On Quasi - f -Power Increasing Sequences

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Abstract

A result concerning absolute indexed Summability factor of an infinite series using Quasi - f - power increasing sequences has been established.

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1. Introduction

A positive sequence (a_n) is said to be almost increasing if there exists a positive sequence (b_n) and two positive constants A and B such that

$$(1.1) \quad Ab_n \leq a_n \leq Bb_n, \text{ for all } n.$$

The sequence (a_n) is said to be quasi- β -power increasing, if there exists a constant K depending upon β with $K \geq 1$ such that

$$(1.2) \quad Kn^\beta a_n \geq m^\beta a_m,$$

for all $n \geq m$. In particular, if $\beta = 0$, then (a_n) is said to be quasi-increasing sequence. It is clear that every almost increasing sequence is a quasi- β -power increasing sequence for any non-negative β . But the converse is not true as $(n^{-\beta})$ is quasi- β -power increasing but not almost increasing.

Let $f = (f_n)$ be a positive sequence of numbers. Then the positive sequence (a_n) is said to be quasi- f -power increasing, if there exists a constant K depending upon f with $K \geq 1$ such that

$$(1.3) \quad Kf_n a_n \geq f_m a_m,$$

for $n \geq m \geq 1$. Clearly, if (α_n) is a quasi- f -power increasing sequence, then the $(\alpha_n f_n)$ is a quasi-increasing sequence.

Let $\sum a_n$ be an infinite series with sequence of partial sums $\{s_n\}$. Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^n p_v \rightarrow \infty, \text{ as } n \rightarrow \infty.$$

Then the sequence-to-sequence transformation

$$(1.4) \quad T_n = \frac{1}{P_n} \sum_{v=0}^n p_v s_v, \quad P_n \neq 0,$$

defines the $(\overline{N}, p_n)$ - mean of the sequence (s_n) generated by the sequence of coefficients $\{p_n\}$.
The series $\sum a_n$ is said to be summable $\left| \overline{N}, p_n \right|_k, k \geq 1([1])$, if

$$(1.5) \quad \sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |T_n - T_{n-1}|^k < \infty .$$

The series $\sum a_n$ is said to be summable $\left| \overline{N}, p_n; \delta \right|_k, k \geq 1, \delta \geq 0$, if

$$(1.6) \quad \sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} |T_n - T_{n-1}|^k < \infty .$$

2. Known Theorems

Dealing with quasi- β -power increasing sequence Bor and Debnath [2] have established the following theorem:

2.1. Theorem:

Let (X_n) be a quasi- β -power increasing sequence for $0 < \beta < 1$ and (λ_n) be a real sequence.

If the conditions

$$(2.1.1) \quad \sum_{n=1}^m \frac{P_n}{n} = O(P_m),$$

$$(2.1.2) \quad \lambda_n X_n = O(1),$$

$$(2.1.3) \quad \sum_{n=1}^m \frac{|t_n|^k}{n} = O(X_m),$$

$$(2.1.4) \quad \sum_{n=1}^m \frac{P_n |t_n|^k}{P_n} = O(X_m)$$

and

$$(2.1.5) \quad \sum_{n=1}^m n X_n |\Delta^2 \lambda_n| < \infty$$

are satisfied, where t_n is the $(C,1)$ mean of the sequence (na_n) . Then the series $\sum a_n \lambda_n$ is summable $\left| \overline{N}, p_n \right|_k, k \geq 1$.

Subsequently Leindler [3] established a similar result reducing certain condition of Bor. He established:

2.2. Theorem:

Let the sequence (X_n) be a quasi- β -power increasing sequence for $0 < \beta < 1$, and the real sequence (λ_n) satisfies the conditions

$$(2.2.1) \quad \sum_{n=1}^m \lambda_n = O(m)$$

and

$$(2.2.2) \quad \sum_{n=1}^m |\Delta \lambda_n| = O(m).$$

Further, suppose the conditions (2.1.3), (2.1.4) and

$$(2.2.3) \quad \sum_{n=1}^m n X_n(\beta) |\Delta \lambda_n| < \infty,$$

hold, where $X_n(\beta) = \max(n^\beta X_n, \log n)$. Then the series $\sum a_n \lambda_n$ is summable $\left| \overline{N}, p_n \right|_k, k \geq 1$.

