On $t$–Derivations of Incline Algebras

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Abstract

In this paper, we introduce the notion of $t$– derivations of incline algebras and investigate some of their properties. Moreover, we show that if a $t$– derivation is nonzero on an integral incline $K$, then it is nonzero on any nonzero ideals of $K$.

Mathematics Subject Classification: 06F35

Keywords: Incline algebras, ideals, $t$- derivations of incline algebras

1 Introduction

Cao et al. [1] introduced the notion of incline algebras in their book: Incline algebra and applications, and was studied by some authors [2-3]. Inclines are a generalization of both Boolean and fuzzy algebras, and a special type of a semiring, and they give a way to combine algebras with ordered structures to express the degree of intensity of binary relations. An incline is a structure which has an associative, commutative addition, and a distributive multiplication such that $x + x = x$, $x + xy = x$ for all $x, y$. It has both a semiring
structure and a poset structure. Inclines can also be used to represent automata and other mathematical systems, in optimization theory, to study inequalities for nonnegative matrices of polynomials. Ahn et al. [6] introduced the notion of quotient incline and obtained the structure of incline algebras. They also introduced the notion of prime and maximal ideals in an incline, and studied some relations between them in incline algebras. Jun and Xin [9] applied the notion of derivation in ring and near-ring theory to BCI-algebras. In this paper, we introduced the concept of \( t \)-derivations for an incline and investigate some of its properties. We show that if \( d_t \) is a \( t \)-derivation of an integral incline \( K \), and \( a \in K \) such that \( a \ast d_t(x) = 0 \) or \( d_t(x) \ast a = 0 \) for all \( x \in K \), then either \( a = 0 \) or \( d_t \) is zero. Also we show that if \( d_{t_1}, d_{t_2} \) are two \( t \)-derivations of an incline and \( d_{t_1}d_{t_2} = 0 \), then \( d_{t_1}d_{t_2} \) is a \( t \)-derivation of \( K \). Finally we prove that if \( M \) is a nonzero ideal of an integral incline \( K \) and \( d_t \) is a nonzero \( t \)-derivation of \( K \), Then \( d_t \) is nonzero on \( M \).

2 Preliminaries

An incline (algebra) is a set \( K \) with two binary operations denoted by + and * satisfying the following axioms for all \( x, y, z \in K \):

\[(K1) \ x + y = y + x,\]
\[(K2) \ x + (y + z) = (x + y) + z,\]
\[(K3) \ x \ast (y \ast z) = (x \ast y) \ast z,\]
\[(K4) \ x \ast (y + z) = (x \ast y) + (x \ast z),\]
\[(K5) \ (y + z) \ast x = (y \ast x) + (z \ast x),\]
\[(K6) \ x + x = x,\]
\[(K7) \ x + (x \ast y) = x,\]
\[(K8) \ y + (x \ast y) = y.\]

Furthermore, an incline algebra \( K \) is said to be commutative if \( x \ast y = y \ast x \) for all \( x, y \in K \).

For convenience, we pronounce + (resp. *) as addition (resp. multiplication). Every distributive lattice is an incline. An incline is a distributive lattice (as a semiring) if and only if \( x \ast x = x \) for all \( x \in K \).

Note that \( x \leq y \iff x + y = y \) for all \( x, y \in K \). It is easy to see that \( \leq \) is a partial order on \( K \) and that for any \( x, y \in K \), the element \( x + y \) is the least upper bound of \( \{x, y\} \). We say that \( \leq \) is induced by operation +. It follows that

1. \( x \ast y \leq x \) and \( y \ast x \leq x \) for all \( x, y \in K \);
2. \( y \leq z \) implies \( x \ast y \leq x \) and \( y \ast x \leq z \ast x \) for any \( x, y, z \in K \);
3. if \( x \leq y, a \leq b \), then \( x + a \leq y + b, x \ast a \leq y \ast b \).

A subincline of an incline \( K \) is a non-empty subset \( M \) of \( K \) which is closed under addition and multiplication. A subincline \( M \) is said to be an ideal of
an incline $K$ if $x \in M$ and $y \leq x$ then $y \in M$. An element $0$ in an incline algebra $K$ is a zero element if $x + 0 = x = 0 + x$ and $x \ast 0 = 0 \ast x = 0$, for any $x \in K$. An element $1$ ( ≠ zero element) in an incline algebra $K$ is called a multiplicative identity if for any $x \in K$, $x \ast 1 = 1 \ast x = x$. A non−zero element $a$ in an incline algebra $K$ with zero element is said to be a left (resp. right) zero divisor if there exists a non−zero $b \in K$ such that $a \ast b = 0$ (resp. $b \ast a = 0$). A zero divisor is an element of $K$ which is both a left zero divisor and a right zero divisor. An incline $K$ with multiplicative identity $1$ and zero element $0$ is called an integral incline if it has no zero divisors.

