On Local Controllability of Nonlinear Hyperbolic Equation

Dieudonné Ampini, Fidèle Moukamba and Yumba Nkasa

Département de Mathématiques
Faculté des Sciences
Université Marien NGOUABI
BP. 69 Brazzaville-République du Congo
moukambaf@yahoo.fr

Abstract

In this paper, we prove local controllability of a one-dimensional nonlinear wave equation, with control functions on the right hand of the equation, using the inversion theorem.

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1 Introduction

In this paper, we present a result related to the following local controllability problem for the nonlinear one-dimensional wave equation. We consider the following nonlinear equation:

$$\begin{aligned}u_{tt}(x, t) - u_{xx}(x, t) &= F(x, t) \\u(x, t)|_{t=0} &= 0, u_t(x, t)|_{t=0} = 0, 0 \leq x \leq 0 \\u(x, t)|_{x=0} &= 0, u(x, t)|_{x=l} = 0, 0 \leq t \leq T, 0 \leq x \leq 0 \\u(x, t)|_{t=T} &= \chi_1(x), u_t(x, t)|_{t=T} = \chi_2(x)\end{aligned}$$

where $F(x, t) = (\frac{x}{l} - 1)f(t) - \frac{x}{l}g(t)$.

The problem of local controllability can be formatted in the following way:

in time $T > 0$, we propose to drive the system from its initial state

$$u(x, t)|_{t=0} = 0, u_t(x, t)|_{t=0} = 0$$

to a given final state

$$u(x, t)|_{t=T} = \chi_1(x), u_t(x, t)|_{t=T} = \chi_2(x)$$

by mean of the action of control functions f and g .

In order words, for fixed $T > 0$, find control functions f and g such that the solution u of the equation satisfies the conditions

$$u(x, t)|_{t=T} = \chi_1(x), u_t(x, t)|_{t=T} = \chi_2(x).$$

To prove the existence theorem of controls, we use the properties of solutions of a linear problem and the local inversion theorem in the linear and quasilinear cases, the same results was established respectively in [1] and [2].

In nonlinear case, similar results have been established in [3].

2 Statement of the problem and notations

$$\Gamma =]\Gamma[\subset \mathbf{R}, I =]0, T[\subset \mathbf{R}, Q_T = \Gamma \times I.$$

$$H^{p,m}(\Gamma) = \{f : \Gamma \longrightarrow \mathbf{R} : f^{(m)} \in L^p(\Gamma)\}$$

$$H^m(\Gamma) = \{f : \Gamma \longrightarrow \mathbf{R} : f^{(m)} \in L^2(\Gamma)\}$$

$$H_0^1(\Gamma) = \overline{D(\Gamma)} \text{ in } H^1(\Gamma).$$

In order to state in a rigorous way the problem, we need to remember some results established in [1] and [2].

Let $u = u(x, t)$. Consider the problem

$$u_{tt} - u_{xx} = F(x, t) \quad (1)$$

$$u|_{t=0} = 0, u_t|_{t=0} = 0, 0 \leq x \leq l \quad (2)$$

$$u|_{x=0} = 0, u|_{x=l} = 0, 0 \leq t \leq T \quad (3)$$

$$u(x, t)|_{t=T} = \chi_1(x), u_t(x, t)|_{t=T} = \chi_2(x) \quad (4)$$

where $F(x, t) = (\frac{x}{l} - 1)f(t) - \frac{x}{l}g(t)$.

$$\chi_1 \in H^5(\Gamma) \cap H_0^1(\Gamma), \chi_2 \in H^4(\Gamma) \cap H_0^1(\Gamma), f \in H^3(I), g \in H^3(I).$$

Definition 1 ([1], p.693) . We say that a function $(x, t) \longmapsto u(x, t)$ belongs to class $W_2^2(Q_T)$, if u and its first partial derivate are continuous in $\overline{Q_T}$, and if this function admits all second generalised partial dervatives belonging to $L_2(\Gamma)$ for all $t \in I$, and to $L^2(I)$ for all $x \in \overline{\Gamma}$.

Definition 2 ([1], p.694) . $(u, f, g) \in W_2^2(Q_T) \times H^3(I) \times H^3(I)$ is said to be a solution of the mixed problem (1) – (4), if:

- (i) (u, f, g) satisfies the equation (1) a.e. in Q_T ;
- (ii) u satisfies the initial conditions (2) a.e. in $\overline{\Gamma}$;
- (iii) u satisfies the boundary conditions (3) a. e. in $\overline{I}$;
- (iv) u satisfies the conditions (4) a.e. in $\overline{\Gamma}$.

Theorem 1 ([2], p. 402) . There exists a unique solution for which u is a solution of the problem (1) – (4).

Remark 1 Let $H = \{u \in W_2^2(Q_T) / \exists F(x, t) \in H^3(Q_T) : (u, f, g) \text{ is a solution of the mixed problem (1) – (4)}\}$,

with the norm $\|u\|_H = \|Lu\|_{H^3(Q_T)}$,

where $L = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$.

Then $L : H \longrightarrow H^3(Q_T)$ is an isomorphism. To prove this fact, see ([4] P. 306).

Note that

$H \subset \{u \in W_2^2(Q_T) / u(x, t)|_{t=0} = 0, u_t(x, t)|_{t=0} = 0, u(x, t)|_{x=0} = 0, u(x, t)|_{x=l} = 0\}$.

This implies that $(H, \|\cdot\|_H)$ is a complete space. We also get the continuous embedding $H \hookrightarrow H^4(Q_T)$. This means that $H \subset H^4(Q_T)$.

3 Formulation of the main result

Let $u \longmapsto \nu(u) \in C^4(\mathbf{R})$; with a function of two real variables, $x \in \Gamma$ and $t \in I$, such that:

$$\nu(0) = 0, \nu'(0) = 0. \quad (5)$$

Consider the problem

$$u_{tt} - u_{xx} + \nu(u(x, t)) = F(x, t) \quad (6)$$

$$u|_{t=0} = 0, u_t|_{t=0} = 0, 0 \leq x \leq l \quad (7)$$

$$u|_{x=0} = 0, u|_{x=l} = 0, 0 \leq t \leq T \quad (8)$$

$$u(x, t)|_{t=T} = \chi_1(x), u_t(x, t)|_{t=T} = \chi_2(x) \quad (9)$$

where $F(x, t) = (\frac{x}{l} - 1)f(t) - \frac{x}{l}g(t)$.

