(\vec{z}) \\ &= \int_{\mathbb{R}^r}\int_{L_2[0,t]} H_1(y,\xi_0,\xi_1,v,\vec{z})H_2(\lambda,v,\vec{z})d\sigma(v)d\rho(\vec{z}), \end{aligned}$$

where $H_3$ is given by (5). By (4), the Morera's theorem and the dominated convergence theorem, we have the existence of $T_\lambda[\Psi|X_1](y,\xi_0,\xi_1)$ on $\mathbb{C}_+$. Let $T_q^{(p)}[\Psi|X_1](y,\xi_0,\xi_1)$ be given by (8) and $\frac{1}{p} + \frac{1}{p'} = 1$. Then

$$\begin{aligned} &\|T_\lambda[\Psi|X_1](\cdot,\xi_0,\xi_1) - T_q^{(p)}[\Psi|X_1](\cdot,\xi_0,\xi_1)\|_{p'} \\ \le\ & \int_{\mathbb{R}^r}\int_{L_2[0,t]} |H_2(\lambda,v,\vec{z}) - H_2(-iq,v,\vec{z})|d|\sigma|(v)d|\rho|(\vec{z}) \end{aligned}$$

which converges to 0 as $\lambda$ approaches to $-iq$ through $\mathbb{C}_+$ by the dominated convergence theorem. Now, the proof is completed. □

**Theorem 2.3** *Let $\phi_1, \phi_2 \in \hat{\mathrm{M}}(\mathbb{R}^r)$ and $\rho_1, \rho_2 \in \mathcal{M}(\mathbb{R}^r)$ be related by (6), respectively, and let $F_1, F_2 \in \mathcal{S}_{w_\varphi}$ and $\sigma_1, \sigma_2 \in \mathcal{M}(L_2[0,t])$ be related by (7), respectively. Let $\Psi_1(x) = F_1(x)\phi_1(\vec{v}, x)$ and $\Psi_2(x) = F_2(x)\phi_2(\vec{v}, x)$ for $w_\varphi$-a.e. $x \in C[0,t]$. Then $w_\varphi$-a.e. $y \in C[0,t]$ and $P_{X_1}$-a.e. $(\xi_0,\xi_1) \in \mathbb{R}^2$,*

$$\begin{aligned} &[(\Psi_1 * \Psi_2)_q|X_1](y,\xi_0,\xi_1) \\ =\ & \int_{\mathbb{R}^{2r}}\int_{(L_2[0,t])^2} H_1\left(y,\xi_0,\xi_1,\frac{1}{\sqrt{2}}v_1,\frac{1}{\sqrt{2}}\vec{z}_1\right) H_1\left(y,-\xi_0,-\xi_1,\frac{1}{\sqrt{2}}v_2,\frac{1}{\sqrt{2}}\vec{z}_2\right) \\ & \times H_2\left(-iq,\frac{1}{\sqrt{2}}(v_1-v_2),\frac{1}{\sqrt{2}}(\vec{z}_1-\vec{z}_2)\right) d\sigma_1(v_1)d\sigma_2(v_2)d\rho_1(\vec{z}_1)d\rho_2(\vec{z}_2) \end{aligned}$$

*for a nonzero real $q$, where $H_1$ and $H_2$ are given by (2) and (3), respectively.*

*Proof.* For $\lambda > 0$, $w_\varphi$-a.e. $y \in C[0,t]$ and $P_{X_1}$-a.e. $(\xi_0,\xi_1) \in \mathbb{R}^2$,

$$[(\Psi_1 * \Psi_2)_\lambda|X_1](y,\xi_0,\xi_1)$$

$$= \int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\int_{L_2[0,t]}\int_{L_2[0,t]} H_1\left(y,\xi_0,\xi_1,\frac{1}{\sqrt{2}}v_1,\frac{1}{\sqrt{2}}\vec{z}_1\right) H_1\Big(y,-\xi_0,-\xi_1,\frac{1}{\sqrt{2}}v_2, \frac{1}{\sqrt{2}}\vec{z}_2\Big)\int_C H_3\Big(\lambda,\frac{1}{\sqrt{2}}(v_1-v_2),\frac{1}{\sqrt{2}}(\vec{z}_1-\vec{z}_2),x\Big)dw_\varphi(x)d\sigma_1(v_1)d\sigma_2(v_2) d\rho_1(\vec{z}_1)d\rho_2(\vec{z}_2)$$

