Duals of g -Frames and g -Frame Sequences

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Abstract

In this paper we study duals of g -frames and g -frame sequences. We define oblique dual g -frames and characterize these kind of duals. Also, we introduce Type *I* and Type *II* duals for g -frame sequences and verify some properties of them. Then, we state some conditions for existence of these duals.

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1 Introduction

The concept of frames (discrete frames) in Hilbert spaces has been introduced by Duffin and Schaeffer ([6]) in 1952 to study some deep problems in nonharmonic Fourier series. After the fundamental paper [5] by Daubechies, Grossmann and Meyer, frame theory began to be widely used. Sun introduced a g -frame and a g -Riesz basis in a complex Hilbert space and discussed some properties of them ([9]). Also, continuous g -frames are introduced in [1]. In this paper we generalize some results in [3] and [4] to g -frames and g -frame sequences.

Throughout this paper, H is a Hilbert space and $\{H_i\}_{i \in I}$ is a family of Hilbert spaces.

Definition 1.1. We call $\{\Lambda_i \in B(H, H_i) : i \in I\}$ a *generalized frame*, or simply a *g-frame*, for H with respect to $\{H_i\}_{i \in I}$, if there are two positive constants A and B such that

$$A\|f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad f \in H. \quad (1.1)$$

We call A and B the lower and upper *g-frame* bounds, respectively.

We call $\{\Lambda_i \in B(H, H_i) : i \in I\}$ a λ -tight *g-frame* if $A = B = \lambda$ and we call it a Parseval *g-frame* if $A = B = 1$. If we have only the second inequality in (1.1), we call it a *g-Bessel* sequence.

For a sequence $\{H_i\}_{i \in I}$, define

$$\left(\sum_{i \in I} \oplus H_i\right)_{l_2} = \left\{ \{f_i\}_{i \in I} \mid f_i \in H_i, \|\{f_i\}_{i \in I}\|_2^2 = \sum_{i \in I} \|f_i\|^2 < \infty \right\}.$$

It is easy to show that with pointwise operations and inner product as

$$\langle \{f_i\}_{i \in I}, \{g_i\}_{i \in I} \rangle = \sum_{i \in I} \langle f_i, g_i \rangle,$$

$\left(\sum_{i \in I} \oplus H_i\right)_{l_2}$ is a Hilbert space.

Definition 1.2. We say $\{\Lambda_i \in B(H, H_i) : i \in I\}$ is a *g-frame sequence* in H , if it is a *g-frame* for $\overline{\text{span}}\{\Lambda_i^*(H_i)\}_{i \in I}$.

We define the synthesis operator for a *g-Bessel* sequence $\{\Lambda_i \in B(H, H_i) : i \in I\}$ as follows:

$$\begin{aligned} T_\Lambda : \left(\sum_{i \in I} \oplus H_i\right)_{l_2} &\longrightarrow H \\ T_\Lambda(\{f_i\}_{i \in I}) &= \sum_{i \in I} \Lambda_i^*(f_i). \end{aligned} \quad (1.2)$$

This series converges unconditionally in H . It is easy to show that the adjoint operator of T_Λ is given by

$$\begin{aligned} T_\Lambda^* : H &\longrightarrow \left(\sum_{i \in I} \oplus H_i\right)_{l_2} \\ T_\Lambda^*(f) &= \{\Lambda_i f\}_{i \in I}. \end{aligned} \quad (1.3)$$

The operator T_Λ^* is called the analysis operator of $\{\Lambda_i\}_{i \in I}$. Also, the *g-frame* operator of $\{\Lambda_i\}_{i \in I}$ is defined as follows:

$$S_\Lambda : H \longrightarrow H$$

$$S_\Lambda = T_\Lambda T_\Lambda^* f = \sum_{i \in I} \Lambda_i^* \Lambda_i f, \quad (1.4)$$

which is a bounded, self-adjoint, positive and invertible operator and

$$AI_H \leq S_\Lambda \leq BI_H,$$

where I is the identity operator on H .

Lemma 1.3. ([2]) *Let H, K be Hilbert spaces, and suppose that $U : K \longrightarrow H$ is a bounded operator with closed range. Then there exists a bounded operator $U^\dagger : H \longrightarrow K$ for which*

$$\ker(U^\dagger) = \text{range}(U)^\perp, \quad \text{range}(U^\dagger) = \ker(U)^\perp, \quad UU^\dagger f = f, \quad f \in \text{range}(U).$$

We call the operator $U^\dagger$ the pseudo-inverse of U . This operator is uniquely determined by these properties.