The main result of this paper is the following.

Theorem 2 *If conditions (5) are satisfied, then there exists a unique solution (u, f, g) on a neighborhood of zero in $H^4(Q_T) \times H^3(I) \times H^4(I)$ for small enough $(\chi_1, \chi_2) \in H^5(\Gamma) \cap H_0^1(\Gamma) \times H^4(\Gamma) \cap H_0^1(\Gamma)$.*

Remark 2 *In order to prove theorem 2, we have to verify the fact that the conditions of the inversion theorem are satisfied; which means that we must establish the differentiability of the involved mappings and specify the functional spaces in which we are going to study problem (6) – (9).*

4 Differentiability of Nemwisk operator

Proposition 3 *The operator $N : H^4(Q_T) \longrightarrow H^3(Q_T)$, $u \longmapsto N(u)$, where $[N(u)](x, t) = \nu(u(x, t))$ is differentiable in $H^4(Q_T)$ and*

$$[N'(u)\hat{u}](x, t) = \nu'(u(x, t))\hat{u}(x, t). \quad (10)$$

Proof. First of all, let us prove that the operator N maps $H^4(Q_T)$ into $H^3(Q_T)$. Suppose that $u \in H^4(Q_T)$. Then according to the embedding theorem, we have

$$u \in C(\overline{Q_T}), \quad u_t \in H^3(Q_T) \subset C(\overline{Q_T}), \quad u_{tt} \in H^2(Q_T) \subset C(\overline{Q_T}) \quad (11)$$

and

$$u_x \in H^3(Q_T) \subset C(\overline{Q_T}), \quad u_{xx} \in H^2(Q_T) \subset C(\overline{Q_T}). \quad (12)$$

In this case $\int_{Q_T} |\nu(u(x, t))|^2 dx dt < \infty$ since $\nu(u) \in C(Q_T)$, which means that $\nu(u(x, t)) \in L_2(Q_T)$. We have

$$\begin{aligned} \int_{Q_T} \left| \frac{\partial \nu}{\partial t}(u(x, t)) \right|^2 dx dt &= \int_{Q_T} |\nu'(u(x, t)) u_t|^2 dx dt \\ &\leq \sup_{(x, t) \in Q_T} |\nu'(u)|^2 \int_{Q_T} |u_t(x, t)|^2 dx dt \\ &= \sup_{(x, t) \in Q_T} |\nu'(u)|^2 \|u_t\|_{L_2(Q_T)}^2 < \infty, \end{aligned}$$

i.e. $\frac{\partial \nu}{\partial t}(u(x, t)) \in L_2(Q_T)$. In the same way, we can estimate $\int_{Q_T} |\frac{\partial \nu}{\partial x}(u(x, t))|^2 dx dt$. We then have:

$$\begin{aligned} &\int_{Q_T} \left| \frac{\partial^2 \nu}{\partial t^2}(u(x, t)) \right|^2 dx dt \\ &= \int_{Q_T} \left| \frac{\partial}{\partial t} \left(\frac{\partial \nu}{\partial t}(u(x, t)) \right) \right|^2 dx dt \\ &= \int_{Q_T} \left| \frac{\partial \nu'}{\partial t}(u(x, t)) u_t(x, t) \right|^2 dx dt \\ &= \int_{Q_T} |\nu''(u(x, t)) u_t^2(x, t) + \nu'(u(x, t)) u_{tt}(x, t)|^2 dx dt \\ &\leq 2 \int_{Q_T} |\nu''(u(x, t)) u_t^2(x, t)|^2 dx dt + 2 \int_{Q_T} |\nu'(u(x, t)) u_{tt}(x, t)|^2 dx dt \\ &\leq 2 \sup_{(x, t) \in Q_T} |\nu''(u)|^2 \int_{Q_T} |u_t(x, t)|^2 dx dt + 2 \sup_{(x, t) \in Q_T} |\nu'(u)|^2 \int_{Q_T} |u_{tt}(x, t)|^2 dx dt \\ &= 2 \sup_{(x, t) \in Q_T} |\nu''(u)|^2 \|u_t\|_{L_4(Q_T)}^4 + 2 \sup_{(x, t) \in Q_T} |\nu'(u)|^2 \|u_{tt}\|_{L_2(Q_T)}^2 < \infty \end{aligned}$$

i.e. $\frac{\partial^2 \nu}{\partial t^2}(u(x, t)) \in L_2(Q_T)$. Similarly, we can estimate $\int_{Q_T} \left| \frac{\partial^2 \nu}{\partial x^2}(u(x, t)) \right|^2 dx dt$. Next,

$$\begin{aligned}
& \int_{Q_T} \left| \frac{\partial^2 \nu}{\partial x \partial t}(u(x, t)) \right|^2 dx dt \\
&= \int_{Q_T} \left| \frac{\partial}{\partial x} \left(\frac{\partial \nu}{\partial t}(u(x, t)) \right) \right|^2 dx dt \\
&= \int_{Q_T} \left| \frac{\partial}{\partial x} (\nu'(u(x, t)) u_t(x, t)) \right|^2 dx dt \\
&= \int_{Q_T} |\nu''(u(x, t)) u_x(x, t) u_t(x, t) + \nu'(u(x, t)) u_{tx}(x, t)|^2 dx dt \\
&\leq 2 \int_{Q_T} |\nu''(u(x, t)) u_x(x, t) u_t(x, t)|^2 dx dt + 2 \int_{Q_T} |\nu'(u(x, t)) u_{tx}(x, t)|^2 dx dt \\
&\leq 2 \sup_{(x, t) \in Q_T} |\nu''(u)|^2 \left(\int_{Q_T} |u_x(x, t)|^4 dx dt \right)^{\frac{1}{2}} \left(\int_{Q_T} |u_t(x, t)|^4 dx dt \right)^{\frac{1}{2}} \\
&\quad + 2 \sup_{(x, t) \in Q_T} |\nu'(u)|^2 \int_{Q_T} |u_{tx}(x, t)|^2 dx dt.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_{Q_T} \left| \frac{\partial^2 \nu}{\partial x \partial t}(u(x, t)) \right|^2 dx dt &\leq 2 \sup_{(x, t) \in Q_T} |\nu''(u)|^2 \|u_x\|_{L_4(Q_T)}^2 \|u_t\|_{L_4(Q_T)}^2 \\
&\quad + 2 \sup_{(x, t) \in Q_T} |\nu'(u)|^2 \|u_{tx}\|_{L_2(Q_T)}^2 < +\infty
\end{aligned}$$

i.e

$$\frac{\partial^2 \nu}{\partial x \partial t}(u(x, t)) \in L_2(Q_T).$$

Similarly, we can estimate $\int_{Q_T} \left| \frac{\partial^2}{\partial t \partial x} \nu(u(x, t)) \right|^2 dx dt$.