**Definition 2.1.** Let $K$ be an incline and $d : K \rightarrow K$ be a function. We call $d$ a derivation of $K$, if it satisfies the following condition for all $x, y \in K$,

$$d(x \ast y) = (dx \ast y) + (x \ast dy).$$

**Definition 3.1.** Let $K$ be an incline algebra. Then for any $t \in K$, we define a self map $dt : K \rightarrow K$ by $dtx = x \ast t$, for all $x \in K$.

**Definition 3.2.** Let $K$ be an incline and $dt : K \rightarrow K$ be a function. We call $dt$ a $t$−derivation of $K$, if it satisfies the following condition for all $x, y \in K$,

$$dt(x \ast y) = (dtx \ast y) + (x \ast dt y).$$

We often abbreviate $dt(x)$ to $dx$.

**Example 3.3.** Let $K$ be a commutative incline algebra and define mapping $dt : K \rightarrow K$ by $dtx = x \ast t$ for all $x \in K$.

Then it is easily checked that $dt$ is a $t$−derivation of $K$.

$$\text{L.H.S} = dt(x \ast y) = (x \ast y) \ast t$$

$$\text{R.H.S} = dx \ast y + x \ast dt y$$

$$= (x \ast t) \ast y + x \ast (y \ast t)$$

$$= x \ast (t \ast y) + x \ast (y \ast t)$$

$$= x \ast (y \ast t) + x \ast (y \ast t)$$

$$= x \ast (y \ast t)$$

$$= (x \ast y) \ast t$$

$$= \text{L.H.S}.$$ 

**Example 3.4.** Let $K$ be an incline with multiplicative identity and $dt : K \rightarrow K$ defined by $dt(x) = x \ast t$ where $t = 1$ then $dt$ is a $t$−derivation of $K$.

$$\text{L.H.S} = dt(x \ast y) = (x \ast y) \ast 1 = x \ast y$$

$$\text{R.H.S} = dx \ast y + x \ast dt y$$

$$= (x \ast 1) \ast y + x \ast (y \ast 1)$$

$$= x \ast y + x \ast y$$

$$= x \ast y$$

$$= \text{L.H.S}.$$
Proposition 3.5. Let $K$ be an incline and $d_t$ be a self map of $K$. Then the following hold for all $x, y \in K$:

(i) $d_t(x * y) \leq d_t x + d_t y$

(ii) $d_t x \leq x$

(iii) If $x \leq y$, then $d_t(x * y) \leq y$

Proof. (i) Let $x, y \in K$,

$$d_t(x * y) = (x * y) * t$$
$$= ((x + x) * y) * t$$
$$= ((x * y) + (x * y)) * t$$
$$\leq (x + y) * t$$
$$= (x * t) + (y * t)$$
$$= d_t x + d_t y$$.

(ii) Let $x \in K$, $d_t x = x * t \leq x$.

(iii) If $x \leq y$, then $x + y = y$.

so from (i), (ii) we get: $d_t(x * y) \leq d_t x + d_t y \leq x + y = y$.

Proposition 3.6. Let $K$ be an incline with zero element and $d_t$ be a $t$–derivation of $K$. Then $d_t 0 = 0$.

Proof. Let $t \in K$, then $d_t 0 = 0 * t = 0$.

Proposition 3.7. Let $K$ be an incline with multiplicative identity and $d_t$ be a $t$–derivation of $K$. Then the following hold for all $x \in K$:

(i) $x * d_t 1 = d_t x$

(ii) If $d_t 1 = 1$, then $d_t$ is an identity derivation.

Proof. (i) Let $x \in K$, $x * d_t 1 = x * (1 * t) = x * t = d_t x$.

(ii) follows directly from (i).

Proposition 3.8. Let $d_t$ be a $t$–derivation of an integral incline $K$, and $a$ be an element of $K$. Then:

(i) If $a * d_t x = 0$ for all $x \in K$ then either $a = 0$ or $d_t$ is zero.