Lemma 1.4. ([2]) *Let $U : K \longrightarrow H$ be a bounded operator with closed range. Then the following holds:*

- (i) *The orthogonal projection of H onto $\text{range}(U)$ is given by $UU^\dagger$.*
- (ii) *The orthogonal projection of K onto $\text{range}(U^\dagger)$ is given by $U^\dagger U$.*
- (iii) *U^* has closed range, and $(U^*)^\dagger = (U^\dagger)^*$.*
- (iv) *On $\text{range}(U)$, the operator $U^\dagger$ is given explicitly by $U^\dagger = U^*(UU^*)^{-1}$.*

Given closed subspaces W and V such that $H = W \oplus V^\perp$ (a direct sum, not necessarily orthogonal), the oblique projection of H on W along $V^\perp$ is defined by

$$E_{WV^\perp}(\omega + v^\perp) = \omega, \quad \omega \in W, \quad v^\perp \in V^\perp.$$

The definition implies that $\text{range}(E_{WV^\perp}) = W$ and $\ker(E_{WV^\perp}) = V^\perp$. The orthogonal projection of H onto a subspace W will be denoted by P_W . Note that $E_{WV^\perp}^2 = E_{WV^\perp}$.

As a consequence of Lemma 2.1 in [3], the oblique projection $E_{VW^\perp}$ is also well defined. Straightforward calculation gives that the adjoint operator associated to the bounded operator $E_{WV^\perp}$ is

$$E_{WV^\perp}^* = E_{VW^\perp}. \quad (1.5)$$

2 Oblique dual g -frames

We start this section by definition of dual g -frames and oblique dual g -frames.

Definition 2.1. *Let $\Lambda = \{\Lambda_i\}_{i \in I}$ be a g -frame sequence and $\Theta = \{\Theta_i\}_{i \in I}$ a g -Bessel sequence in H and $H_\Lambda = \overline{\text{span}}\{\Lambda_i^*(H_i)\}_{i \in I}$. $\{\Theta_i\}_{i \in I}$ is an alternative dual of $\{\Lambda_i\}_{i \in I}$, or simply a dual, if*

$$T_\Lambda T_\Theta^*|_{H_\Lambda} = I|_{H_\Lambda}.$$

Definition 2.2. Let $\{\Lambda_i\}_{i \in I}$ be a g -frame for a closed subspace W of H and $\{\Theta_i\}_{i \in I}$ a g -frame for closed subspace V of H such that $H = W \oplus V^\perp$. $\{\Theta_i\}_{i \in I}$ is an oblique dual g -frame of $\{\Lambda_i\}_{i \in I}$ on V when

$$f = \sum_{i \in I} \Lambda_i^* \Theta_i(f), \quad f \in W.$$

Lemma 2.3. Assume that $\{\Lambda_i\}_{i \in I}$ and $\{\Theta_i\}_{i \in I}$ are g -Bessel sequences in H and let $W = \overline{\text{span}}\{\Lambda_i^*(H_i)\}_{i \in I}$, $V = \overline{\text{span}}\{\Theta_i^*(H_i)\}_{i \in I}$. Suppose that $H = W \oplus V^\perp$, then the following are equivalent:

- (i) $f = \sum_{i \in I} \Lambda_i^* \Theta_i(f)$, $f \in W$.
- (ii) $E_{WV^\perp} = \sum_{i \in I} \Lambda_i^* \Theta_i$.
- (iii) $E_{VW^\perp} = \sum_{i \in I} \Theta_i^* \Lambda_i$.
- (iv) $\langle E_{VW^\perp} f, g \rangle = \sum_{i \in I} \langle \Lambda_i(f), \Theta_i(g) \rangle$, $f, g \in H$.
- (v) $\langle E_{WV^\perp} f, g \rangle = \sum_{i \in I} \langle \Theta_i(f), \Lambda_i(g) \rangle$, $f, g \in H$.