Finally, let us estimate $\int_{Q_T} \left| \frac{\partial^3}{\partial t^3} \nu(u(x, t)) \right|^2 dx dt$

$$\begin{aligned}
& \int_{Q_T} \left| \frac{\partial^3}{\partial t^3} \nu(u(x, t)) \right|^2 dx dt \\
&= \int_{Q_T} \left| \frac{\partial}{\partial t} \left(\frac{\partial^2 \nu}{\partial t^2} (u(x, t)) \right) \right|^2 dx dt \\
&= \int_{Q_T} \left| \frac{\partial}{\partial t} [\nu''(u(x, t)) u_t^2(x, t) + \nu'(u(x, t)) u_{tt}(x, t)] \right|^2 dx dt \\
&= \int_{Q_T} |\nu'''(u) u_t^3 + 3\nu''(u) u_t u_{tt} + \nu'(u) u_{ttt}|^2 dx dt \\
&\leq 3 \left(\sup_{Q_T} |\nu'''(u)|^2 \int_{Q_T} u_t^6 dx dt + 3 \sup_{Q_T} |\nu''(u)|^2 \int_{Q_T} u_t^2 u_{tt}^2 dx dt \right. \\
&\quad \left. + \sup_{Q_T} |\nu'(u)|^2 \int_{Q_T} u_{ttt}^2 dx dt \right).
\end{aligned}$$

This implies

$$\begin{aligned}
\int_{Q_T} \left| \frac{\partial^3 \nu}{\partial t^3} (u(x, t)) \right|^2 dx dt &\leq 3 \sup_{(x, t) \in Q_T} |\nu'''(u)|^2 \|u_t\|_{L_6(Q_T)}^6 \\
&\quad + 3 \sup_{(x, t) \in Q_T} |\nu''(u)|^2 \|u_t\|_{L_4(Q_T)}^2 \|u_{tt}\|_{L_4(Q_T)}^2 \\
&\quad + \sup_{(x, t) \in Q_T} |\nu'(u)|^2 \|u_{ttt}\|_{L_2(Q_T)}^2 \|u_t\|_{L_4(Q_T)}^2 < +\infty
\end{aligned}$$

i.e.

$$\frac{\partial^3}{\partial t^3} \nu(u(x, t)) \in L_2(Q_T).$$

Similarly, we can estimate $\int_{Q_T} \left| \frac{\partial^3 \nu}{\partial x^3} (u(x, t)) \right|^2 dx dt$.

We can, in the same way, estimate the following expressions:

$$\begin{aligned}
& \int_{Q_T} \left| \frac{\partial}{\partial x} \left(\frac{\partial^2 \nu}{\partial t^2} (u(x, t)) \right) \right|^2 dx dt, \quad \int_{Q_T} \left| \frac{\partial}{\partial t} \left(\frac{\partial^2 \nu}{\partial x^2} (u(x, t)) \right) \right|^2 dx dt, \\
& \int_{Q_T} \left| \frac{\partial^2}{\partial x^2} \left(\frac{\partial \nu}{\partial t} (u(x, t)) \right) \right|^2 dx dt, \quad \int_{Q_T} \left| \frac{\partial^2}{\partial t^2} \left(\frac{\partial \nu}{\partial x} (u(x, t)) \right) \right|^2 dx dt, \\
& \int_{Q_T} \left| \frac{\partial}{\partial x} \left(\frac{\partial}{\partial t} \left(\frac{\partial \nu}{\partial x} (u(x, t)) \right) \right) \right|^2 dx dt, \quad \int_{Q_T} \left| \frac{\partial}{\partial t} \left(\frac{\partial}{\partial x} \left(\frac{\partial \nu}{\partial t} (u(x, t)) \right) \right) \right|^2 dx dt.
\end{aligned}$$

All these results together with (11) and (12) imply that $N(u) \in H^3(Q_T)$.

Now suppose that $u, u_k, \hat{u}_k \in H^4(Q_T)$, $\hat{u}_k \neq 0$, $u_k \rightarrow u$ and $\hat{u}_k \rightarrow 0$ in $H^4(Q_T)$, when $k \rightarrow \infty$.

Let

$$[R(u_k, \hat{u}_k)](x, t) = \nu(u_k(x, t) + \hat{u}_k(x, t)) - \nu(u_k(x, t)) - \nu'(u(x, t))\hat{u}_k(x, t) \quad (13)$$

and let us prove that

$$\lim_{k \rightarrow \infty} \frac{R(u_k, \hat{u}_k)}{\|\hat{u}_k\|_{H^4(Q_T)}} \rightarrow 0 \quad (14)$$

in $H^3(Q_T)$.

Considering subsequences, we can assume that $u_k \rightarrow u$, $\hat{u}_k \rightarrow 0$,

$u_{kt} \rightarrow u_t$, $u_{ktt} \rightarrow u_{tt}$, $\hat{u}_{kt} \rightarrow 0$ and $\hat{u}_{ktt} \rightarrow 0$ almost everywhere on Q_T . Since $u \mapsto \nu'(u)$ is continuous, $u \mapsto \nu'(u)$ is uniformly continuous on every segment. Moreover, since $u_k \rightarrow u$ in $H^4(Q_T)$, from (11), (12) we get $u_k \rightarrow u$, $\hat{u}_k \rightarrow 0$ in $C(\overline{Q_T})$. This implies that there exist constants C_{16} and C_{17} such that $\|u_k\|_{C(\overline{Q_T})} \leq C_{16}$ and $\|\hat{u}_k\|_{C(\overline{Q_T})} \leq C_{17}$.

