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Strong Convergence Theorems for a Countable Family of Total Quasi-ϕ-Asymptotically Nonexpansive Nonself Mappings

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Abstract

In this paper, we introduce a class of total quasi-ϕ-asymptotically nonexpansive nonself mappings which contain several kinds of mappings as its special cases, and obtain some strong convergence theorems for this type of mappings in Banach spaces under some mild control conditions. The results presented in this paper improve and extend some recent corresponding results.

Mathematics Subject Classifications: 47H09, 47J25

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1 Introduction

Let E be a real Banach space with the dual E^* and C be a nonempty closed convex subset of E. We denote by R^+ and R the set of all nonnegative real numbers and the set

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of all real numbers, respectively. Also, we denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{x^* \in E^* :< x, x^* >= \|x\|^2 = \|x^*\|^2\}, \forall x \in E, \tag{1.1}$$

where $< \cdot, \cdot >$ denotes the generalized duality pairing. Recall that if E is smooth, then J is single-valued and norm-to weak* continuous, and that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subsets of E. We shall denote by J the single-value duality mapping.

A Banach space E is said to be strictly convex if $\frac{\|x+y\|}{2} \leq 1$ for all $x, y \in U = \{z \in E : \|z\| = 1\}$ with $x \neq y$. E is said to be uniformly convex if, for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $\frac{\|x+y\|}{2} \leq 1 - \delta$ for all $x, y \in U$ with $\|x - y\| \geq \varepsilon$. E is said to be smooth if the limit

$$\lim_{n \to \infty} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in U$. E is said to be uniformly smooth if the above limit exists uniformly in $x, y \in U$.

Remark 1.1. The following basic properties of Banach space E can be founded in Cioranescu [1]

(i) If E is a uniformly smooth Banach space, then J is uniformly continuous on each bounded subset of E;

(ii) If E is a reflexive and strictly convex Banach space, then J^{-1} is norm-weak*-continuous;

(iii) If E is a smooth, reflexive and strictly convex Banach space, then the normalized duality mapping $J : E \to 2^{E^*}$ is single-valued, one-to one, and surjective;

(iv) A Banach space E is uniformly smooth if and only if E^* is uniformly convex;

(v) Each uniformly convex Banach space E has the Kadec-Klee property, that is, for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. [see, 1,2] for more details.

A subset C of E is said to be retract of E, if there exists a continuous mapping $P : E \to C$ such that $Px = x$, for all $x \in C$. It is well known that every nonempty closed convex subset of a uniformly convex Banach space is a retract of E. A mapping $P : E \to C$ is said to be a retraction if $P^2 = P$. It follows that if a mapping P is a retraction, then $Py = y$ for all y in the range of P. A mapping $P : E \to C$ is said to be a nonexpansive retraction, if it is nonexpansive and it is a retraction from E to C.

Next, we assume that E is a smooth, reflexive and strictly convex Banach space. Consider the functional defined as in [3,4] by

$$\phi(x, y) = \|x\|^2 - 2 < x, Jy > + \|y\|^2, \quad \forall x, y \in E. \tag{1.2}$$

It is clear that in a Hilbert space H, (1.2) reduces to $\phi(x,y) = \|x-y\|^2, \forall x,y \in H$.

It is obvious from the definition of ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x,y) \leq (\|x\| + \|y\|)^2, \quad \forall x,y \in E. \tag{1.3}$$

and

$$\phi(x, J^{-1}(\lambda Jy + (1-\lambda)Jz) \leq \lambda\phi(x,y) + (1-\lambda)\phi(x,z), \quad \forall x,y \in E. \tag{1.4}$$

Following Alber [3], the generalized projection $\Pi_C : E \to C$ is defined by

$$\Pi_C(x) = arginf_{y \in C}\phi(y,x), \quad \forall x \in E. \tag{1.5}$$

That is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem $\phi(\overline{x}, x) = \inf_{y \in C} \phi(y,x)$.

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J (see,e.g., [1-5]). In Hilbert space H, $\Pi_C = P_C$.

Let C be a nonempty closed convex subset of E, and let T be a mapping from C into itself and $F(T)$ be the set of fixed points of T. A point $p \in C$ is called an asymptotically fixed point of T [6] if exists a sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup p$ and $\|x_n - Tx_n\| \to 0$. The set of asymptotical fixed points of T will be denoted by $\widehat{F}(T)$. A point $p \in C$ is said to be a strong asymptotic fixed point of T, if there exists a sequence $\{x_n\} \subset C$ such that $x_n \to p$ and $\|x_n - Tx_n\| \to 0$. The set of strong asymptotical fixed points of T will be denoted by $\widetilde{F}(T)$.