In case that equivalent conditions are satisfied, $\{\Theta_i\}_{i \in I}$ is an oblique dual g -frame of $\{\Lambda_i\}_{i \in I}$ on V and $\{\Lambda_i\}_{i \in I}$ is an oblique dual g -frame of $\{\Theta_i\}_{i \in I}$ on W . Furthermore, $\{\Lambda_i\}_{i \in I}$ and $\{\Theta_i P_W\}_{i \in I}$ are dual g -frames for W (in sense of classical definition) and $\{\Theta_i\}_{i \in I}$ and $\{\Lambda_i P_V\}_{i \in I}$ are dual g -frames for V .

Proof. (i) $\rightarrow$ (ii): For each $f \in H$ we can write $f = w + v^\perp$ such that $w \in W$ and $v^\perp \in V^\perp$. We have

$$E_{WV^\perp} f = w = \sum_{i \in I} \Lambda_i^* \Theta_i(w) = \sum_{i \in I} \Lambda_i^* \Theta_i(w) + \sum_{i \in I} \Lambda_i^* \Theta_i(v^\perp) = \sum_{i \in I} \Lambda_i^* \Theta_i(f).$$

(ii) $\rightarrow$ (i): It is obvious.

(ii) $\rightarrow$ (iii): Now, Let T and U denote the synthesis operators of $\{\Lambda_i\}_{i \in I}$ and $\{\Theta_i\}_{i \in I}$, respectively. So by (ii), $TU^* = E_{WV^\perp}$. By (1.5), $UT^* = E_{VW^\perp}$, so (iii) is shown.

(iii) $\rightarrow$ (ii): It is clear by (1.5).

(iii) $\rightarrow$ (iv): It is obvious.

(ii) $\rightarrow$ (v): It is obvious.

(iv) $\rightarrow$ (iii): Let $f \in H$. Since $\{\Lambda_i\}_{i \in I}$ and $\{\Theta_i\}_{i \in I}$ are g -Bessel sequences, $\sum_{i \in I} \Theta_i^* \Lambda_i$ is well defined. By (iv) we have

$$\langle E_{VW^\perp}(f) - \sum_{i \in I} \Theta_i^* \Lambda_i(f), g \rangle = 0, \quad g \in H.$$

Therefore, (iii) follows.

Similarly, (v) implies (ii).

If the equivalent conditions are satisfied, by (iv) and since $E_{VW^\perp} f = f$, $f \in V$, we have

$$\|f\|^2 = \sum_{i \in I} \langle \Lambda_i(f), \Theta_i(g) \rangle = \sum_{i \in I} \langle \Lambda_i P_V(f), \Theta_i(g) \rangle, \quad f \in V.$$

So,

$$\|f\|^2 = \sum_{i \in I} \langle \Lambda_i P_V(f), \Theta_i(g) \rangle \leq \sum_{i \in I} \|\Lambda_i P_V(f)\|^2 \sum_{i \in I} \|\Theta_i(f)\|^2, \quad f \in V.$$

Since $\{\Theta_i\}_{i \in I}$ (respectively $\{\Lambda_i P_V\}_{i \in I}$) is a g -Bessel sequence in H , $\{\Lambda_i P_V\}_{i \in I}$ (respectively $\{\Theta_i\}_{i \in I}$) satisfies the lower g -frame condition for all $f \in V$. Therefore both are g -frames for V . Also, by (iii) they are dual g -frames in the classical sense. Similarly, $\{\Lambda_i\}_{i \in I}$ and $\{\Theta_i P_W\}_{i \in I}$ are dual g -frames for W . By (ii) and (iii) and Definition 2.2, the statement about the relevant g -frames being oblique duals follows. $\square$

Proposition 2.4. *Let $\{\Lambda_i\}_{i \in I}$ and $\{\Theta_i\}_{i \in I}$ be g -Bessel sequences in H and*

$$f = \sum_{i \in I} \Lambda_i^* \Theta_i(f), \quad f \in W.$$

Let U be any closed subspace of H for which $H = W \oplus U^\perp$. Then $\{\Theta_i E_{WU^\perp}\}_{i \in I}$ is an oblique dual g -frame of $\{\Lambda_i\}_{i \in I}$ on U .