$$= \int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\int_{L_2[0,t]}\int_{L_2[0,t]} H_1\left(y,\xi_0,\xi_1,\frac{1}{\sqrt{2}}v_1,\frac{1}{\sqrt{2}}\vec{z}_1\right) H_1\Big(y,-\xi_0,-\xi_1,\frac{1}{\sqrt{2}}v_2, \frac{1}{\sqrt{2}}\vec{z}_2\Big) H_2\left(\lambda,\frac{1}{\sqrt{2}}(v_1-v_2),\frac{1}{\sqrt{2}}(\vec{z}_1-\vec{z}_2)\right) d\sigma_1(v_1)d\sigma_2(v_2)d\rho_1(\vec{z}_1)d\rho_2(\vec{z}_2)$$

by Lemma 2.1, where $H_3$ is given by (5). By (4), the Morera's theorem and the dominated convergence theorem, we have the result. □

# 3 The position-independent transforms

In this section, we evaluate the present position-independent conditional Fourier-Feynman transform and conditional convolution product of the functions as given in the previous section. For these purpose, we need the following lemma by the well-known integration formula $\int_{\mathbb{R}}\exp\{-au^2+ibu\}du = (\frac{\pi}{a})^{\frac{1}{2}}\exp\{-\frac{b^2}{4a}\}$ for $a\in\mathbb{C}_+$ and $b\in\mathbb{R}$.

**Lemma 3.1** *For $y\in C[0,t]$, $\xi_0,\xi_1\in\mathbb{R}$, $v\in L_2[0,t]$ and $\vec{z}\in\mathbb{R}^r$, let $H_1(y,\xi_0,\xi_1,v,\vec{z})$ be given by (2) and for $\lambda\in\tilde{\mathbb{C}}_+$, let*

$$H_4(\lambda,v,\vec{z}) = \exp\Big\{-\frac{t}{2\lambda}\Big[\frac{1}{t}\int_0^t v(s)ds + \langle\vec{V}_t,\vec{z}\rangle_{\mathbb{R}}\Big]^2\Big\}. \tag{9}$$

*Then for $\lambda>0$,*

$$\Big(\frac{\lambda}{2\pi t}\Big)^{\frac{1}{2}}\int_{\mathbb{R}} H_1(y,\xi_0,\xi_1,v,\vec{z})\exp\Big\{-\frac{\lambda(\xi_1-\xi_0)^2}{2t}\Big\}d\xi_1 = H_1(y,0,0,v,\vec{z})H_4(\lambda,v,\vec{z}).$$

Using the same process as used in the proof of Theorem 2.2, we can easily prove the following theorem by (4), Theorem 2.2, Lemma 3.1, the Morera's theorem and the dominated convergence theorem.

**Theorem 3.2** *Under the assumptions as given in Theorem 2.2, we have for a nonzero real $q$, $w_\varphi$-a.e. $y\in C[0,t]$ and $P_{X_0}$-a.e. $\xi_0\in\mathbb{R}$*

$$T_q^{(p)}[\Psi|X_0](y,\xi_0) = \int_{\mathbb{R}^r}\int_{L_2[0,t]} H_1(y,0,0,v,\vec{z})H_2(-iq,v,\vec{z})H_4(-iq,v,\vec{z})d\sigma(v)d\rho(\vec{z})$$

*where $H_1$, $H_2$ and $H_4$ are given by (2), (3) and (9), respectively.*

**Theorem 3.3** *Let $\Psi_1$ and $\Psi_2$ be as given in Theorem 2.3. Then for a nonzero real $q$, $w_\varphi$-a.e. $y \in C[0,t]$ and $P_{X_0}$-a.e. $\xi_0 \in \mathbb{R}$,*

$$\begin{aligned} &[(\Psi_1 * \Psi_2)_q|X_0](y,\xi_0) \\ = &\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\int_{L_2[0,t]}\int_{L_2[0,t]} H_1\Big(y,0,0,\frac{1}{\sqrt{2}}(v_1+v_2),\frac{1}{\sqrt{2}}(\vec{z}_1+\vec{z}_2)\Big) \\ &\times H_2\Big(-iq,\frac{1}{\sqrt{2}}(v_1-v_2),\frac{1}{\sqrt{2}}(\vec{z}_1-\vec{z}_2)\Big)H_4\Big(-iq,\frac{1}{\sqrt{2}}(v_1-v_2), \\ &\frac{1}{\sqrt{2}}(\vec{z}_1-\vec{z}_2)\Big)d\sigma_1(v_1)d\sigma_2(v_2)d\rho_1(\vec{z}_1)d\rho_2(\vec{z}_2)\end{aligned}$$