Recently, extending the above results to quasi- f -power increasing sequence, Sulaiman [5] have established the following theorem:

2.3. Theorem:

Let $f = (f_n) = (n^\beta \log^\gamma n)$, $0 \leq \beta < 1, \gamma \geq 0$ be a sequence. Let (X_n) be a quasi- f -power sequence and (λ_n) a sequence of constants satisfying the conditions

$$(2.3.1) \quad \lambda_n \rightarrow 0 \text{ as } n \rightarrow \infty,$$

$$(2.3.2) \quad \sum_{n=1}^{\infty} nX_n \|\Delta \lambda_n\| < \infty,$$

$$(2.3.3) \quad |\lambda_n|X_n = O(1),$$

$$(2.3.4) \quad \sum_{n=1}^{\infty} \frac{1}{nX_n^{k-1}} |t_n|^k = O(X_m)$$

and

$$(2.3.5) \quad \sum_{n=1}^{\infty} \frac{p_n}{P_n} \frac{1}{X_n^{k-1}} |t_n|^k = O(X_m),$$

where t_n is the $(C,1)$ mean of the sequence (na_n) . Then the series $\sum a_n \lambda_n$ is summable $|\overline{N}, p_n|_k, k \geq 1$.

In what follows in this paper we prove the following theorem.

3. Main Theorem

Let $f = (f_n) = (n^\beta \log^\gamma n)$ be a sequence and (X_n) be a quasi- f -power sequence. Let (λ_n) a sequence of constants such that

$$(3.1) \quad \lambda_n \rightarrow 0, \text{ as } n \rightarrow \infty,$$

$$(3.2) \quad \sum_{n=1}^{\infty} nX_n \|\Delta \lambda_n\| < \infty,$$

$$(3.3) \quad |\lambda_n|X_n = O(1),$$

$$(3.4) \quad \sum_{n=v+1}^m \left(\frac{P_n}{p_n} \right)^{\delta k-1} \frac{1}{P_{n-1}} = O \left(\frac{P_m}{p_m} \right)^{\delta k-1},$$

$$(3.5) \quad \sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k-1} \frac{|t_n|^k}{X_v^{k-1}} = O(X_m),$$

$$(3.6) \quad \sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k} \frac{|t_n|^k}{nX_n^{k-1}} = O(X_m).$$

Then the series $\sum a_n \lambda_n$ is summable $\left| \overline{N}, p_n; \delta \right|_k, k \geq 1, \delta \geq 0$.

In order to prove the theorem we require the following lemma.

4. Lemma:

Let $f = (f_n) = (n^\beta \log^\gamma n), 0 \leq \beta < 1, \gamma \geq 0$ be a sequence and (X_n) be a quasi - f - power increasing sequence. Let (λ_n) be a sequence of constants satisfying (3.1) and (3.2). then

$$(4.1) \quad n X_n |\Delta \lambda_n| = O(1)$$

and

$$(4.2) \quad \sum_{n=1}^m X_n |\Delta \lambda_n| < \infty.$$

4.1. Proof of Lemma:

As $\Delta \lambda_n \rightarrow 0$ and $n^\beta \log^\gamma n X_n$ is non-decreasing, we have

$$\begin{aligned} n X_n |\Delta \lambda_n| &= n^{1-\beta} \log^{-\gamma} n (n^\beta \log^\gamma n X_n) \sum_{v=n}^{\infty} \Delta |\Delta \lambda_v| \\ &= O(1) n^{1-\beta} \log^{-\gamma} n \sum_{v=n}^{\infty} v^\beta \log^\gamma v X_v |\Delta \lambda_v| \\ &= O(1) \sum_{v=n}^{\infty} v^{1-\beta} \log^{-\gamma} v v^\beta \log^\gamma v X_v |\Delta \lambda_v| \\ &= O(1) \sum_{v=n}^{\infty} v X_v |\Delta \lambda_v| = O(1). \end{aligned}$$

This establishes (4.1).

Next

$$\begin{aligned} \sum_{n=1}^m X_n |\Delta \lambda_n| &= \sum_{n=1}^{m-1} \left(\sum_{r=1}^n X_r \right) \Delta |\Delta \lambda_r| + \left(\sum_{r=1}^n X_r \right) |\Delta \lambda_r| \\ &= O(1) \sum_{n=1}^{m-1} \left(\sum_{r=1}^n r^{-\beta} \log^{-\gamma} r r^\beta \log^\gamma r X_r \right) |\Delta \lambda_r| \\ &\quad + O(1) \left(\sum_{r=1}^m r^{-\beta} \log^{-\gamma} r r^\beta \log^\gamma r X_r \right) |\Delta \lambda_m| \end{aligned}$$