Proof. By assumption, we have

$$f = \sum_{i \in I} \Lambda_i^* \Theta_i E_{WU^\perp} f, \quad f \in W.$$

Also, for each $f \in U^\perp$,

$$\langle f, E_{UW^\perp} \Theta_i^* \Lambda_i g \rangle = 0, \quad g \in H.$$

Then

$$\sum_{i \in I} \langle f, E_{UW^\perp} \Theta_i^* \Lambda_i g \rangle = 0, \quad f \in U^\perp, \quad g \in H.$$

Thus

$$E_{WU^\perp} f = \sum_{i \in I} \Lambda_i^* \Theta_i E_{WU^\perp} f, \quad f \in H. \quad (2.1)$$

Let T and V be the synthesis operators of $\{\Lambda_i\}_{i \in I}$ and $\{\Theta_i\}_{i \in I}$, respectively. By (2.1),

$$E_{WU^\perp} = TV^* E_{WU^\perp}, \quad (2.2)$$

equivalently, $E_{UW^\perp} = E_{UW^\perp} VT^*$. For each $f \in H$, we have

$$E_{UW^\perp} f = \sum_{i \in I} E_{UW^\perp} \Theta_i^* \Lambda_i f,$$

in particular,

$$f = \sum_{i \in I} E_{UW^\perp} \Theta_i^* \Lambda_i f, \quad f \in U.$$

Then for each $f \in U$,

$$\|f\|^2 = \sum_{i \in I} \langle E_{UW^\perp} \Theta_i^* \Lambda_i f, f \rangle = \sum_{i \in I} \langle \Lambda_i f, \Theta_i E_{WU^\perp} f \rangle. \quad (2.3)$$

By (2.3), $\{\Theta_i E_{WU^\perp}\}_{i \in I}$ satisfies the lower g -frame condition on U . Hence it is a g -frame for U . Via (2.2), it is clear that $\{\Theta_i E_{WU^\perp}\}_{i \in I}$ is an oblique dual g -frame of $\{\Lambda_i\}_{i \in I}$ on U . $\square$

3 Types I and Type II duals of g -frame sequences

In this section, we introduce the Type I and Type II duals of a g -frame sequence and then verify some properties of these types of duals.

Definition 3.1. Let $\Lambda = \{\Lambda_i\}_{i \in I}$ be a g -frame sequence and $\Theta = \{\Theta_i\}_{i \in I}$ a g -Bessel sequence in H .

(i) $\{\Theta_i\}_{i \in I}$ is a Type I dual of $\{\Lambda_i\}_{i \in I}$, if $\{\Theta_i\}_{i \in I}$ is a dual and

$$\text{range}(T_\Theta) \subseteq \text{range}(T_\Lambda).$$

(ii) $\{\Theta_i\}_{i \in I}$ is a Type II dual of $\{\Lambda_i\}_{i \in I}$, if $\{\Theta_i\}_{i \in I}$ is a dual and

$$\text{range}(T_\Theta^*) \subseteq \text{range}(T_\Lambda^*).$$

Lemma 3.2. Let $\{\Lambda_i\}_{i \in I}$ be a g -frame sequence with canonical dual g -frame sequence $\{\tilde{\Lambda}_i\}_{i \in I}$ (for more details about the canonical dual g -frame sequence, see [8]). Then:

(i) $H_{\tilde{\Lambda}} = \overline{\text{span}}\{\tilde{\Lambda}_i^*(H_i)\}_{i \in I} = \text{range}(T_\Lambda) = \text{range}(T_{\tilde{\Lambda}}) = \overline{\text{span}}\{\Lambda_i^*(H_i)\}_{i \in I} = H_\Lambda$.

(ii) $T_{\tilde{\Lambda}}^* = T_\Lambda^* S_\Lambda^\dagger = T_\Lambda^* (T_\Lambda T_\Lambda^*)^\dagger$.

(iii) $\text{range}(T_\Lambda^*) = \text{range}(T_{\tilde{\Lambda}}^*)$ and this is a closed subspace of H .

Proof. (i). We have $\{\tilde{\Lambda}_i\}_{i \in I} = \{\Lambda_i S_\Lambda^\dagger\}_{i \in I}$, where $S_\Lambda = T_\Lambda T_\Lambda^*$. By Corollary 2.3 and Lemma 2.6 in [8],

$$\begin{aligned} H_{\tilde{\Lambda}} &= \overline{\text{span}}\{\tilde{\Lambda}_i^*(H_i)\}_{i \in I} = H_{\tilde{\Lambda}} = \overline{\text{span}}\{(S_\Lambda^\dagger)^* \Lambda_i^*(H_i)\}_{i \in I} \\ &= (S_\Lambda^\dagger)^* \overline{\text{span}}\{\Lambda_i^*(H_i)\}_{i \in I} = (S_\Lambda^\dagger)^*(V) = V. \end{aligned}$$

By Proposition 2.1 and Corollary 2.5 in [8], $\text{range}(T_\Lambda) = H_\Lambda$.