*where $H_1$, $H_2$ and $H_4$ are given by (2), (3) and (9), respectively.*

*Proof.* Note that for $y \in C[0,t]$, for $\xi_0,\xi_1 \in \mathbb{R}$, for $v_1,v_2 \in L_2[0,t]$ and for $\vec{z}_1,\vec{z}_2 \in \mathbb{R}^r$

$$\begin{aligned} &H_1\left(y,\xi_0,\xi_1\frac{1}{\sqrt{2}}v_1,\frac{1}{\sqrt{2}}\vec{z}_1\right)H_1\Big(y,-\xi_0,-\xi_1\frac{1}{\sqrt{2}}v_2,\frac{1}{\sqrt{2}}\vec{z}_2\Big) \\ = &H_1\Big(y,0,0,\frac{1}{\sqrt{2}}(v_1+v_2),\frac{1}{\sqrt{2}}(\vec{z}_1+\vec{z}_2)\Big) \\ &\times H_1\Big(0,\xi_0,\xi_1,\frac{1}{\sqrt{2}}(v_1-v_2),\frac{1}{\sqrt{2}}(\vec{z}_1-\vec{z}_2)\Big).\end{aligned}$$

By Theorem 2.3 and Lemma 3.1, we have for $\lambda > 0$

$$\begin{aligned} &[(\Psi_1 * \Psi_2)_\lambda|X_0](y,\xi_0) \\ = &\Big(\frac{\lambda}{2\pi t}\Big)^{\frac{1}{2}}\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\int_{L_2[0,t]}\int_{L_2[0,t]} H_2\left(\lambda,\frac{1}{\sqrt{2}}(v_1-v_2),\frac{1}{\sqrt{2}}(\vec{z}_1-\vec{z}_2)\right) \\ &\times\int_{\mathbb{R}} H_1\left(y,\xi_0,\xi_1,\frac{1}{\sqrt{2}}v_1,\frac{1}{\sqrt{2}}\vec{z}_1\right)H_1\Big(y,-\xi_0,-\xi_1,\frac{1}{\sqrt{2}}v_2,\frac{1}{\sqrt{2}}\vec{z}_2\Big) \\ &\times\exp\Big\{-\frac{\lambda(\xi_1-\xi_0)^2}{2t}\Big\}d\xi_1 d\sigma_1(v_1)d\sigma_2(v_2)d\rho_1(\vec{z}_1)d\rho_2(\vec{z}_2) \\ = &\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\int_{L_2[0,t]}\int_{L_2[0,t]} H_1\Big(y,0,0,\frac{1}{\sqrt{2}}(v_1+v_2),\frac{1}{\sqrt{2}}(\vec{z}_1+\vec{z}_2)\Big) \\ &\times H_2\Big(\lambda,\frac{1}{\sqrt{2}}(v_1-v_2),\frac{1}{\sqrt{2}}(\vec{z}_1-\vec{z}_2)\Big)H_4\Big(\lambda,\frac{1}{\sqrt{2}}(v_1-v_2),\frac{1}{\sqrt{2}}(\vec{z}_1-\vec{z}_2)\Big) \\ &d\sigma_1(v_1)d\sigma_2(v_2)d\rho_1(\vec{z}_1)d\rho_2(\vec{z}_2).\end{aligned}$$

By (4), the Morera's theorem and the dominated convergence theorem, we have the result. □

**Remark 3.4** *Note that $T_q^{(p)}[\Psi|X_0](y,\xi_0) = T_q^{(p)}[\Psi|X_0](y,0)$ and $[(\Psi_1*\Psi_2)_q|X_0](y,\xi_0) = [(\Psi_1*\Psi_2)_q|X_0](y,0)$ for $P_{X_0}$-a.e. $\xi_0 \in \mathbb{R}$. These mean that they are independent of the initial positions of the continuous paths, but depend only on the present time $t$.*

# 4 Relationships between the transforms and convolutions

In this section, we investigate the inverse conditional transform of the conditional Fourier-Feynman transforms of the functions as given in the previous sections. We also show that the analytic conditional Fourier-Feynman transforms of the conditional convolution products for the functions, can be expressed as the products of the analytic conditional Fourier-Feynman transform of each function.