A mapping $T : C \to C$ is said to be nonexpansive , if

$$\|Tx - Ty\| \leq \|x-y\|, \forall x,y \in C. \tag{1.6}$$

A mapping $T : C \to C$ is said to be relatively nonexpansive [7-9], if $F(T) \neq \emptyset$, $F(T) = \widehat{F}(T)$ and $\phi(p,Tx) \leq \phi(p,x)$, $\forall x \in C$, $\forall p \in F(T)$.

A mapping $T : C \to C$ is said to be quasi-ϕ-nonexpansive, if $F(T) \neq \emptyset$ and $\phi(p,Tx) \leq \phi(p,x)$, $\forall x \in C$, $\forall p \in F(T)$.

A mapping $T : C \to C$ is said to be asymptotically nonexpansive [10], if there exists a sequence $\{k_n\} \subset [1,\infty)$ satisfying $lim_{n\to\infty}k_n = 1$ such that for any $x,y \in C$

$$\|T^n x - T^n y\| \leq k_n \|x-y\|. \tag{1.7}$$

A mapping $T : C \to E$ is said to be asymptotically nonexpansive nonself mapping [11], if there exists a real sequence $\{k_n\} \subset [1,\infty)$ satisfying $lim_{n\to\infty}k_n = 1$ such that for any $x,y \in C$

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x-y\|. \tag{1.8}$$

Inspired by the work of Chidume et al. [11], very recently, Chang et al.[12] introduced the concept of quasi-ϕ-asymptotically nonexpansive nonself mapping and obtained some strong convergence theorems for this type of mappings.

A mapping $T : C \to E$ is said to be a quasi-ϕ-nonexpansive nonself mapping, if $F(T) \neq \emptyset$ and

$$\phi(p, T(PT)^{n-1}x) \leq \phi(p, x), \forall x \in C, \forall n \geq 1, \forall p \in F(T) \tag{1.9}$$

P is the nonexpansive retraction from E to C.

A mapping $T : C \to E$ is said to be quasi-ϕ-asymptotically nonexpansive nonself mapping, if $F(T) \neq \emptyset$ and there exists a real sequence $\{k_n\} \subset [1, \infty)$ satisfying $lim_{n\to\infty} k_n = 1$ such that

$$\phi(p, T(PT)^{n-1}x) \leq k_n \phi(p, x),\ \forall n \geq 1,\ \forall x \in C,\ \forall p \in F(T), \tag{1.10}$$

P is the nonexpansive retraction from E to C.

A countable family of nonself mappings $\{T_i\} : C \to E$ is said to be uniformly quasi-ϕ-asymptotically nonexpansive nonself mapping, if $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and there exists a real sequence $\{k_n\} \subset [1, \infty)$ satisfying $lim_{n\to\infty} k_n = 1$ such that

$$\phi(p, T_i(PT_i)^{n-1}x) \leq k_n \phi(p, x),\ \forall n \geq 1,\ \forall x \in C,\ \forall p \in \cap_{i=1}^{\infty} F(T_i). \tag{1.11}$$

Iterative approximation of fixed points for asymptotically nonexpansive self or nonself mappings, relatively nonexpansive, quasi-ϕ-nonexpansive and quasi-ϕ-asymptotically nonexpansive self mappings have been studied extensively by many authors[13-24]. Recently, Chang et al. [25,26] introduced the concept of total quasi-ϕ-asymptotically nonexpansive self mapping and obtained some weak and strong convergence theorems to a fixed point of total quasi-ϕ-asymptotically nonexpansive self mapping.

Inspired and motivated by the recent work of Su et al. [14], Kizitune et al. [15], Yildirim et al. [16], Wang [21,22], Chang et al. [12,13,24,25], etc., in this paper, we introduce the concept of total quasi-ϕ-asymptotically nonexpansive nonself mapping which contains several kinds of mappings as its special cases, and prove some strong convergence theorems of common fixed points for a countable family of total quasi-ϕ-asymptotically nonexpansive nonself mappings in Banach spaces under some mild control conditions. The results presented in this paper improve and extend some recent corresponding results in [11-26].

Definition 1.2. Let $P : E \to C$ be the nonexpansive retraction.