$$\begin{aligned}
&= O(1) \sum_{n=1}^{m-1} \left(n^\beta \log^{-\gamma} n X_n \right) \left| \Delta \right| \left| \Delta \lambda_v \right| \left| \sum_{r=1}^n r^{-\beta-\epsilon} \log^{-\gamma} r r^\epsilon \right| \\
&\quad + O(1) m^\beta X_m \left| \Delta \lambda_m \right| \log^{-\gamma} m \sum_{r=1}^m r^{-\beta-\epsilon} \log^{-\gamma} r r^\epsilon, \epsilon < 1 - \beta. \\
&= O(1) \sum_{n=1}^{m-1} \left(n^\beta \log^{-\gamma} n X_n \right) \left| \Delta \right| \left| \Delta \lambda_v \right| n^\epsilon \log^{-\gamma} n \sum_{r=1}^n r^{-\beta-\epsilon} \\
&\quad + O(1) m^\beta X_m \left| \Delta \lambda_m \right| \log^{-\gamma} m m^\epsilon \sum_{r=1}^m r^{-\beta-\epsilon} \\
&= O(1) \sum_{n=1}^m n^{\beta+\epsilon} X_n \left| \Delta \right| \left| \Delta \lambda_v \right| \left\| \int_1^n u^{-\beta-\epsilon} du \right\| + O(1) m^{\beta+\epsilon} X_m \left| \Delta \lambda_m \right| \left\| \int_1^m u^{-\beta-\epsilon} du \right\| \\
&= O(1) \sum_{n=1}^m n X_n \left| \Delta \right| \left| \Delta \lambda_v \right| + O(1) m X_m \left| \Delta \lambda_m \right| \\
&= O(1).
\end{aligned}$$

This establishes (4.2).

5. Proof of Theorem:

Let (T_n) be the sequence of $(\overline{N}, p_n)$ mean of the series $\sum_{n=1}^{\infty} a_n \lambda_n$, then

$$\begin{aligned}
T_n &= \frac{1}{P_n} \sum_{v=0}^n p_v \sum_{r=0}^v a_r \lambda_r \\
&= \frac{1}{P_n} \sum_{v=0}^n (p_n - p_{v-1}) a_v \lambda_v
\end{aligned}$$

Hence for $n \geq 1$

$$\begin{aligned}
T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^n p_{v-1} a_v \lambda_v \\
&= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^n v a_v \left(\frac{1}{v} p_{v-1} \lambda_v \right) \\
&= \frac{(n+1)}{n} \frac{p_n}{P_n} t_n \lambda_n + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_{v-1} t_v \lambda_v \frac{v+1}{v} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v t_v \frac{v+1}{v} \Delta \lambda_v \\
&= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v t_v \frac{\lambda_{v+1}}{v}
\end{aligned}$$

$$= T_{n1} + T_{n2} + T_{n3} + T_{n4} \text{ (say).}$$

In order to prove the theorem, using Minkowski's inequality it is enough to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} |T_{nj}| < \infty, \quad j = 1, 2, 3, 4.$$

Applying Hölder's inequality, we have

$$\begin{aligned} \sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} |T_{n1}|^k &= \sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} \left| \frac{n+1}{n} \frac{P_n}{p_n} t_n \lambda_n \right|^k \\ &= O(1) \sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{\delta k - 1} \frac{|t_n|^k}{X_n^{k-1}} (X_n |\lambda_n|)^{k-1} |\lambda_n| \\ &= O(1) \sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{\delta k - 1} \frac{|t_n|^k}{X_n^{k-1}} |\lambda_n| \\ &= O(1) \sum_{n=1}^{m-1} \left(\sum_{v=1}^n \left(\frac{P_v}{p_v} \right)^{\delta k - 1} \frac{|t_v|^k}{X_v^{k-1}} \right) \Delta |\lambda_v| + O(1) \sum_{v=1}^m \left(\frac{P_v}{p_v} \right)^{\delta k - 1} \frac{|t_v|^k}{X_v^{k-1}} |\lambda_m| \\ &= O(1) \sum_{n=1}^{m-1} X_n \Delta |\lambda_n| + O(1) X_m |\lambda_m| \\ &= O(1). \end{aligned}$$

Next

$$\begin{aligned} \sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} |T_{n2}|^k &= \sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} \left| \frac{P_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_{v-1} t_v \lambda_v \frac{v+1}{v} \right|^k \\ &= O(1) \sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_{v-1} |t_v|^k |\lambda_v|^k \left(\sum_{v=1}^{n-1} \frac{P_v}{P_{n-1}} \right)^{k-1} \\ &= O(1) \sum_{v=1}^m P_{v-1} |t_v|^k |\lambda_v|^k \sum_{n=v+1}^m \left(\frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{P_{n-1}} \\ &= O(1) \sum_{v=1}^n \left(\frac{P_v}{p_v} \right)^{\delta k - 1} |t_v|^k |\lambda_v|^k \\ &= O(1), \text{ as in the case of } T_{n1}. \end{aligned}$$