**Theorem 4.1** *Let the assumptions and notations be as given in Theorem 2.2. For* $(\xi_0,\xi_1),(\zeta_0,\zeta_1)\in\mathbb{R}^2$ *and* $y\in C[0,t]$*, define*

$$\Psi_{\xi_0,\xi_1,\zeta_0,\zeta_1}(y)=\int_{\mathbb{R}^r}\int_{L_2[0,t]}H_1(y,\zeta_0+\xi_0,\zeta_1+\xi_1,v,\vec{z})d\sigma(v)d\rho(\vec{z}),$$

*where* $H_1$ *is given by (2). Then, for* $P_{X_1}$*-a.e.* $(\xi_0,\xi_1),(\zeta_0,\zeta_1)\in\mathbb{R}^2$

$$\|T_{\overline{\lambda}}[T_\lambda[\Psi|X_1](\cdot,\xi_0,\xi_1)|X_1](\cdot,\zeta_0,\zeta_1)-\Psi_{\xi_1,\xi_0,\zeta_0,\zeta_1}\|_p\to 0$$

*as* $\lambda$ *approaches to* $-iq$ *through* $\mathbb{C}_+$*.*

*Proof.* For $\lambda_1>0$, $\lambda\in\mathbb{C}_+$, $w_\varphi$-a.e. $y\in C[0,t]$ and $P_{X_1}$-a.e. $(\xi_0,\xi_1),(\zeta_0,\zeta_1)\in\mathbb{R}^2$

$$\begin{aligned}
&I^{\lambda_1}_{T_\lambda[\Psi|X_1](\cdot,\xi_0,\xi_1)}(y,\zeta_0,\zeta_1)\\
=&\int_{\mathbb{R}^r}\int_{L_2[0,t]}\int_C H_1(y,\zeta_0+\xi_0,\zeta_1+\xi_1,v,\vec{z})H_2(\lambda.v.\vec{z})H_3(\lambda_1,v,\vec{z},x)dw_\varphi(x)\\
&d\sigma(v)d\rho(\vec{z})\\
=&\int_{\mathbb{R}^r}\int_{L_2[0,t]}H_1(y,\zeta_0+\xi_0,\zeta_1+\xi_1,v,\vec{z})H_2(\lambda.v.\vec{z})H_2(\lambda_1,v,\vec{z})d\sigma(v)d\rho(\vec{z})
\end{aligned}$$

by Lemma 2.1 and Theorem 2.2, where $H_1$, $H_2$ and $H_3$ are given by (2), (3) and (5), respectively. By (4) and the Morera's Theorem, we have the existence of $T_{\lambda_1}[T_\lambda[\Psi|X_1](\cdot,\xi_0,\xi_1)|X_1](y,\zeta_0,\zeta_1)$ for $\lambda_1\in\mathbb{C}_+$. Now, we have for $\lambda\in\mathbb{C}_+$

$$\begin{aligned}
&T_{\overline{\lambda}}[T_\lambda[\Psi|X_1](\cdot,\xi_0,\xi_1)|X_1](y,\zeta_0,\zeta_1)\\
=&\int_{\mathbb{R}^r}\int_{L_2[0,t]}H_1(y,\zeta_0+\xi_0,\zeta_1+\xi_1,v,\vec{z})H_2\left(\frac{|\lambda|^2}{2\mathrm{Re}\lambda},v,\vec{z}\right)d\sigma(v)d\rho(\vec{z})
\end{aligned}$$

so that

$$\begin{aligned}
&\|T_{\overline{\lambda}}[T_\lambda[\Psi|X_1](\cdot,\xi_0,\xi_1)|X_1](y,\zeta_0,\zeta_1)-\Psi_{\xi_0,\xi_1,\zeta_0,\zeta_1}\|_p\\
\le&\int_{\mathbb{R}^r}\int_{L_2[0,t]}\Big[1-H_2\Big(\frac{|\lambda|^2}{2\mathrm{Re}\lambda},v,\vec{z}\Big)\Big]d|\sigma|(v)d|\rho|(\vec{z})
\end{aligned}$$

which converges to 0 as $\lambda$ approaches to $-iq$ through $\mathbb{C}_+$ by the dominated convergence theorem. $\square$

**Theorem 4.2** *Let $1 \le p \le \infty$. Under the assumptions as given in Theorem 3.2, we have for $P_{X_0}$-a.e. $\xi_0, \zeta_0 \in \mathbb{R}$*

$$\|T_{\overline{\lambda}}[T_\lambda[\Psi|X_0](\cdot,\xi_0)|X_0](\cdot,\zeta_0) - \Psi\|_p \to 0$$