(ii) If $d_t x * a = 0$ for all $x \in K$ then either $a = 0$ or $d_t$ is zero.

Proof. (i) Let $a * d_t x = 0$ for all $x \in K$.

Let $y \in K$, replace $x$ by $x * y$ then

$0 = a * d_t (x * y) = a * (d_t x * y) + a * (x * d_t y) = a * (x * d_t y)$.

Putting $x = 1$ we get that $a * d_t y = 0$, But $K$ has no zero divisors, so $a = 0$ or $d_t y = 0$ for all $y \in K$. 

Thus we have that \( a = 0 \) or \( d_t \) is zero.

(ii) Similar to (i).

**Definition 3.9.** Let \( d_t \) be a \( t \)-derivation of an incline \( K \).
If \( x \leq y \) implies \( d_t x \leq d_t x \), for all \( x, y \in K \), \( d_t \) is called an isotone \( t \)-derivation.

**Proposition 3.10.** Let \( K \) be an incline and \( d_t \) be a \( t \)-derivation of \( K \).
If \( d_t(x + y) = d_t x + d_t y \), for all \( x, y \in K \), then the following hold for all \( x, y \in K \):

(i) \( d_t(x * y) \leq d_t x \),
(ii) \( d_t(x * y) \leq d_t y \),
(iii) \( d_t \) is an isotone \( t \)-derivation.

**Proof.** (i) Let \( x, y \in K \), then by (1) we have:
\[
d_t(x * y) = (x * y) * t \leq x * t = d_t x .
\]
Hence \( d_t(x * y) \leq d_t x \).

(ii) Similar to (i).

(iii) Let \( x \leq y \), then \( x + y = y \), and so \( d_t y = d_t(x + y) = d_t x + d_t y \)
Hence \( d_t x \leq d_t y \).

**Proposition 3.11.** Let \( K \) be an incline with zero element and \( d_t \) be a \( t \)-derivation of \( K \).
Denote \( d_t^{-1}(0) = \{ x \in K \mid d_t x = 0 \} \).
If \( d_t(x + y) = d_t x + d_t y \), for all \( x, y \in K \), then \( d_t^{-1}(0) \) is an ideal of \( K \).

**Proof.** Let \( x, y \in d_t^{-1}(0) \), thus \( d_t x = d_t y = 0 \).
From the hypotheses we get
\[
d_t(x + y) = d_t x + d_t y = 0 + 0 = 0, x + y \in d_t^{-1}(0)
\]
also \( d_t(x * y) = (x * y) * t = x * (y * t) = x * d_t y = x * 0 = 0 \),
and so \( x * y \in d_t^{-1}(0) \). Then \( d_t^{-1}(0) \) is a subincline of \( K \).
Now let \( x \in K \) and \( y \in d_t^{-1}(0) \) such that \( x \leq y \), thus \( d_t y = 0 \) and \( x + y = y \).
\[
0 = d_t y = d_t(x + y) = d_t x + d_t y = d_t x + 0 = d_t x ,
\]
then \( x \in d_t^{-1}(0) \).
Hence \( d_t^{-1}(0) \) is an ideal of \( K \).

**Theorem 3.12.** Let \( d_t \) be a \( t \)-derivation of an integral incline \( K \).
Define \( d_t^2(x) = d_t(d_t x) \) for all \( x \in K \).
If \( d_t^2 = 0 \), then \( d_t \) is zero.
Proof. Let \( x, y \in K \), then
\[
0 = d_t^2(x * y) = d_t(d_t x * y + x * d_t y)
\]
\[
= d_t^2 x * y + d_t x * d_t y + d_t x * d_t y + x * d_t^2
\]
\[
= d_t x * d_t y + d_t x * d_t y.
\]
From (K6) we get that \( d_t x * d_t y = 0 \),
Since \( K \) has no zero divisors we have that,
\( d_t x = 0 \) for all \( x \in K \) or \( d_t y = 0 \) for all \( y \in K \),
In two cases we have \( d_t = 0 \).

\[\Box\]

**Theorem 3.13.** Let \( K \) be an incline and \( d_{t_1}d_{t_2} \) \( t \)-derivations of \( K \).
Define \( d_{t_1}d_{t_2}(x) = d_{t_1}(d_{t_2}x) \) for all \( x \in K \).
If \( d_{t_1}d_{t_2} = 0 \), then \( d_{t_2}d_{t_1} \) is a \( t \)-derivation of \( K \).