Next

$$\sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} |T_{n3}|^k = \sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} \left| \frac{P_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v t_v \frac{v+1}{v} \Delta \lambda_v \right|^k$$

$$\begin{aligned}
&= O(1) \sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{\delta k-1} \frac{1}{P_{n-1}^k} \sum_{v=1}^{n-1} P_v^k \frac{|t_v|^k}{X_v^{k-1}} |\Delta \lambda_v| \left(\sum_{v=1}^{n-1} X_v |\Delta \lambda_v| \right)^{k-1} \\
&= O(1) \sum_{v=1}^m P_v^k \frac{|t_v|^k}{X_v^{k-1}} |\Delta \lambda_v| \sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n} \right)^{\delta k-1} \frac{1}{P_{n-1}^k} . \\
&= O(1) \sum_{v=1}^n \left(\frac{P_n}{p_n} \right)^{\delta k} \frac{1}{v} \frac{|t_v|^k}{X_v^{k-1}} (v |\Delta \lambda_v|) \\
&= O(1) \sum_{v=1}^{m-1} \left(\sum_{r=1}^v \frac{P_r}{p_r} \right)^{\delta k} \frac{1}{r} \frac{|t_r|^k}{X_r^{k-1}} \Delta(v |\Delta \lambda_v|) + O(1) \sum_{r=1}^m \left(\frac{P_r}{p_r} \right)^{\delta k} \frac{1}{r} \frac{|t_r|^k}{X_r^{k-1}} (m |\Delta \lambda_m|) \\
&\quad + O(1) \left(\frac{P_m}{p_m} \right)^{\delta k} m X_m |\Delta \lambda_m| . \\
&= O(1) \sum_{v=1}^{m-1} (X_v - |\Delta \lambda_v| + (v+1) |\Delta \lambda_v|) + O(1) \left(\frac{P_m}{p_m} \right)^{\delta k} m X_m |\Delta \lambda_m| . \\
&= O(1) \sum_{v=1}^n X_v |\Delta \lambda_v| + O(1) \sum_{v=1}^n v X_v |\Delta \lambda_v| + O(1) \left(\frac{P_m}{p_m} \right)^{\delta k} m X_m |\Delta \lambda_m| \\
&= O(1) .
\end{aligned}$$

Finally

$$\begin{aligned}
\sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{\delta k+k-1} |T_{n4}|^k &= \sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{\delta k+k-1} \left| \frac{P_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v t_v \frac{\lambda_{v+1}}{v} \right|^k \\
&= O(1) \sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{\delta k-1} \frac{1}{P_{n-1}^k} \sum_{v=1}^{n-1} \frac{P_v}{v} |t_v|^k |\lambda_v|^k \left(\sum_{v=1}^{n-1} \frac{P_v}{v} \right)^{k-1} \\
&= O(1) \sum_{v=1}^m \frac{P_v}{v} |t_v|^k |\lambda_v|^k \sum_{n=v+1}^m \left(\frac{P_n}{p_n} \right)^{\delta k-1} \frac{1}{P_{n-1}^k} \\
&= O(1) \sum_{v=1}^m \left(\frac{P_v}{p_v} \right)^{\delta k} \frac{|t_v|^k}{X_v^{k-1}} (X_v |\lambda_v|)^{k-1} |\lambda_v| \\
&= O(1) \sum_{v=1}^m \left(\frac{P_v}{p_v} \right)^{\delta k} \frac{|t_v|^k |\lambda_v|}{v X_v^{k-1}} \\
&= O(1) \sum_{v=1}^{m-1} \left(\sum_{r=1}^v \left(\frac{P_r}{p_r} \right)^{\delta k} \frac{|t_r|^k}{r X_r^{k-1}} \right) |\Delta \lambda_v| + O(1) \sum_{r=1}^m \left(\frac{P_r}{p_r} \right)^{\delta k} \frac{|t_r|^k}{r X_r^{k-1}}
\end{aligned}$$

$$\begin{aligned}
&= O(1) \sum_{\nu=1}^m X_{\nu} |\Delta \lambda_{\nu}| + O(1) X_m |\lambda_m| \\
&= O(1).
\end{aligned}$$

This completes the proof of the theorem.

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