Thus,

$$\begin{aligned} & \int_{Q_T} |R(u_k, \hat{u}_k)|^2 dx dt \\ &= \int_{Q_T} |\nu(u_k + \hat{u}_k) - \nu(u_k) - \nu'(u) \hat{u}_k|^2 dx dt \\ &= \int_{Q_T} \left(\int_0^1 [\nu'(u_k + \theta \hat{u}_k) - \nu'(u)] \hat{u}_k d\theta \right)^2 dx dt \\ &\leq \int_{Q_T} \int_0^1 [\nu'(u_k + \theta \hat{u}_k) - \nu'(u)]^2 \hat{u}_k^2 d\theta \int_0^1 1^2 d\theta dx dt \\ &= \int_0^1 \int_{Q_T} [\nu'(u_k + \theta \hat{u}_k) - \nu'(u)]^2 \hat{u}_k^2 dx dt d\theta \\ &\leq \sup_{(x,t) \in Q_T} |\hat{u}_k|^2 \int_0^1 \left(\int_{Q_T} |\nu'(u_k + \theta \hat{u}_k) - \nu'(u)|^2 dx dt \right) d\theta \\ &= \|\hat{u}_k\|_{C(\overline{Q_T})}^2 \int_0^1 \left(\int_{Q_T} |\nu'(u_k + \theta \hat{u}_k) - \nu'(u)|^2 dx dt \right) d\theta \\ &\leq \|\hat{u}_k\|_{H^4(Q_T)}^2 C_{18}^2 \int_0^1 \left(\int_{Q_T} |\nu'(u_k + \theta \hat{u}_k) - \nu'(u)|^2 dx dt \right) d\theta. \end{aligned}$$

We can see that

$$|\nu'(u_k + \theta \hat{u}_k) - \nu'(u)|^2 \rightarrow 0$$

almost everywhere on Q_T and that

$$|\nu'(u_k + \theta \hat{u}_k) - \nu'(u)|^2 \leq C_{18}$$

uniformly with respect to $\theta \in [0, 1]$. Using twice Lebesgue theorem, we obtain:

$$\lim_{k \rightarrow \infty} \frac{\|R(u_k, \hat{u}_k)\|_{L_2(Q_T)}}{\|\hat{u}_k\|_{H^4(Q_T)}} \rightarrow 0$$

Next, using the inequality $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$, we have

$$\begin{aligned} & \int_{Q_T} \left| \frac{\partial}{\partial t} [R(u_k, \hat{u}_k)](x, t) \right|^2 dx dt \\ &= \int_{Q_T} \left[\frac{\partial}{\partial t} (\nu(u_k + \hat{u}_k) - \nu(u_k) - \nu'(u) \hat{u}_k) \right]^2 dx dt \\ &= \int_{Q_T} \left(\frac{\partial}{\partial t} \int_0^1 [\nu'(u_k + \theta \hat{u}_k) - \nu'(u)] \hat{u}_k d\theta \right)^2 dx dt \\ &= \int_{Q_T} \left(\int_0^1 \{ [\nu''(u_k + \theta \hat{u}_k)(u_{kt} + \theta \hat{u}_{kt}) - \nu''(u) u_t] \hat{u}_k \right. \\ &\quad \left. + [\nu'(u_k + \theta \hat{u}_k) - \nu'(u)] \hat{u}_{kt} \} d\theta \right)^2 dx dt \\ &= \int_{Q_T} \left(\int_0^1 \{ [\nu''(u_k + \theta \hat{u}_k) - \nu''(u)](u_{kt} + \theta \hat{u}_{kt}) \hat{u}_k \right. \\ &\quad \left. + \nu''(u)(u_{kt} + \theta \hat{u}_{kt} - u_t) \hat{u}_k \right. \\ &\quad \left. + [\nu'(u_k + \theta \hat{u}_k) - \nu'(u)] \hat{u}_{kt} \} d\theta \right)^2 dx dt \\ &\leq \int_{Q_T} \int_0^1 \{ \dots \}^2 d\theta dx dt = \int_0^1 \int_{Q_T} \{ \dots \}^2 dx dt d\theta \\ &\leq 3 \left[\int_0^1 \int_{Q_T} |\nu''(u_k + \theta \hat{u}_k) - \nu''(u)|^2 |u_{kt} + \theta \hat{u}_{kt}|^2 \hat{u}_k^2 dx dt d\theta \right. \\ &\quad \left. + \int_0^1 \int_{Q_T} |\nu''(u)|^2 |(u_{kt} + \theta \hat{u}_{kt} - u_t)|^2 \hat{u}_k^2 dx dt d\theta \right. \\ &\quad \left. + \int_0^1 \int_{Q_T} |\nu'(u_k + \theta \hat{u}_k) - \nu'(u)|^2 \hat{u}_{kt}^2 dx dt d\theta \right] \end{aligned}$$