*as $\lambda$ approaches to $-iq$ through $\mathbb{C}_+$.*

*Proof.* For $\xi_0, \zeta_0, \zeta_1 \in \mathbb{R}$, $\lambda_1 > 0$, $\lambda \in \mathbb{C}_+$ and $y \in C[0,t]$

$$\begin{aligned} I^{\lambda_1}_{T_\lambda[\Psi|X_0](\cdot,\xi_0)}(y,\zeta_0,\zeta_1) &= \int_{\mathbb{R}^r}\int_{L_2[0,t]} H_1(y,\zeta_0,\zeta_1,v,\vec{z})H_2(\lambda_1,v,\vec{z}) \\ &\quad \times H_2(\lambda,v,\vec{z})H_4(\lambda,v,\vec{z})d\sigma(v)d\rho(\vec{z}) \end{aligned}$$

by Theorems 2.2 and 3.2, so that by Lemma 3.1

$$\begin{aligned} K^{\lambda_1}_{T_\lambda[\Psi|X_0](\cdot,\xi_0)}(y,\zeta_0) &= \int_{\mathbb{R}^r}\int_{L_2[0,t]} H_1(y,0,0,v,\vec{z})H_2(\lambda_1,v,\vec{z}) \\ &\quad \times H_2(\lambda,v,\vec{z})H_4(\lambda_1,v,\vec{z})H_4(\lambda,v,\vec{z})d\sigma(v)d\rho(\vec{z}), \end{aligned}$$

where $H_1$, $H_2$ and $H_4$ are given by (2), (3) and (9), respectively. By (4), the Morera's theorem and the dominated convergence theorem, we have the existence of $T_{\lambda_1}[T_\lambda[\Psi|X_0](\cdot,\xi_0)|X_0](y,\zeta_0)$ with respect to $\lambda_1 \in \mathbb{C}_+$ as the analytic extension of $K^{\lambda_1}_{T_\lambda[\Psi|X_0](\cdot,\xi_0)}(y,\zeta_0)$. Now, for $\lambda \in \mathbb{C}_+$ and $y \in C[0,t]$

$$\begin{aligned} &T_{\overline{\lambda}}[T_\lambda[\Psi|X_0](\cdot,\xi_0)|X_0](y,\zeta_0) \\ &= \int_{\mathbb{R}^r}\int_{L_2[0,t]} H_1(y,0,0,v,\vec{z})H_2\Big(\frac{|\lambda|^2}{2\mathrm{Re}\lambda},v,\vec{z}\Big)H_4\Big(\frac{|\lambda|^2}{2\mathrm{Re}\lambda},v,\vec{z}\Big)d\sigma(v)d\rho(\vec{z}) \end{aligned}$$

so that we have

$$\begin{aligned} &\|T_{\overline{\lambda}}[T_\lambda[\Psi|X_0](\cdot,\xi_0)|X_0](y,\zeta_0) - \Psi\|_p \\ &\le \int_{\mathbb{R}^r}\int_{L_2[0,t]} \Big[1 - H_2\Big(\frac{|\lambda|^2}{2\mathrm{Re}\lambda},v,\vec{z}\Big)H_4\Big(\frac{|\lambda|^2}{2\mathrm{Re}\lambda},v,\vec{z}\Big)\Big]d|\sigma|(v)d|\rho|(\vec{z}) \end{aligned}$$

which converges to 0 as $\lambda$ approaches to $-iq$ through $\mathbb{C}_+$ by the dominated convergence theorem. $\square$

**Theorem 4.3** *Let $1 \le p \le \infty$. Furthermore, let $\Psi_1$ and $\Psi_2$ be as given in Theorem 2.3. Then for a nonzero real $q$, $w_\varphi$-a.e. $y \in C[0,t]$ and $P_{X_1}$-a.e. $(\xi_0,\xi_1),(\zeta_0,\zeta_1) \in \mathbb{R}^2$,*

$$\begin{aligned} &T_q^{(p)}[[(\Psi_1 * \Psi_2)_q|X_1](\cdot,\xi_0,\xi_1)|X_1](y,\zeta_0,\zeta_1) \\ &= \Big[T_q^{(p)}[\Psi_1|X_1]\Big(\frac{1}{\sqrt{2}}y,\frac{1}{\sqrt{2}}(\zeta_0+\xi_0),\frac{1}{\sqrt{2}}(\zeta_1+\xi_1)\Big)\Big] \\ &\quad \times \Big[T_q^{(p)}[\Psi_2|X_1]\Big(\frac{1}{\sqrt{2}}y,\frac{1}{\sqrt{2}}(\zeta_0-\xi_0),\frac{1}{\sqrt{2}}(\zeta_1-\xi_1)\Big)\Big]. \qquad (10) \end{aligned}$$