(1) A mapping $T : C \to E$ is said to be total quasi-ϕ-asymptotically nonexpansive nonself mapping, if $F(T) \neq \emptyset$ and there exists nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n \to 0$, $\mu_n \to 0$ $(as\ n \to \infty)$ and a strictly increasing continuous function $\xi : R^+ \to R^+$ with $\xi(0) = 0$ such that

$$\phi(p, T(PT)^{n-1}x) \leq \phi(p, x) + \nu_n \xi(\phi(p, x)) + \mu_n,\ \forall n \geq 1,\ \forall x \in C,\ \forall p \in F(T); \tag{1.12}$$

(2) A countable family of nonself mappings $\{T_i\} : C \to E$ is said to be uniformly total quasi-ϕ-asymptotically nonexpansive nonself mapping, if $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and there

exists nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n \to 0$, $\mu_n \to 0$ ($as\ n \to \infty$) and a strictly increasing continuous function $\xi : R^+ \to R^+$ with $\xi(0) = 0$ such that

$$\phi(p, T_i(PT_i)^{n-1}x) \leq \phi(p, x) + \nu_n \xi(\phi(p, x)) + \mu_n,\ \forall n \geq 1,\ \forall x \in C,\ \forall p \in \cap_{i=1}^{\infty} F(T_i); \tag{1.13}$$

(3) A countable family of nonself mappings $\{T_i\} : C \to E$ is said to be asymptotically nonexpansive nonself mapping, if there exists a real sequence $\{k_n\} \subset [1, \infty)$ satisfying $lim_{n\to\infty} k_n = 1$ such that for any $x, y \in C$

$$\|T_i(PT_i)^{n-1}x - T_i(PT_i)^{n-1}y\| \leq k_n \|x - y\|. \tag{1.14}$$

(4) A nonself mapping $T : C \to E$ is said to be uniformly L-Lipschitz continuous, if there exists a constant $L > 0$ such that for any $x, y \in C$

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\|. \tag{1.15}$$

Remark 1.3. From the definition, it is easy to know that

(i) Taking $\xi(t) = t$, $t \geq 0$, $\nu_n = (k_n - 1)$ and $\mu_n = 0$, then $\nu_n \to 0$ ($as\ n \to \infty$), and (1.10) can be rewritten as

$$\phi(p, T(PT)^{n-1}x) \leq \phi(p, x) + \nu_n \xi(\phi(p, x)) + \mu_n,\ \forall n \geq 1,\ x \in C,\ p \in F(T). \tag{1.16}$$

This implies that each quasi-ϕ-asymptotically nonexpansive nonself mapping must be a total quasi-ϕ-asymptotically nonexpansive nonself mapping, but the converse is not true.

(ii) The class of quasi-ϕ-asymptotically nonexpansive nonself mappings contains properly the class of quasi-ϕ-nonexpansive nonself mappings as a subclass, but the converse is not true.

(iii) if E is a real Hilbert space, then a mapping $T : C \to E$ is a total quasi-ϕ-asymptotically nonexpansive nonself mapping, if $F(T) \neq \emptyset$ and there exists nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n \to 0$, $\mu_n \to 0$ ($as\ n \to \infty$) and a strictly increasing continuous function $\xi : R^+ \to R^+$ with $\xi(0) = 0$ such that

$$\|p - T(P_C T)^{n-1}x\|^2 \leq \|p - x\|^2 + \nu_n \xi(\|p - x\|^2) + \mu_n,\ \forall n \geq 1,\ \forall x \in C,\ \forall p \in F(T),$$

where P_C is the metric projection from H onto C.

2 Preliminaries

Throughout this paper, Let E be a real Banach space with the dual E^* and C be a nonempty closed convex subset of E. We denote the strong convergence, weak convergence of a sequence $\{x_n\}$ to a point $x \in E$ by $x_n \to x$, $x_n \rightharpoonup x$, respectively, and $F(T)$ is the fixed point set of a mapping T.