$$\begin{aligned}
&\leq 3[\|\hat{u}_k\|_{C(Q_T)}^2 \int_0^1 \int_{Q_T} |\nu''(u_k + \theta \hat{u}_k) - \nu''(u)|^2 |u_{kt} + \theta \hat{u}_{kt}|^2 dx dt d\theta \\
&\quad + \|\hat{u}_k\|_{C(Q_T)}^2 \int_0^1 \int_{Q_T} |\nu''(u)|^2 |(u_{kt} + \theta \hat{u}_{kt} - u_t)|^2 dx dt d\theta \\
&\quad + \int_0^1 \int_{Q_T} |\nu'(u_k + \theta \hat{u}_k) - \nu'(u)|^2 \hat{u}_{kt}^2 dx dt d\theta] \\
&\leq 3[\|\hat{u}_k\|_{C(Q_T)}^2 \int_0^1 \left(\int_{Q_T} |\nu''(u_k + \theta \hat{u}_k) - \nu''(u)|^4 dx dt \right)^{\frac{1}{2}} \\
&\quad + \left(\int_{Q_T} |u_{kt} + \theta \hat{u}_{kt}|^4 dx dt \right)^{\frac{1}{2}} d\theta + \|\hat{u}_k\|_{C(Q_T)}^2 \\
&\quad + \int_0^1 \left(\int_{Q_T} |\nu''(u)|^4 dx dt \right)^{\frac{1}{2}} \left(\int_{Q_T} |(u_{kt} + \theta \hat{u}_{kt} - u_t)|^4 dx dt \right)^{\frac{1}{2}} d\theta \\
&\quad + \int_0^1 \left(\int_{Q_T} |\nu'(u_k + \theta \hat{u}_k) - \nu'(u)|^4 dx dt \right)^{\frac{1}{2}} \left(\int_{Q_T} \hat{u}_{kt}^4 dx dt \right)^{\frac{1}{2}} d\theta] \\
&\leq C_{19} \left[\int_0^1 \left(\int_{Q_T} |\nu''(u_k + \theta \hat{u}_k) - \nu''(u)|^4 dx dt \right)^{\frac{1}{2}} d\theta \|\hat{u}_k\|_{C(Q_T)}^2 \right. \\
&\quad + \|\hat{u}_k\|_{C(Q_T)}^2 \int_0^1 \left(\int_{Q_T} |(u_{kt} + \theta \hat{u}_{kt} - u_t)|^4 dx dt \right)^{\frac{1}{2}} d\theta \\
&\quad + \int_0^1 \left(\int_{Q_T} |\nu'(u_k + \theta \hat{u}_k) - \nu'(u)|^4 dx dt \right)^{\frac{1}{2}} \left(\int_{Q_T} \hat{u}_{kt}^4 dx dt \right)^{\frac{1}{2}} d\theta \Big] \\
&= C_{19} \left[\int_0^1 \left(\int_{Q_T} |\nu''(u_k + \theta \hat{u}_k) - \nu''(u)|^4 dx dt \right)^{\frac{1}{2}} d\theta \|\hat{u}_k\|_{C(Q_T)}^2 \right. \\
&\quad + \|\hat{u}_k\|_{C(Q_T)}^2 \int_0^1 \left(\int_{Q_T} |(u_{kt} + \theta \hat{u}_{kt} - u_t)|^4 dx dt \right)^{\frac{1}{2}} d\theta \\
&\quad + \int_0^1 \left(\int_{Q_T} |\nu'(u_k + \theta \hat{u}_k) - \nu'(u)|^4 dx dt \right)^{\frac{1}{2}} d\theta \|\hat{u}_{kt}\|_{L^4(Q_T)}^2 \Big]
\end{aligned}$$

Let us note that, in view of (11) and (12) in all of three obtained expressions, the last term can be estimated by $\|\hat{u}_k\|_{H^4(Q_T)}^2$. In view of Lebesgue theorem, the other terms are integrals converging to zero when $k \rightarrow \infty$.

Indeed, $|\nu'''(u_k + \theta \hat{u}_k) - \nu'''(u)|^4 \rightarrow 0$ almost everywhere with respect to $(x, t) \in Q_T$ and

$$|\nu''(u_k + \theta \hat{u}_k) - \nu''(u)|^4 \leq C_{20}$$

uniformly with respect to $\theta \in [0, 1]$. In view of (11) and (15),

$$|(u_{kt} + \theta \hat{u}_{kt})^2 - u_t^2|^4 \rightarrow 0$$

a.e. with respect to $(x, t) \in Q_T$ and

$$|(u_{kt} + \theta \hat{u}_{kt})^2 - u_t^2|^4 \in L_1(Q_T)$$

uniformly with respect to $\theta \in [0, 1]$. In view of (11) and (15),

$$|\nu''(u_k + \theta \hat{u}_k) - \nu''(u)|^4 \rightarrow 0$$

a.e. with respect to $(x, t) \in Q_T$ and

$$|\nu''(u_k + \theta \hat{u}_k) - \nu''(u)|^4 \leq C_{21}$$

uniformly with respect to $\theta \in [0, 1]$.

Similarly $\int_{Q_T} \left| \frac{\partial}{\partial x} [R(u_k, \hat{u}_k)](x, t) \right|^2 dx dt$ can be estimated.

We next have:

$$\begin{aligned} & \int_{Q_T} \left| \frac{\partial^2}{\partial t^2} [R(u_k, \hat{u}_k)](x, t) \right|^2 dx dt \\ &= \int_{Q_T} \left\{ \frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} (R(u_k, \hat{u}_k)) \right) (x, t) \right\}^2 dx dt \\ &= \int_{Q_T} \left(\frac{\partial}{\partial t} \int_0^1 \{ [\nu''(u_k + \theta \hat{u}_k)(u_{kt} + \theta \hat{u}_{kt}) - \nu''(u)u_t] \hat{u}_k \right. \\ & \quad \left. + [\nu'(u_k + \theta \hat{u}_k) - \nu'(u)] \hat{u}_{kt} \} d\theta \right)^2 dx dt \\ &= \int_{Q_T} \left(\int_0^1 \{ [\nu'''(u_k + \theta \hat{u}_k)(u_{kt} + \theta \hat{u}_{kt})^2 \right. \\ & \quad + (\nu''(u_k + \theta \hat{u}_k)(u_{ktt} + \theta \hat{u}_{ktt}) - \nu'''(u)u_t^2 - \nu''(u)u_{tt}) \hat{u}_k \\ & \quad + [\nu''(u_k + \theta \hat{u}_k)(u_{kt} + \theta \hat{u}_{kt}) - \nu''(u)u_t] \hat{u}_{kt} \\ & \quad + [\nu''(u_k + \theta \hat{u}_k)(u_{kt} + \theta \hat{u}_{kt}) - \nu''(u)u_t] \hat{u}_{kt} \\ & \quad \left. + [\nu'(u_k + \theta \hat{u}_k) - \nu'(u)] \hat{u}_{ktt} \} d\theta \right)^2 dx dt \\ &= \int_{Q_T} \left(\int_0^1 \{ [\nu'''(u_k + \theta \hat{u}_k)(u_{kt} + \theta \hat{u}_{kt})^2 \right. \\ & \quad + (\nu''(u_k + \theta \hat{u}_k)(u_{ktt} + \theta \hat{u}_{ktt}) - \nu'''(u)u_t^2 - \nu''(u)u_{tt}) \hat{u}_k \\ & \quad + 2[(\nu''(u_k + \theta \hat{u}_k)(u_{kt} + \theta \hat{u}_{kt}) - \nu''(u)u_t) \hat{u}_{kt} \\ & \quad \left. + [\nu'(u_k + \theta \hat{u}_k) - \nu'(u)] \hat{u}_{ktt} \} d\theta \right)^2 dx dt \end{aligned}$$