(ii). By Lemma 2.6 and Corollary 2.3 and Proposition 2.1 and Corollary 2.2 in [8], we have

$$T_\Lambda^* = U_\Lambda = \tilde{\mathcal{U}}_\Lambda = \mathcal{U}\mathcal{S}_\Lambda^{-1}P = \mathcal{U}\iota_V\mathcal{S}_\Lambda^{-1}P = \mathcal{U}P\iota_V\mathcal{S}^{-1}P = T_\Lambda^*S^\dagger,$$

where $\iota_V : V \longrightarrow H$ is the inclusion operator and $\mathcal{U}_\Lambda$, $\tilde{\mathcal{U}}_\Lambda$ and $\mathcal{S}_\Lambda$ are defined in [8].

(iii). By Proposition 2.1(iv) and Corollary 2.2(iii) in [8],

$$\begin{aligned} \text{range}(T_\Lambda^*) &= \text{range}(U_\Lambda) = \text{range}(\mathcal{U}_\Lambda) \\ &= \text{range}(\mathcal{U}_\Lambda)\mathcal{S}_\Lambda^{-1} = \text{range}(\mathcal{U}_\Lambda) \\ &= \text{range}(U_\Lambda) = \text{range}(T^*), \end{aligned}$$

where $\mathcal{U}_\Lambda$ and $\mathcal{S}_\Lambda$ are defined in [8]. $\square$

Theorem 3.3. *Let $\Lambda = \{\Lambda_i\}_{i \in I}$ be a g -frame sequence and $\Theta = \{\Theta_i\}_{i \in I}$ a g -Bessel sequence in H .*

(i) *If Θ is a Type I dual of Λ then Θ is a g -frame sequence and Λ is a Type I dual of Θ .*

In this case, Λ and Θ are oblique duals, $H_\Theta = \text{range}(T_\Theta) = \text{range}(T_\Lambda) = H_\Lambda$ and

$$H = \text{range}(T_\Lambda) \oplus \text{range}(T_\Theta)^\perp.$$

(ii) *If Θ is a Type II dual of Λ then Θ is a g -frame sequence and Λ is a Type II dual of Θ .*

In this case, Λ and Θ are oblique duals, $\text{range}(T_\Theta^) = \text{range}(T_\Lambda^*)$ and*

$$H = \text{range}(T_\Lambda) \oplus \text{range}(T_\Theta)^\perp.$$

Proof. (i). By assumption, we have

$$f = T_\Lambda T_\Theta^*(f) = \sum_{i \in I} \Lambda_i^* \Theta_i(f), \quad f \in H_\Lambda.$$

So for each $f \in H_\Lambda$,

$$\|f\|^2 = \sum_{i \in I} \langle \Theta_i f, \Lambda_i f \rangle \leq \left(\sum_{i \in I} \|\Theta_i f\|^2 \right)^{\frac{1}{2}} \left(\sum_{i \in I} \|\Lambda_i f\|^2 \right)^{\frac{1}{2}}.$$

Since Λ is a g -frame sequence, so Θ is a g -frame sequence. Also,

$$T_\Theta T_\Lambda^*|_{H_\Theta} = I|_{H_\Theta}.$$

Therefore Λ is a Type *I* dual of Θ . The other parts are obvious.
(ii). Let $\tilde{\Lambda}$ be the canonical dual g -frame sequence of Λ . Then

$$\begin{aligned} T_{\Theta}^* &= P_{range(T_{\Theta}^*)} T_{\Theta}^* \\ &= P_{range(T_{\Lambda}^*)} T_{\Theta}^* \quad (\text{since } \Theta \text{ is a Type II dual}) \\ &= T_{\Lambda}^{\dagger} T_{\Lambda} T_{\Theta}^* \quad (\text{Lemma 1.3 and Lemma 1.4}) \\ &= T_{\Lambda}^{\dagger} \quad (\text{since } \Theta \text{ is a dual of } \Lambda) \\ &= T_{\Lambda}^{\dagger} T_{\Lambda} T_{\tilde{\Lambda}}^* \quad (\text{since } \tilde{\Lambda} \text{ is a dual of } \Lambda) \\ &= P_{range(T_{\Lambda}^*)} T_{\tilde{\Lambda}}^* \quad (\text{Lemma 1.3 and Lemma 1.4}). \end{aligned}$$