*Proof.* Let $\lambda \in \mathbb{C}_+^{\sim}$. For $\lambda_1 > 0$, $w_\varphi$-a.e. $y \in C[0,t]$ and $P_{X_1}$-a.e. $(\xi_0, \xi_1), (\zeta_0, \zeta_1) \in \mathbb{R}^2$, we have by Theorem 2.3

$$\begin{aligned}
&I^{\lambda_1}_{[(\Psi_1 * \Psi_2)_\lambda | X_1](\cdot, \xi_0, \xi_1)}(y, \zeta_0, \zeta_1) \\
= &\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\int_{L_2[0,t]}\int_{L_2[0,t]}\int_C H_1\Big(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\zeta_0 + \xi_0), \frac{1}{\sqrt{2}}(\zeta_1 + \xi_1), v_1, \vec{z}_1\Big) H_1\Big(\frac{1}{\sqrt{2}} \\
&\times y, \frac{1}{\sqrt{2}}(\zeta_0 - \xi_0), \frac{1}{\sqrt{2}}(\zeta_1 - \xi_1), v_2, \vec{z}_2\Big) H_2\Big(\lambda, \frac{1}{\sqrt{2}}(v_1 - v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 - \vec{z}_2)\Big) H_3 \\
&\Big(\lambda_1, \frac{1}{\sqrt{2}}(v_1 + v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 + \vec{z}_2), x\Big) dw_\varphi(x) d\sigma_1(v_1) d\sigma_2(v_2) d\rho_1(\vec{z}_1) d\rho_2(\vec{z}_2),
\end{aligned}$$

where $H_1$, $H_2$ and $H_3$ are given by (2), (3) and (5), respectively. By Lemma 2.1,

$$\begin{aligned}
&I^{\lambda_1}_{[(\Psi_1 * \Psi_2)_\lambda | X_1](\cdot, \xi_0, \xi_1)}(y, \zeta_0, \zeta_1) \\
= &\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\int_{L_2[0,t]}\int_{L_2[0,t]} H_1\Big(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\zeta_0 + \xi_0), \frac{1}{\sqrt{2}}(\zeta_1 + \xi_1), v_1, \vec{z}_1\Big) H_1\Big(\frac{1}{\sqrt{2}}y, \\
&\frac{1}{\sqrt{2}}(\zeta_0 - \xi_0), \frac{1}{\sqrt{2}}(\zeta_1 - \xi_1), v_2, \vec{z}_2\Big) H_2\Big(\lambda, \frac{1}{\sqrt{2}}(v_1 - v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 - \vec{z}_2)\Big) H_2\Big(\lambda_1, \\
&\frac{1}{\sqrt{2}}(v_1 + v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 + \vec{z}_2)\Big) d\sigma_1(v_1) d\sigma_2(v_2) d\rho_1(\vec{z}_1) d\rho_2(\vec{z}_2). \qquad (11)
\end{aligned}$$

By (4), the Morera's theorem and the dominated convergence theorem, we have the analytic extension $T_{\lambda_1}[[(\Psi_1 * \Psi_2)_\lambda | X_1](\cdot, \xi_0, \xi_1) | X_1](y, \zeta_0, \zeta_1)$ of (11) as the function of $\lambda_1 \in \mathbb{C}_+$. Let $T_q^{(p)}[[(\Psi_1 * \Psi_2)_\lambda | X_1](\cdot, \xi_0, \xi_1) | X_1](y, \zeta_0, \zeta_1)$ be given by (11), where $\lambda_1$ is replaced by $-iq$, and let $\frac{1}{p} + \frac{1}{p'} = 1$. By (4)

$$\begin{aligned}
&\|T_{\lambda_1}[[(\Psi_1 * \Psi_2)_\lambda | X_1](\cdot, \xi_0, \xi_1) | X_1](y, \zeta_0, \zeta_1) \\
&\qquad - T_q^{(p)}[[(\Psi_1 * \Psi_2)_\lambda | X_1](\cdot, \xi_0, \xi_1) | X_1](y, \zeta_0, \zeta_1)\|_{p'} \\
\le &\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\int_{L_2[0,t]}\int_{L_2[0,t]} \Big| H_2\Big(\lambda_1, \frac{1}{\sqrt{2}}(v_1 + v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 + \vec{z}_2)\Big) - H_2\Big(-iq, \\
&\frac{1}{\sqrt{2}}(v_1 + v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 + \vec{z}_2)\Big)\Big| d|\sigma_1|(v_1) d|\sigma_2|(v_2) d|\rho_1|(\vec{z}_1) d|\rho_2|(\vec{z}_2)
\end{aligned}$$