We get

$$\begin{aligned}
& \int_{Q_T} \left| \frac{\partial^2}{\partial t^2} [R(u_k, \hat{u}_k)](x, t) \right|^2 dx dt \\
= & \int_{Q_T} \left(\int_0^1 \{ [\nu'''(u_k + \theta \hat{u}_k) - \nu'''(u)](u_{kt} + \theta \hat{u}_{kt})^2 \hat{u}_k \right. \\
& + \nu'''(u) [(u_{kt} + \theta \hat{u}_{kt})^2 - u_t^2] \hat{u}_k + 2[\nu''(u_k + \theta \hat{u}_k) - \nu''(u)](u_{kt} \\
& + \theta \hat{u}_{kt}) \hat{u}_{kt} + 2\nu''(u)(u_{kt} + \theta \hat{u}_{kt} - u_t) \hat{u}_{kt} \\
& + [\nu''(u_k + \theta \hat{u}_k) - \nu''(u)](u_{ktt} + \theta \hat{u}_{ktt}) \hat{u}_k \\
& \left. + \nu''(u)(u_{ktt} + \theta \hat{u}_{ktt} - u_{tt}) \hat{u}_k + [\nu'(u_k + \theta \hat{u}_k) - \nu'(u)] \hat{u}_{ktt} \} d\theta \right)^2 dx dt \\
\leq & \int_{Q_T} \int_0^1 \{ \dots \}^2 d\theta dx dt = \int_0^1 \int_{Q_T} \{ \dots \}^2 dx dt d\theta \\
\leq & 7 \left[\int_0^1 \int_{Q_T} |\nu'''(u_k + \theta \hat{u}_k) - \nu'''(u)|^2 |(u_{kt} + \theta \hat{u}_{kt})|^2 \hat{u}_k^2 dx dt d\theta \right. \\
& + 4 \int_0^1 \int_{Q_T} |\nu''(u_k + \theta \hat{u}_k) - \nu''(u)|^2 |u_{kt} + \theta \hat{u}_{kt}|^2 \hat{u}_{kt}^2 dx dt d\theta \\
& + 4 \int_0^1 \int_{Q_T} |\nu''(u)|^2 |u_{kt} + \theta \hat{u}_{kt} - u_t|^2 \hat{u}_{kt}^2 dx dt d\theta \\
& + \int_0^1 \int_{Q_T} |\nu''(u_k + \theta \hat{u}_k) - \nu''(u)|^2 |u_{ktt} + \theta \hat{u}_{ktt}|^2 \hat{u}_k^2 dx dt d\theta \\
& + \int_0^1 \int_{Q_T} |\nu''(u)|^2 |u_{ktt} + \theta \hat{u}_{ktt} - u_{tt}|^2 \hat{u}_k^2 dx dt d\theta \\
& \left. + \int_0^1 \int_{Q_T} |\nu'(u_k + \theta \hat{u}_k) - \nu'(u)|^2 \hat{u}_{ktt}^2 dx dt d\theta \right]
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
& \int_{Q_T} \left| \frac{\partial^2}{\partial t^2} [R(u_k, \hat{u}_k)](x, t) \right|^2 dx dt \\
\leq & 7 [\|\hat{u}_k\|_{C(Q_T)}^2 \int_0^1 \left(\int_{Q_T} |\nu'''(u_k + \theta \hat{u}_k) - \nu'''(u)|^4 dx dt \right)^{\frac{1}{2}} \times \left(\int_{Q_T} |u_{kt} + \theta \hat{u}_{kt}|^8 dx dt \right)^{\frac{1}{2}} d\theta \\
& + \|\hat{u}_k\|_{C(Q_T)}^2 \int_0^1 \left(\int_{Q_T} |\nu'''(u)|^4 dx dt \right)^{\frac{1}{2}} \times \left(\int_{Q_T} |(u_{kt} + \theta \hat{u}_{kt})^2 - u_t^2|^4 dx dt \right)^{\frac{1}{2}} d\theta \\
& + 4 \|\hat{u}_{kt}\|_{L_4(Q_T)}^2 \sup_{(x,t) \in Q_T} |u_{kt} + \theta \hat{u}_{kt}|^2 \times \int_0^1 \left(\int_{Q_T} |\nu''(u_k + \theta \hat{u}_k) - \nu''(u)|^4 dx dt \right)^{\frac{1}{2}} d\theta \\
& + 4 \|\hat{u}_{kt}\|_{L_4(Q_T)}^2 \sup_{Q_T} |\nu''(u)|^2 \int_0^1 \left(\int_{Q_T} |u_{kt} + \theta \hat{u}_{kt} - u_t|^4 dx dt \right)^{\frac{1}{2}} d\theta \\
& + \|\hat{u}_{kt}\|_{C(Q_T)}^2 \int_0^1 \left(\int_{Q_T} |\nu''(u_k + \theta \hat{u}_k) - \nu''(u)|^4 dx dt \right)^{\frac{1}{2}} \cdot \left(\int_{Q_T} |u_{kt} + \theta \hat{u}_{kt}|^4 dx dt \right)^{\frac{1}{2}} d\theta \\
& + \|\hat{u}_k\|_{C(Q_T)}^2 \int_0^1 \left(\int_{Q_T} |\nu''(u)|^4 dx dt \right)^{\frac{1}{2}} \left(\int_{Q_T} |u_{kt} + \theta \hat{u}_{kt} - u_t|^4 dx dt \right)^{\frac{1}{2}} d\theta \\
& + \int_0^1 \left(\int_{Q_T} |\nu'(u_k + \theta \hat{u}_k) - \nu'(u)|^4 dx dt \right)^{\frac{1}{2}} \left(\int_{Q_T} |\hat{u}_{kt}|^4 dx dt \right)^{\frac{1}{2}} d\theta \\
\leq & c_{22} \left[\int_0^1 \left(\int_{Q_T} |\nu'''(u_k + \theta \hat{u}_k) - \nu'''(u)|^4 dx dt \right)^{\frac{1}{2}} \cdot \|\hat{u}_k\|_{C(Q_T)}^2 \right. \\
& + \int_0^1 \left(\int_{Q_T} |(u_{kt} + \theta \hat{u}_{kt})^2 - u_t^2|^4 dx dt \right)^{\frac{1}{2}} d\theta \cdot \|\hat{u}_k\|_{C(Q_T)}^2 \\
& + \int_0^1 \left(\int_{Q_T} |\nu''(u_k + \theta \hat{u}_k) - \nu''(u)|^4 dx dt \right)^{\frac{1}{2}} d\theta \cdot \|\hat{u}_k\|_{L_4(Q_T)}^2 \\
& + \int_0^1 \left(\int_{Q_T} |u_{kt} + \theta \hat{u}_{kt} - u_t|^4 dx dt \right)^{\frac{1}{2}} d\theta \cdot \|\hat{u}_k\|_{C(Q_T)}^2 \\
& + \int_0^1 \left(\int_{Q_T} |\nu''(u_k + \theta \hat{u}_k) - \nu''(u)|^4 dx dt \right)^{\frac{1}{2}} d\theta \cdot \|\hat{u}_k\|_{C(Q_T)}^2 \\
& + \int_0^1 \left(\int_{Q_T} |u_{kt} + \theta \hat{u}_{kt} - u_t|^4 dx dt \right)^{\frac{1}{2}} d\theta \cdot \|\hat{u}_k\|_{C(Q_T)}^2 \\
& \left. + \int_0^1 \left(\int_{Q_T} |\nu'(u_k + \theta \hat{u}_k) - \nu'(u)|^4 dx dt \right)^{\frac{1}{2}} d\theta \cdot \|\hat{u}_{kt}\|_{L_4(Q_T)}^2 \right].
\end{aligned}$$