Lemma 2.1[12]. *Let E be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property, and C be a nonempty closed convex subset of E. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in C such that $x_n \to p$ and $\phi(x_n, y_n) \to 0$, where ϕ is the function defined by (1.2), then $y_n \to p$.*

Lemma 2.2 [3]. *Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. Then the following conclusions hold:*

(a) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \le \phi(x, y),\ \forall\, x \in C,\ \forall y \in E;$

(b) *If $x \in E$ and $z \in C$, then $z = \Pi_C x$ iff $< z - y, Jx - Jz > \ge 0,\ \forall y \in C;$*

(C) *For $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$;*

Lemma 2.3 *Let E be a real uniformly smooth and strictly convex and reflexive Banach space, and C be a nonempty closed convex subset of E. Let $T : C \to E$ be a total quasi-ϕ-asymptotically nonexpansive nonself mapping with nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ and a strictly increasing continuous functions $\xi : R^+ \to R^+$ such that $\nu_n \to 0$, $\mu_n \to 0$ (as $n \to \infty$) and $\xi(0) = 0$. If the fixed point set $F(T)$ of T is nonempty, and $\mu_1 = 0$, $\nu_1 = 0$, then $F(T)$ is a closed and convex subset of C.*

proof. Let $\{u_n\}$ be a sequence in $F(T)$ with $u_n \to u$ (as $n \to \infty$), we prove that $u \in F(T)$. In fact, since $T : C \to E$ is a total quasi-ϕ-asymptotically nonexpansive nonself mapping, we have

$$\begin{aligned} \phi(u, Tu) &= \lim_{n\to\infty} \phi(u_n, Tu) \\ &\le \lim_{n\to\infty} [\phi(u_n, u) + \nu_1 \zeta(\phi(u_n, u))] \\ &= 0. \end{aligned}$$

Hence, $\phi(u, Tu) = 0$. By Lemma 2.2, we have $u \in F(T)$. This implies that $F(T)$ is closed.

Next we prove that $F(T)$ is convex. For any $p, q \in F(T)$, $t \in (0, 1)$, putting $z =$

$tp+(1-t)q$, we prove that $z \in F(T)$. In fact, in view of the definition of $\phi(x,y)$ we have

$$\begin{aligned}
\phi(z,Tz) &= \|z\|^2 - 2 < z, JTz > + \|Tz\|^2 \\
&= \|z\|^2 - 2t < p, JTz > -2(1-t) < q, JTz > + \|Tz\|^2 \\
&= \|z\|^2 + t\phi(p,Tz) + (1-t)\phi(q,Tz) - t\|p\|^2 - (1-t)\|q\|^2 \\
&\leq \|z\|^2 + t\phi(p,z) + (1-t)\phi(q,z) - t\|p\|^2 - (1-t)\|q\|^2 \\
&= \|z\|^2 + t[\|p\|^2 - 2 < p, Jz > + \|z\|^2] + (1-t)[\|q\|^2 - 2 < q, Jz > + \|z\|^2] \\
&\quad -t\|p\|^2 - (1-t)\|q\|^2 \\
&= \|z\|^2 - 2t < p, Jz > -2(1-t) < q, Jz > + \|z\|^2 \\
&= \|z\|^2 - 2 < tp + (1-t)q, Jz > + \|z\|^2 \\
&= 0.
\end{aligned}$$

So, $Tz = z$, that is, $z \in F(T)$. This completes the proof of Lemma 2.3.

Remark 2.4. In fact, a total quasi-ϕ-asymptotically nonexpansive nonself mapping reduces to a quasi-ϕ-asymptotically nonexpansive nonself mapping as $\mu_n = 0$ and $\nu_n = 0$, $n = 1, 2, \cdots$. Therefore, from Lemma 2.3, we know that the fixed point set of quasi-ϕ-asymptotically nonexpansive nonself mapping is a closed and convex subset of C. But in [12], they just proved that the fixed point set of quasi-ϕ-asymptotically nonexpansive nonself mapping is a closed subset of C.

3 Main Results

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex Banach space E. Let $T_i : C \to E$, $i = 1, 2, \cdots$ be a countable family of uniformly total quasi-ϕ-asymptotically nonexpansive nonself mappings with nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ and a strictly increasing continuous function $\zeta : R^+ \to R^+$ such that $\mu_1 = 0$, $\nu_1 = 0$, $\nu_n \to 0$, $\mu_n \to 0$ (as $n \to \infty$) and $\zeta(0) = 0$, and for each $i \geq 1$, T_i be uniformly L_i-Lipschitz continuous. $\{x_n\}$ is defined by*