Hence, $T_{\Theta}^*|_{H_{\Lambda}} = T_{\tilde{\Lambda}}^*|_{H_{\Lambda}}$. Now, since T_{Λ}^* has closed range (by Corollary 2.5 in [8]),

$$T_{\Lambda}^* : H_{\Lambda} = range(T_{\Lambda}) \rightarrow range(T_{\Lambda}^*)$$

is a bijection. Likewise

$$T_{\tilde{\Lambda}}^* : H_{\Lambda} = range(T_{\tilde{\Lambda}}) \rightarrow range(T_{\tilde{\Lambda}}^*)$$

is a bijection. Therefore

$$\begin{aligned} range(T_{\Lambda}^*) &= range(T_{\tilde{\Lambda}}^*) \\ &= range(T_{\tilde{\Lambda}}^*|_{H_{\Lambda}}) \\ &= range(T_{\Theta}^*|_{H_{\Lambda}}) \\ &\subseteq range(T_{\Theta}^*) \\ &\subseteq range(T_{\Lambda}^*) \quad (\text{since } \Theta \text{ is a Type II dual}). \end{aligned}$$

Hence

$$range(T_{\Theta}^*) = range(T_{\Lambda}^*)$$

is closed, so $range(T_{\Theta})$ is closed. Consequently, by Lemma 1.2 in [7], Θ is a g -frame sequence in H and

$$H_{\Theta} = \overline{span}\{\Theta_i^*(H_i)\}_{i \in I} = range(T_{\Theta}).$$

Now, we claim that $T_{\Lambda} T_{\Theta}^* = P_{H_{\Lambda}, H_{\Theta}^{\perp}}$. Since $T_{\Lambda} T_{\Theta}^* = I|_{H_{\Lambda}}$ and $range(T_{\Lambda}) = H_{\Lambda}$, we have both $(T_{\Lambda} T_{\Theta}^*)^2 = T_{\Lambda} T_{\Theta}^*$ and $range(T_{\Lambda} T_{\Theta}^*) = H_{\Lambda}$.

If $f \in \ker(T_{\Lambda} T_{\Theta}^*)$ then $T_{\Lambda} T_{\Theta}^*(f) = 0$, so

$$T_{\Theta}^* f \in range(T_{\Theta}^*) \cap \ker(T_{\Lambda}) = range(T_{\Lambda}^*) \cap \ker(T_{\Lambda}) = 0.$$

Thus $\ker(T_{\Lambda} T_{\Theta}^*) \subseteq \ker(T_{\Theta}^*)$ and the opposite inclusion is trivial. Therefore

$$\ker(T_{\Lambda} T_{\Theta}^*) = \ker(T_{\Theta}^*) = range(T_{\Theta}^*)^{\perp} = H^{\perp}.$$

So $T_\Lambda T_\Theta^* = P_{H_\Lambda, H_\Theta^\perp}$, also the fact that this oblique projection exists implies that

$$H = \text{range}(T_\Lambda) \oplus \text{range}(T_\Theta)^\perp.$$

□

Theorem 3.4. *Let U and V be closed subspaces of H and $\{\Lambda_i\}_{i \in I}$ be a g -frame for U . Then the following statements are equivalent:*

- (i) $H = U \oplus V^\perp$.
- (ii) There is a g -frame $\{\Theta_i\}_{i \in I}$ for V which is a Type II dual of $\{\Lambda_i\}_{i \in I}$.
- (iii) There is a g -frame $\{\Theta_i\}_{i \in I}$ for V which is an oblique dual of $\{\Lambda_i\}_{i \in I}$.

In case these hold, $\{\Lambda_i P\}_{i \in I}$ is a g -frame for V and we can take $\{\Theta_i\}_{i \in I}$ to be the canonical dual g -frame of $\{\Lambda_i P\}_{i \in I}$ in V , where $P = P_V|_U$.