Let us note that, in view of (11) and (12), the last term can be estimated by $\|\hat{u}_k\|_{H^4(Q_T)}^2$. In view of Lebesgue theorem, the other terms are integrals converging to zero.

Indeed, $|\nu'''(u_k + \theta \hat{u}_k) - \nu'''(u)|^4 \rightarrow 0$ almost everywhere with respect to $(x, t) \in Q_T$ and $|\nu'''(u_k + \theta \hat{u}_k) - \nu'''(u)|^4 \leq C_{23}$

for all $\theta \in [0, 1]$. In view of (11) and (15), $|(u_{kt} + \theta \hat{u}_{kt})^2 - u_t^2|^4 \rightarrow 0$ a.e. with respect to $(x, t) \in Q_T$ and $|(u_{kt} + \theta \hat{u}_{kt})^2 - u_t^2|^4 \in L_1(Q_T)$ for all $\theta \in [0, 1]$. In view of (11) and (15), $|u_{kt} + \theta \hat{u}_{kt} - u_t|^4 \rightarrow 0$ a.e. with respect to $(x, t) \in Q_T$ and $|u_{kt} + \theta \hat{u}_{kt} - u_t|^4 \in L_1(Q_T)$ for all $\theta \in [0, 1]$. In view of (11) and (15), $|u_{ktt} + \theta \hat{u}_{ktt} - u_{tt}|^4 \rightarrow 0$ a.e. with respect to $(x, t) \in Q_T$ and $|u_{ktt} + \theta \hat{u}_{ktt} - u_{tt}|^4 \in L_1(Q_T)$ for all $\theta \in [0, 1]$. In view of (11) and (15) $|\nu'(u_k + \theta \hat{u}_k) - \nu'(u)|^4 \rightarrow 0$ a.e. with respect to $(x, t) \in Q_T$

and $|\nu'(u_k + \theta \hat{u}_k) - \nu'(u)|^4 \in L_1(Q_T)$ for all $\theta \in [0, 1]$. Similarly $\int_{Q_T} \left| \frac{\partial^2}{\partial x^2} [R(u_k, \hat{u}_k)](x, t) \right|^2 dx dt$ can be estimated. In the same way, we are able to estimate the following terms:

$$\int_{Q_T} \left| \frac{\partial^2}{\partial x \partial t} [R(u_k, \hat{u}_k)](x, t) \right|^2 dx dt, \int_{Q_T} \left| \frac{\partial^2}{\partial t \partial x} [R(u_k, \hat{u}_k)](x, t) \right|^2 dx dt, \int_{Q_T} \left| \frac{\partial^3}{\partial t^3} [R(u_k, \hat{u}_k)](x, t) \right|^2 dx dt$$

$$\int_{Q_T} \left| \frac{\partial^3}{\partial x^3} [R(u_k, \hat{u}_k)](x, t) \right|^2 dx dt, \int_{Q_T} \left| \frac{\partial^3}{\partial t^2 \partial x} [R(u_k, \hat{u}_k)](x, t) \right|^2 dx dt, \int_{Q_T} \left| \frac{\partial^3}{\partial t \partial x^2} [R(u_k, \hat{u}_k)](x, t) \right|^2 dx dt,$$

$$\int_{Q_T} \left| \frac{\partial^3}{\partial t \partial x \partial t} [R(u_k, \hat{u}_k)](x, t) \right|^2 dx dt, \int_{Q_T} \left| \frac{\partial^3}{\partial x \partial t \partial x} [R(u_k, \hat{u}_k)](x, t) \right|^2 dx dt, \int_{Q_T} \left| \frac{\partial^3}{\partial x \partial t^2} [R(u_k, \hat{u}_k)](x, t) \right|^2 dx dt$$

and

$$\int_{Q_T} \left| \frac{\partial^3}{\partial x^2 \partial t} [R(u_k, \hat{u}_k)](x, t) \right|^2 dx dt,$$

which all contain terms $\nabla'''(u)$ and $\nu^{(4)}(u)$. We then deduce (14). Hence, the operator N is differentiable, and the proof of proposition is complete. ■

5 Proof of the main result

Consider the operator $\xi : H \rightarrow H^3(Q_T)$ defined by $\xi(u) = LU + NU$.