$$\begin{cases}
x_1 \in E \quad \textit{chosen arbitrary}, \quad C_1 = C, \\
y_{n,i} = J^{-1}[\alpha_n Jx_1 + (1-\alpha_n)(\beta_n Jx_n + (1-\beta_n)JT_i(PT_i)^{n-1}x_n)], \ i \geq 1, \\
C_{n+1} = \{\nu \in C_n : \sup_{i\geq 1} \phi(\nu, y_{n,i}) \leq \alpha_n \phi(\nu, x_1) + (1-\alpha_n)\phi(\nu, x_n) + \xi_n\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_1, \quad n \geq 1,
\end{cases} \tag{3.1}$$

where $\xi_n = \nu_n sup_{p\in\Theta}\zeta(\phi(p,x_n)) + \mu_n$, $\Pi_{C_{n+1}}$ is the generalized projection of E onto C_{n+1}, $\{\beta_n\}$ is a sequence in $(0,1)$ and $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying the following conditions:

(1) $lim_{n\to\infty}\alpha_n = 0$;

(2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

If $\Theta := \cap_{i=1}^{\infty} F(T_i)$ is a nonempty and bounded subset in C, Then the sequences $\{x_n\}$ converges strongly to $q \in \Theta$ where $q = \Pi_\Theta x_1$.

Proof. We shall divide the proof into five steps.

Step 1. We first show that Θ and C_n are closed and convex for each $n \geq 1$.
It follows from Lemma 2.3 that $F(T_i)$, $i \geq 1$ is closed and convex subset of C. Therefore Θ is closed and convex in C.

Again by the assumption, $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for some $n \geq 2$. Since for any $\nu \in C_n$, we known

$$\begin{aligned} C_{n+1} &= \{\nu \in C_n : \sup_{i\geq 1} \phi(\nu, y_{n,i}) \leq \alpha_n \phi(\nu, x_1) + (1-\alpha_n)\phi(\nu, x_n) + \xi_n\} \\ &= \bigcap_{i\geq 1} \{\nu \in C : \phi(\nu, y_{n,i}) \leq \alpha_n \phi(\nu, x_1) + (1-\alpha_n)\phi(\nu, x_n) + \xi_n\} \bigcap C_n \\ &= \bigcap_{i\geq 1} \{\nu \in C : 2\alpha_n < \nu, Jx_1 > +2(1-\alpha_n) < \nu, Jx_n > -2 < \nu, Jy_{n,i} > \\ &\quad \leq \alpha_n \|x_1\|^2 + (1-\alpha_n)\|x_n\|^2 - \|y_{n,i}\|^2 + \xi_n\} \bigcap C_n. \end{aligned} \tag{3.2}$$

This shows that C_{n+1} is closed and convex. The desired conclusion is proved.

Step 2. We show that $\Theta \subset C_n$ for all $n \geq 1$.
It is obvious that $\Theta \subset C_1 = C$. Suppose that $\Theta \subset C_n$ for some $n \geq 2$. Letting

$$\omega_{n,i} = J^{-1}(\beta_n J x_n + (1-\beta_n) J T_i (P T_i)^{n-1} x_n). \tag{3.3}$$

For any given $z \in \Theta \subset C_n$, it follows from (1.4) that

$$\begin{aligned} \phi(z, y_{n,i}) &= \phi(z, J^{-1}[\alpha_n J x_1 + (1-\alpha_n) J\omega_{n,i}] \\ &\leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, \omega_{n,i}), \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \phi(z, \omega_{n,i}) &= \phi(z, J^{-1}(\beta_n J x_n + (1-\beta_n) J T_i (P T_i)^{n-1} x_n)) \\ &\leq \beta_n \phi(z, x_n) + (1-\beta_n)\phi(z, T_i (P T_i)^{n-1} x_n) \\ &\leq \beta_n \phi(z, x_n) + (1-\beta_n)[\phi(z, x_n) + \nu_n \zeta \phi(z, x_n) + \mu_n] \\ &= \phi(z, x_n) + (1-\beta_n)[\nu_n \zeta \phi(z, x_n) + \mu_n]. \end{aligned} \tag{3.5}$$

Hence,

$$\begin{aligned} sup_{i\geq 1} \phi(z, y_{n,i}) &\leq \alpha_n \phi(z, x_1) + (1-\alpha_n)[\phi(z, x_n) + (1-\beta_n)[\nu_n \zeta \phi(z, x_n) + \mu_n]] \\ &= \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) + (1-\alpha_n)(1-\beta_n)[\nu_n \zeta \phi(z, x_n) + \mu_n] \\ &\leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) + \nu_n sup_{p\in\Theta} \zeta(\phi(p, x_n)) + \mu_n \\ &\leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) + \xi_n, \end{aligned} \tag{3.6}$$

where $\xi_n = \nu_n sup_{p\in\Theta}\zeta(\phi(p,x_n)) + \mu_n$. This show that $z \in C_{n+1}$ implies that $\Theta \subset C_{n+1}$ and hence $\Theta \subset C_n$ for all $n \geq 1$. Since Θ is nonempty, C_n is a nonempty closed convex subset of E and hence Π_{C_n} exist for all $n \geq 1$. This implies that the sequence $\{x_n\}$ is well defined.