which converges to 0 as $\lambda_1$ approaches to $-iq$ through $\mathbb{C}_+$ by the dominated convergence theorem. This shows that the existence of $T_q^{(p)}[[(\Psi_1 * \Psi_2)_\lambda | X_1](\cdot, \xi_0, \xi_1) | X_1](y, \zeta_0, \zeta_1)$. Now, we have by (11)

$$\begin{aligned}
&T_q^{(p)}[[(\Psi_1 * \Psi_2)_q | X_1](\cdot, \xi_0, \xi_1) | X_1](y, \zeta_0, \zeta_1) \\
= &\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\int_{L_2[0,t]}\int_{L_2[0,t]} H_1\Big(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\zeta_0 + \xi_0), \frac{1}{\sqrt{2}}(\zeta_1 + \xi_1), v_1, \vec{z}_1\Big)
\end{aligned}$$

$$\times H_1\Big(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\zeta_0-\xi_0), \frac{1}{\sqrt{2}}(\zeta_1-\xi_1), v_2, \vec{z}_2\Big)H_2(-iq, v_1, \vec{z}_1))H_2(-iq, v_2, \vec{z}_2)d\sigma_1(v_1)d\sigma_2(v_2)d\rho_1(\vec{z}_1)d\rho_2(\vec{z}_2)$$

which completes the proof by Theorem 2.2. □

Note that by the same method as used in the proof of Theorem 4.3, we can obtain (10), where $-iq$ is replaced by $\lambda \in \mathbb{C}_+$.

Now, we have the final result of our work.

**Theorem 4.4** *Under the assumptions as given in Theorem 4.3, we have for a nonzero real $q$, $w_\varphi$-a.e. $y \in C[0,t]$ and $P_{X_0}$-a.e. $\xi_0, \zeta_0 \in \mathbb{R}$*

$$\begin{aligned} &T_q^{(p)}[[(\Psi_1 * \Psi_2)_q|X_0](\cdot,\xi_0)|X_0](y,\zeta_0) \\ = &\Big[T_q^{(p)}[\Psi_1|X_0]\Big(\frac{1}{\sqrt{2}}y,0\Big)\Big]\Big[T_q^{(p)}[\Psi_2|X_0]\Big(\frac{1}{\sqrt{2}}y,0)\Big)\Big]. \end{aligned}$$

*Proof.* Let $\lambda \in \mathbb{C}_+^\sim$. For $\lambda_1 > 0$, $w_\varphi$-a.e. $y \in C[0,t]$ and $P_{X_0}$-a.e. $\xi_0, \zeta_0 \in \mathbb{R}$,

$$\begin{aligned} &K^{\lambda_1}_{[(\Psi_1*\Psi_2)_\lambda|X_0](\cdot,\xi_0)}(y,\zeta_0) \\ = &\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\int_{L_2[0,t]}\int_{L_2[0,t]} H_1\Big(y,0,0,\frac{1}{\sqrt{2}}(v_1+v_2),\frac{1}{\sqrt{2}}(\vec{z}_1+\vec{z}_2)\Big)H_2\Big(\lambda_1, \\ &\frac{1}{\sqrt{2}}(v_1+v_2),\frac{1}{\sqrt{2}}(\vec{z}_1+\vec{z}_2)\Big)H_4\Big(\lambda_1,\frac{1}{\sqrt{2}}(v_1+v_2),\frac{1}{\sqrt{2}}(\vec{z}_1+\vec{z}_2)\Big)H_2\Big(\lambda, \\ &\frac{1}{\sqrt{2}}(v_1-v_2),\frac{1}{\sqrt{2}}(\vec{z}_1-\vec{z}_2)\Big)H_4\Big(\lambda,\frac{1}{\sqrt{2}}(v_1-v_2),\frac{1}{\sqrt{2}}(\vec{z}_1-\vec{z}_2)\Big)d\sigma_1(v_1) \\ &d\sigma_2(v_2)d\rho_1(\vec{z}_1)d\rho_2(\vec{z}_2) \end{aligned}$$

by Lemma 3.1, Theorems 3.2 and 3.3, where $H_1$, $H_2$ and $H_4$ are given by (2), (3) and (9), respectively. By (4), the Morera's theorem and the dominated convergence theorem, we have the analytic extension $T_{\lambda_1}[[(\Psi_1*\Psi_2)_\lambda|X_0](\cdot,\xi_0)|X_0](y,\zeta_0)$ of $K^{\lambda_1}_{[(\Psi_1*\Psi_2)_\lambda|X_0](\cdot,\xi_0)}(y,\zeta_0)$ as the function of $\lambda_1 \in \mathbb{C}_+$. Let $T_q^{(p)}[[(\Psi_1 * \Psi_2)_\lambda|X_0](\cdot,\xi_0)|X_0](y,\zeta_0)$ be given by the right-hand side of the above equality, where $\lambda_1$ is replaced by $-iq$, and let $\frac{1}{p}+\frac{1}{p'}=1$. Then we have