We know that ξ is differentiable. Moreover, $\xi(0) = 0$, and, since $\nu'(0) = 0$, $\xi'(0) = L \neq 0$.

Hence according to the theorem 1 and, by the inversion theorem, there exist open neighborhoods of zeros $U_1 \subset H$ and $V \subset H^3(Q_T)$ such that the operator ξ is C^1 -diffeomorphism from U_1 to V_1 ; and the operator $\eta = \xi^{-1} : V_1 \longrightarrow U_1$ is also a C^1 -diffeomorphism. The operator $P : H^3(Q_T) \times H^3(Q_T) \longrightarrow H$ defined by

$$P[(f, g)](x, t) = \eta\left[\left(\frac{x}{l} - 1\right)f(0) - \frac{x}{l}g(0)\right]$$

is obviously differentiable on some neighborhood of zero in $H^3(Q_T) \times H^3(Q_T)$. Let V_2 be this neighborhood $V_2 \subset V_1$.

We wish to show the existence of $(f, g) \in V_2$ such that

$u(x, t) = \eta\left[\left(\frac{x}{l} - 1\right)f(t) - \frac{x}{l}g(t)\right] = P(f, g)(x, t)$, which is a solution of (6) – (8), also satisfies condition (9) of problem (6) – (9)

$$u(x, T) = \chi_1(x) \implies \frac{\partial u}{\partial t}(x, T) = \chi_2(T)$$

and

$$\frac{\partial u}{\partial t}(x, T) = \chi_2(T) \implies \frac{\partial^2 u}{\partial x^2}(x, T) = \frac{\partial^2}{\partial x^2}\chi_2(x)$$

we have, using (2):

$$F(x, T) - \frac{\partial^2}{\partial t^2}[P(f, g)](x, T) - N[P(f, g)](x, T) = -\frac{\partial^2}{\partial x^2}\chi_1(x)$$

and

$$\frac{\partial}{\partial t}F(x, T) - \frac{\partial^3}{\partial t^3}[P(f, g)](x, T) - \frac{\partial}{\partial t}N[P(f, g)](x, T) = -\frac{\partial^2}{\partial x^2}\chi_2(x).$$

Consider the operators

$$M_1 : H^3(I) \times H^3(I) \longrightarrow H^3(\Gamma),$$

$$M_2 : H^3(I) \times H^3(I) \longrightarrow H^3(\Gamma)$$

$$M_1(f, g)(x, T) = F(x, T) - \frac{\partial^2}{\partial t^2}[P(f, g)](x, T) - N[P(f, g)](x, T)$$

$$M_2(f, g)(x, T) = \frac{\partial}{\partial t}F(x, T) - \frac{\partial^3}{\partial t^3}[P(f, g)](x, T) - \frac{\partial}{\partial t}N[P(f, g)](x, T).$$

It follows from foregoing results that M_1 and M_2 are differentiable; and straightforward calculation shows that

$$M'_k(0, 0) \neq 0 \quad \forall k.$$

Hence, by the inversion theorem, open neighborhoods of zeros $U \subset U_1 \subset H$ and $V \subset V_2 \subset H^3(I) \times H^3(I)$ such that the operator

$M : H^3(I) \times H^3(I) \longrightarrow H^3(\Gamma) \times H^3(\Gamma)$ defined by

$M(f, g) = (M_1(f, g), M_2(f, g))$ is also C^1 -diffeomorphism from V to U .

$(\chi_1, \chi_2) \in H^5(\Gamma) \times H^4(\Gamma) \implies (\chi_1'', \chi_2'') \in H^3(\Gamma) \times H^2(\Gamma)$ And one can show quite straightforwadly, by using Poicaré inequality, that, if $(\chi_1'', \chi_2'') \in H^3(\Gamma) \times H^2(\Gamma)$, then $\exists$ a constant $\gamma > 0$ such that

$$\|(\chi_1'', \chi_2'')\|_{H^3 \times H^2} < \gamma \implies \begin{cases} (\chi_1'', \chi_2'') \in V \subset V_2 \\ (\chi_1, \chi_2) \in V \subset V_2 \end{cases}.$$

Hence, $\forall (\chi_1, \chi_2) \in H^5(\Gamma) \cap H_0^1(\Gamma) \times H^4(\Gamma) \cap H_0^1(\Gamma)$, choose that constant γ such that $\|(\chi_1'', \chi_2'')\|_{H^3 \times H^2} < \gamma$, so that $\exists$ a unique $(f, g) \in U \subset U_1 \subset H$ such that

$$M(f, g) = (M_1(f, g), M_2(f, g)) = \left(-\frac{d^2\chi_1}{dx^2}, -\frac{d^2\chi_2}{dx^2}\right).$$

Let

$$u(x, t) = P(f, g)(x, t) = \eta\left[\left(\frac{x}{l} - 1\right)f(t) - \frac{x}{l}g(t)\right]$$

It's then enough to show that u solves the problem (6) – (9). This is quite straightforward, and the proof is complete.

It's obvious that u satisfies (6) – (8), by definition of η . Let's show (9). We have:
 $M_1(f, g)(x, T) = -\frac{d^2\chi_1}{dx^2} \implies$

$$F(x, T) - \frac{\partial^2}{\partial t^2}[P(f, g)](x, T) + N[P(f, g)](x, T) = -\frac{d^2\chi_1}{dx^2}$$

$P(f, g)$ is a solution (6) – (8), we have

$$\frac{\partial^2}{\partial t^2}[P(f, g)](x, t) - \frac{\partial^2}{\partial x^2}[P(f, g)](x, t) + N[P(f, g)](x, t) = F(x, t)$$

and therefore, as $u(x, t) = P(f, g)(x, t)$,

$$-\frac{\partial^2 u}{\partial x^2}(x, T) = -\frac{d^2\chi_1}{dx^2}(x) \implies u(x, T) = \chi_1(x)$$

by (8) and since $\chi_1 \in H_0^1(\Gamma)$. The other condition (9) is established in the same way. The proof is thus complete.

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