Moreover, by the assumptions on $\{\nu_n\}$, $\{\mu_n\}$ and Θ, from (1.3) we have

$$\xi_n = \nu_n \sup_{p\in\Theta}\zeta(\phi(p,x_n)) + \mu_n \to 0,\ n \to \infty. \tag{3.7}$$

Step 3. we prove that $\{x_n\}$ is a Cauchy sequence.

It follows from (3.1) and Lemma 2.2 that

$$\begin{aligned}\phi(x_n,x_1) &\leq \phi(\Pi_{C_n}x_1,x_1)\\ &\leq \phi(z,x_1)-\phi(z,x_n)\\ &\leq \phi(z,x_1),\ \forall\ z\in C_n, \forall\ n\geq 1.\end{aligned} \tag{3.8}$$

From definition of C_{n+1} that $x_n = \Pi_{C_n}x_1$ and $x_{n+1} = \Pi_{C_{n+1}}x_1$, we have

$$\phi(x_n,x_1) \leq \phi(x_{n+1},x_1), \forall\ n \geq 1. \tag{3.9}$$

Therefore, $\{\phi(x_n,x_1)\}$ is nondecreasing and bounded. So, $\{\phi(x_n,x_1)\}$ is a convergent sequence, without loss of generality, we can assume that $\lim_{n\to\infty}\phi(x_n,x_1) = d \geq 0$. Hence for any positive integer m, $m \geq 1$, using Lemma 2.2 we have

$$\phi(x_m,x_n) = \phi(x_m,\Pi_{C_n}x_1) \leq \phi(x_m,x_1) - \phi(x_n,x_1). \tag{3.10}$$

Since $\lim_{n\to\infty}\phi(x_n,x_1)$ exists, we obtain that

$$lim_{m,n\to\infty}\phi(x_m,x_n) = 0. \tag{3.11}$$

Thus, it follows from Lemma 2.1 that

$$\lim_{m,n\to\infty}\|x_m - x_n\| = 0. \tag{3.12}$$

This implies that the sequence $\{x_n\}$ is a Cauchy sequence in C. Since C is a nonempty closed subset of Banach space E, there exists a q in C such that

$$x_n \to q\ (n \to \infty). \tag{3.13}$$

Step 4. We prove that $q \in \Theta := \cap_{i=1}^{\infty}F(T_i)$.

Since $x_{n+1} \in C_{n+1}$, it follows from (3.7) and (3.12) that

$$\sup_{i\geq 1}\phi(x_{n+1},y_{n,i}) \leq \alpha_n\phi(x_{n+1},x_1) + (1-\alpha_n)\phi(x_{n+1},x_n) + \xi_n \to 0, (as\ n \to \infty). \tag{3.14}$$

Since $x_n \to q$, by Lemma 2.1

$$\lim_{n\to\infty} y_{n,i} = q. \tag{3.15}$$

Since $\{x_n\}$ is bounded, $\{T_i\}_{i=1}^{\infty}$ is uniformly total quasi-ϕ-asymptotically nonexpansive nonself mapping and $\nu_n \to 0$, $\mu_n \to 0$ (as $n \to \infty$), $\zeta(0) = 0$, for any given $q \in \Theta$, we have

$$\phi(q, T_i(PT_i)^{n-1}x_n) \leq \phi(q, x_n) + \nu_n \zeta(\phi(q, x_n)) + \mu_n, \ \forall \ n \geq 1. \tag{3.16}$$

This implies that $\{T_i(PT_i)^{n-1}x_n\}$ is uniformly bounded. Since

$$\begin{aligned} \|\omega_{n,i}\| &= \|J^{-1}(\beta_n Jx_n + (1-\beta_n)JT_i(PT_i)^{n-1}x_n)\| \\ &\leq \beta_n \|x_n\| + (1-\beta_n)\|T_i(PT_i)^{n-1}x_n)\| \\ &\leq \|x_n\| + \|T_i(PT_i)^{n-1}x_n)\|. \end{aligned} \tag{3.17}$$