**Proof.** Let \( x, y \in K \), then
\[
0 = d_{t_1}d_{t_2}(x * y)
\]
\[
= d_{t_1}(d_{t_2}x * y + x * d_{t_2} y)
\]
\[
= d_{t_1}d_{t_2}x * y + d_{t_2}x * d_{t_1} y + d_{t_1}x * d_{t_2} y + x * d_{t_1}d_{t_2} y
\]
\[
= d_{t_2}x * d_{t_1} y + d_{t_1}x * d_{t_1} y.
\]
Then,
\[
d_{t_2}d_{t_1}(x * y) = d_{t_2}(d_{t_1}x * y + x * d_{t_1} y)
\]
\[
= d_{t_2}d_{t_1} x * y + d_{t_1}x * d_{t_2} y + d_{t_2}x * d_{t_1} y + x * d_{t_2}d_{t_1} y
\]
\[
= d_{t_2}d_{t_1} x * y + x * d_{t_2}d_{t_1} y.
\]
This implies that \( d_{t_2}d_{t_1} \) is a \( t \)-derivation.

\[\Box\]

**Theorem 3.14.** Let \( M \) be a nonzero ideal of an integral incline \( K \).
If \( d_t \) is a nonzero \( t \)-derivation of \( K \), then \( d_t \) is nonzero on \( M \).

**Proof.** Assume that \( d_t = 0 \) on \( M \) and \( x \in M \), then \( d_t x = 0 \).
Let \( y \in K \), since \( x * y \leq x \) and \( M \) is an ideal of \( K \), thus we have \( x * y \in M \).
Therefore \( d_t(x * y) = 0 \), then we get that
\[
0 = d_t(x * y) = d_t x * y + x * d_t y = 0 + x * d_t y = x * d_t y.
\]
But \( K \) has no zero divisors, so \( x = 0 \) for all \( x \in M \) or \( d_t y = 0 \) for all \( y \in K \),
Since \( M \neq 0 \), we get \( d_t y = 0 \) for all \( y \in K \) this contradicts \( d_t \neq 0 \) on \( K \).

\[\Box\]

**Theorem 3.15.** If \( d_t \) is a nonzero \( t \)-derivation and \( M \neq 0 \) is an ideal of an integral incline \( K \), then \( d_t^2(M) \neq 0 \).
Proof. Assume that \( d_t^2(M) = 0 \). Then for \( x, y \in M \) we have

\[
0 = d_t^2(x \ast y) \\
= d_t(d_t(x \ast y + x \ast d_t y) \\
= d_t(d_t x \ast y) + d_t(x \ast d_t y) \\
= (d_t^2 x \ast y + d_t x \ast d_t y) + (d_t x \ast d_t y + x \ast d_t^2 y) \\
= d_t x \ast d_t y + d_t x \ast d_t y \\
= d_t x \ast d_t y
\]

Since \( K \) is an integral incline we have \( d_t(x) = 0 \) or \( d_t(y) = 0 \).

Hence \( d_t(M) = 0 \). By Theorem 3.13 we get \( d_t \) is zero \( t \)-derivation on \( K \). This contradicts with hypothesis that \( d_t \neq 0 \).

Hence \( d_t^2 \) must be different than zero on \( M \).

\[ \square \]

**Theorem 3.16.** Let \( d_t \) be a nonzero \( t \)-derivation of an integral incline \( K \).

If \( M \) is a nonzero ideal of \( K \), and \( a \in K \) such that \( a \ast d_t(M) = 0 \), then \( a = 0 \).

**Proof.** By Theorem 3.13 we know that there is an element \( x \) in \( M \) such that \( d_t(x) \neq 0 \).

Assume that \( M \) is a nonzero ideal of \( K \), and \( a \in K \) such that \( a \ast d_t(M) = 0 \).

Then for \( x, y \in M \) we can write

\[
0 = a \ast d_t(x \ast y) \\
= a \ast (d_t(x \ast y + x \ast d_t y) \\
= a \ast (d_t x \ast y) + a \ast (x \ast d_t y) \\
= (a \ast d_t x) \ast y + a \ast (x \ast d_t y) \\
= 0 + a \ast (x \ast d_t y) \\
= a \ast (x \ast d_t y).
\]

Since \( K \) is an integral incline, \( d_t \) is a nonzero \( t \)-derivation of \( K \) and \( M \) is a nonzero ideal of \( K \) we have \( a = 0 \).

\[ \square \]

**References**


Received: December, 2012