$$\begin{aligned} &\|T_{\lambda_1}[[(\Psi_1*\Psi_2)_\lambda|X_0](\cdot,\xi_0)|X_0](y,\zeta_0) \\ &\qquad -T_q^{(p)}[[(\Psi_1*\Psi_2)_\lambda|X_0](\cdot,\xi_0)|X_0](y,\zeta_0)\|_{p'} \\ \le &\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\int_{L_2[0,t]}\int_{L_2[0,t]}\Big|H_2\Big(\lambda_1,\frac{1}{\sqrt{2}}(v_1+v_2),\frac{1}{\sqrt{2}}(\vec{z}_1+\vec{z}_2)\Big)H_4\Big(\lambda_1,\frac{1}{\sqrt{2}}(v_1 \\ &+v_2),\frac{1}{\sqrt{2}}(\vec{z}_1+\vec{z}_2)\Big) - H_2\Big(-iq,\frac{1}{\sqrt{2}}(v_1+v_2),\frac{1}{\sqrt{2}}(\vec{z}_1+\vec{z}_2)\Big)H_4\Big(-iq, \\ &\frac{1}{\sqrt{2}}(v_1+v_2),\frac{1}{\sqrt{2}}(\vec{z}_1+\vec{z}_2)\Big)\Big|d|\sigma_1|(v_1)d|\sigma_1|(v_2)d|\rho_1|(\vec{z}_1)d|\rho_2|(\vec{z}_2) \end{aligned}$$

which converges to 0 as $\lambda$ approaches to $-iq$ through $\mathbb{C}_+$ by the dominated convergence theorem. This shows that the existence of $T_q^{(p)}[[(\Psi_1 * \Psi_2)_\lambda|X_0](\cdot,\xi_0)|X_0]\ (y,\zeta_0)$. By simple calculations, we have from (2), (3) and (9)

$$\begin{aligned}
&T_\lambda[[(\Psi_1 * \Psi_2)_\lambda|X_0](\cdot,\xi_0)|X_0](y,\zeta_0)\\
=\ &\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\int_{L_2[0,t]}\int_{L_2[0,t]} H_1\Big(y,0,0,\frac{1}{\sqrt{2}}(v_1+v_2),\frac{1}{\sqrt{2}}(\vec{z}_1+\vec{z}_2)\Big)H_2\Big(\lambda,\\
&\frac{1}{\sqrt{2}}(v_1+v_2),\frac{1}{\sqrt{2}}(\vec{z}_1+\vec{z}_2)\Big)H_4\Big(\lambda,\frac{1}{\sqrt{2}}(v_1+v_2),\frac{1}{\sqrt{2}}(\vec{z}_1+\vec{z}_2)\Big)H_2\Big(\lambda,\\
&\frac{1}{\sqrt{2}}(v_1-v_2),\frac{1}{\sqrt{2}}(\vec{z}_1-\vec{z}_2)\Big)H_4\Big(\lambda,\frac{1}{\sqrt{2}}(v_1-v_2),\frac{1}{\sqrt{2}}(\vec{z}_1-\vec{z}_2)\Big)d\sigma_1(v_1)\\
&d\sigma_2(v_2)d\rho_1(\vec{z}_1)d\rho_2(\vec{z}_2)\\
=\ &\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\int_{L_2[0,t]}\int_{L_2[0,t]} H_1\Big(\frac{1}{\sqrt{2}}y,0,0,v_1,\vec{z}_1\Big)H_1\Big(\frac{1}{\sqrt{2}}y,0,0,v_2,\vec{z}_2\Big)H_2(\lambda,\\
&v_1,\vec{z}_1)H_2(\lambda,v_2,\vec{z}_2)H_4(\lambda,v_1,\vec{z}_1)H_4(\lambda,v_2,\vec{z}_2)d\sigma_1(v_1)d\sigma_2(v_2)d\rho_1(\vec{z}_1)d\rho_2(\vec{z}_2).
\end{aligned}$$

Now, we have the result by Theorem 3.2. □

Note that by the same method as used in the proof of Theorem 4.4, we can obtain the same equality in the theorem, where $-iq$ is replaced by $\lambda \in \mathbb{C}_+$.

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