This implies that $\{\omega_{n,i}\}$ is also uniformly bounded. So, we have

$$\lim_{n\to\infty} \|Jy_{n,i} - J\omega_{n,i}\| = \lim_{n\to\infty} \alpha_n \|Jx_1 - J\omega_{n,i}\| = 0, \ for\ all\ i \geq 1. \tag{3.18}$$

Since J^{-1} is uniformly continuous on each bounded subset of E^*, it follows from (3.15) and (3.18) that

$$\lim_{n\to\infty} \omega_{n,i} = q, \ for\ all\ i \geq 1. \tag{3.19}$$

Since J is uniformly continuous on each bounded subset of E, we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} \|J\omega_{n,i} - Jq\| \\ &= \lim_{n\to\infty} \|\beta_n Jx_n + (1-\beta_n)JT_i(PT_i)^{n-1}x_n - Jq\| \\ &= \lim_{n\to\infty} \|\beta_n(Jx_n - Jq) + (1-\beta_n)(JT_i(PT_i)^{n-1}x_n - Jq)\| \\ &= \lim_{n\to\infty} (1-\beta_n)\|JT_i(PT_i)^{n-1}x_n - Jq\|. \end{aligned} \tag{3.20}$$

Therefore, we have

$$\lim_{n\to\infty} \|JT_i(PT_i)^{n-1}x_n - Jq\| = 0. \tag{3.21}$$

Since J is uniformly continuous, this shows that

$$\lim_{n\to\infty} T_i(PT_i)^{n-1}x_n = q, \ \forall \ i \geq 1 \tag{3.22}$$

Again by the assumption that for each $i \geq 1$, T_i is uniformly L_i-Lipschitz continuous, hence we have

$$\begin{aligned} &\|T_i(PT_i)^n x_n - T_i(PT_i)^{n-1}x_n\| \\ &\quad \leq \|T_i(PT_i)^n x_n - T_i(PT_i)^n x_{n+1}\| + \|T_i(PT_i)^n x_{n+1} - x_{n+1}\| \\ &\quad\quad + \|x_{n+1} - x_n\| + \|x_n - T_i(PT_i)^{n-1}x_n\| \\ &\quad \leq (L_i+1)\|x_{n+1} - x_n\| + \|T_i(PT_i)^n x_{n+1} - x_{n+1}\| + \|x_n - T_i(PT_i)^{n-1}x_n\|. \end{aligned} \tag{3.23}$$

Since $x_n \to q$, this together with (3.22) and (3.23), yields

$$\lim_{n\to\infty} \|T_i(PT_i)^n x_n - T_i(PT_i)^{n-1}x_n\| = 0. \tag{3.24}$$

and

$$\lim_{n\to\infty} T_i(PT_i)^n x_n = q. \tag{3.25}$$

thus,

$$\lim_{n\to\infty} T_iP(T_i(PT_i)^{n-1}x_n) = q. \tag{3.26}$$

In view of the continuity of T_iP, it yields that $T_iPq = q$. Since $q \in C$, $Pq = q$. this shows that $q = T_iq$. By the arbitrariness of $i \geq 1$, we have $q \in \Theta$.

Step 5. We prove that $x_n \to q = \Pi_\Theta x_1$.

Let $z = \Pi_\Theta x_1$. From $x_n = \Pi_{C_n} x_1$ and $z \in \Theta \subset C_n$, we have

$$\phi(x_n, x_1) \leq \phi(z, x_1), \ \forall n \geq 1. \tag{3.27}$$

This implies that

$$\phi(q, x_1) = lim_{n\to\infty}\phi(x_n, x_1) \leq \phi(z, x_1). \tag{3.28}$$

By definition of $z = \Pi_\Theta x_1$, we have $z = q$. Therefore, $x_n \to q = \Pi_\Theta x_1$. This completes the proof.