Proof. (i) $\rightarrow$ (ii): Suppose that $H = U \oplus V^\perp$ and $\{\Lambda_i\}_{i \in I}$ be a g -frame for U . We will construct a g -frame for V that is a Type II dual of $\{\Lambda_i\}_{i \in I}$. Recall from Proposition 2.1 in [4] that $P := P_V|_U : U \rightarrow V$ is an isomorphism, so $P^{-1}V \rightarrow U$ is also an isomorphism. Therefore, $\{\Lambda_i P\}_{i \in I}$ is a g -frame for V . Let $\{\Theta_i\}_{i \in I}$ be the canonical dual g -frame of $\{\Lambda_i P\}_{i \in I}$ in V , that is, $\{\Theta_i\}_{i \in I} = \{\Lambda_i S^{-1}\}_{i \in I}$. Given any $u \in U$ we have $P_V u = Pu$. Therefore,

$$\sum_{i \in I} \Lambda_i^* \Theta_i(u) = P^{-1} \left(\sum_{i \in I} P \Lambda_i^* \Theta_i(P_V u) \right) = P^{-1} \left(\sum_{i \in I} (\Lambda_i P)^* \Theta_i(Pu) \right) = P^{-1} P(u).$$

Hence $\{\Theta_i\}_{i \in I}$ is a dual of $\{\Lambda_i\}_{i \in I}$ and furthermore $\{\Lambda_i\}_{i \in I}$ is a g -frame for V . It remains to show that $\text{range}(T_\Theta^*) \subseteq \text{range}(T_\Lambda^*)$. Since $\text{range}(T_\Lambda) \subseteq U$, if we consider T_Λ to be a mapping of $\left(\sum_{i \in I} \oplus H_i \right)_{l_2}$ into U then the synthesis operator of $\Lambda P = \{\Lambda_i P\}_{i \in I}$ is $T_{\Lambda P} = P^* T_\Lambda$. The corresponding g -frame operator $S_{\Lambda P} : V \rightarrow V$ for ΛP is

$$S_{\Lambda P} = T_{\Lambda P} (T_{\Lambda P})^* = P^* T_\Lambda T_\Lambda^* P$$

and this is an invertible mapping of V onto itself. Therefore the canonical dual g -frame of ΛP is given by

$$\{\Theta_i\}_{i \in I} = \{\Lambda_i P S_{\Lambda P}^{-1}\}_{i \in I} = \{\Lambda_i S_\Lambda^{-1} P^{-1}\}_{i \in I}.$$

Further, by Lemma 3.2, the synthesis operator for $\{\Theta_i\}_{i \in I}$ is

$$T_\Theta = S_{\Lambda P}^{-1} T_{\Lambda P} = P^{-1} S_\Lambda^{-1} (P^*)^{-1} P^* T_\Lambda = P^{-1} S_\Lambda^{-1} T_\Lambda.$$

Since S_Λ and P are both invertible, we conclude that $\ker(T_\Theta) = \ker(T_\Lambda)$ and consequently $\text{range}(T_\Theta^*) = \text{range}(T_\Lambda^*)$.

(ii) $\rightarrow$ (iii): This is obvious via Theorem 3.3.

(iii) $\rightarrow$ (i): Suppose $\{\Theta_i\}_{i \in I}$ is an oblique dual of $\{\Lambda_i\}_{i \in I}$ such that $\{\Theta_i\}_{i \in I}$ is also a g -frame for V . Set $T = T_\Lambda T_\Theta^*$. Since $\{\Theta_i\}_{i \in I}$ is dual to $\{\Lambda_i\}_{i \in I}$ and $U = H_\Lambda = \text{range}(T_\Lambda)$, we have $T|_U = I|_U$. We show that $T = P_{U, V^\perp}$. By definition of dual, we have $T^2 = T$ and $\text{range}(T) = U$. Also, since $\{\Theta_i\}_{i \in I}$ is an oblique dual, $\{\Lambda_i\}_{i \in I}$ is dual to $\{\Theta_i\}_{i \in I}$ and therefore $T^*|_V = T_\Lambda T_\Theta^*|_V = I|_V$. Hence $\text{range}(T^*) = V$. So $\ker(T) = V^\perp$. This shows that $T = P_{U, V^\perp}$ and therefore, $H = U \oplus V^\perp$. $\square$

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