Corollary 3.2. *Let C, E, $\{\alpha_n\}$, $\{\beta_n\}$ be the same as in Theorem 3.1. Let $T_i : C \to E$, $i = 1, 2, \cdots$ be a countable family of uniformly quasi-ϕ-asymptotically nonexpansive nonself mappings with sequence $\{k_n\} \subset [1, \infty)$ satisfying $lim_{n\to\infty}k_n = 1$ and for each $i \geq 1$, T_i be uniformly L_i-Lipschitz continuous. $\{x_n\}$ is defined by*

$$\begin{cases} x_1 \in E \quad \text{chosen arbitrary}, \quad C_1 = C, \\ y_{n,i} = J^{-1}[\alpha_n Jx_1 + (1-\alpha_n)(\beta_n Jx_n + (1-\beta_n)JT_i(PT_i)^{n-1}x_n)], \ i \geq 1, \\ C_{n+1} = \{\nu \in C_n : \sup_{i\geq 1}\phi(\nu, y_{n,i}) \leq \alpha_n\phi(\nu, x_1) + (1-\alpha_n)\phi(\nu, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_1, \quad \forall n \geq 1, \end{cases} \tag{3.29}$$

where $\xi_n = (k_n - 1)sup_{p\in\Theta}\phi(p, x_n))$, $\Pi_{C_{n+1}}$ is the generalized projection of E onto C_{n+1}. If $k_1 = 0$, $\Theta := \cap_{i=1}^\infty F(T_i)$ is a nonempty and bounded subset in C, Then the sequence $\{x_n\}$ converges strongly to $q \in \Theta$ where $q = \Pi_\Theta x_1$.

Proof. Since $\{T_i\}_{i=1}^\infty$ is a countable family of uniformly quasi-ϕ-asymptotically nonexpansive nonself mappings, by Remark 1.3, it is a countable family of uniformly total quasi-ϕ-asymptotically nonexpansive nonself mappings with nonnegative sequences $\{\nu_n = (k_n - 1)\}$, $\{\mu_n = 0\}$ and a strictly increasing and continuous function $\zeta(t) = t$, $t \geq 0$. Hence $\xi_n = (k_n - 1)sup_{p\in\Theta}\phi(p, x_n) \to 0$, (as $n \to \infty$). Therefore, all conditions in Theorem 3.1 are satisfied. The conclusion of Corollary 3.2 can be obtained from Theorem 3.1 immediately.

Corollary 3.3. *Let C, E, $\{\alpha_n\}$, $\{\beta_n\}$ be the same as in Theorem 3.1. Let $T_i : C \to E$, $i = 1, 2, \cdots$ be a countable family of quasi-ϕ-nonexpansive nonself mappings such that $\Theta := \cap_{i=1}^\infty F(T_i)$ is a nonempty and convex subset in C and for each $i \geq 1$, T_i be uniformly*

L_i-Lipschitzian nonself mapping. $\{x_n\}$ is defined by

$$\begin{cases} x_1 \in E \quad chosen\ arbitrary, \quad C_1 = C, \\ y_{n,i} = J^{-1}[\alpha_n Jx_1 + (1-\alpha_n)(\beta_n Jx_n + (1-\beta_n)JT_i x_n)],\ i \geq 1, \\ C_{n+1} = \{\nu \in C_n : \sup_{i\geq 1} \phi(\nu, y_{n,i}) \leq \alpha_n \phi(\nu, x_1) + (1-\alpha_n)\phi(\nu, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1. \end{cases} \tag{3.29}$$

Then the sequence $\{x_n\}$ converges strongly to $q \in \Theta$, where $q = \Pi_\Theta x_1$.

Proof. Since $\{T_i\}_{i=1}^{\infty}$ is a countable family of quasi-ϕ- nonexpansive nonself mappings, by Remark 1.3, it is a countable family of uniformly quasi-ϕ-asymptotically nonexpansive nonself mappings with sequence $\{k_n = 1\}$. Hence $\xi_n = (k_n - 1)sup_{p\in\Theta}\phi(p, x_n) = 0$. Therefore the conditions appearing in Theorem 3.1 "Θ is a bounded subset in C" is no use here. Therefore all conditions in Theorem 3.1 are satisfied. By the similar methods as given in the proof of Theorem 3.1, The conclusion of Corollary 3.3 can be obtained from Theorem 3.1 immediately.

Remark 3.4. *In Theorem 3.1, we extend the mappings from relatively nonexpansive mappings, asymptotically nonexpansive, quasi-ϕ-nonexpansive and quasi-ϕ-asymptotically nonexpansive self or nonself mappings and total quasi-ϕ-asymptotically nonexpansive self mappings to a countable family of total quasi-ϕ-asymptotically nonexpansive nonself mappings. So, our results improve and extend the corresponding results in [11-26].*

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