On Quasi-\( f \)-Power Increasing Sequences

Mahendra Misra
P.G. Department of Mathematics
N.C. College (Autonomous), Jajpur, Odisha
Mahendramisra2007@gmail.com

B. P. Padhy
Roland Institute of Technology
Golanthara-761008, Odisha, India
iraady@gmail.com

Dattaram Bisoyi
Department of Mathematics
L.N. Mahavidyalaya
Kodala, Ganjam, Odisha, India
dbisoyi2@gmail.com

U. K. Misra
Department of Mathematics
National Institute of Science and Technology
Pallur Hills, Golanthara-761008, Odisha, India
umakanta_misra@yahoo.com

Abstract

A result concerning absolute indexed Summability factor of an infinite series using Quasi-\( f \) - power increasing sequences has been established.

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1. Introduction

A positive sequence \((a_n)\) is said to be almost increasing if there exists a positive sequence \((b_n)\) and two positive constants \(A\) and \(B\) such that

\[(1.1) \quad Ab_n \leq a_n \leq Bb_n, \quad \text{for all } n.\]

The sequence \((a_n)\) is said to be quasi-\(\beta\)-power increasing, if there exists a constant \(K\) depending upon \(\beta\) with \(K \geq 1\) such that

\[(1.2) \quad Kn^{\beta}a_n \geq m^{\beta}a_m,\]

for all \(n \geq m\). In particular, if \(\beta = 0\), then \((a_n)\) is said to be quasi-increasing sequence. It is clear that every almost increasing sequence is a quasi-\(\beta\)-power increasing sequence for any non-negative \(\beta\). But the converse is not true as \((n^{-\beta})\) is quasi-\(\beta\)-power increasing but not almost increasing.

Let \(f = (f_n)\) be a positive sequence of numbers. Then the positive sequence \((a_n)\) is said to be quasi-\(f\)-power increasing, if there exists a constant \(K\) depending upon \(f\) with \(K \geq 1\) such that

\[(1.3) \quad Kf_n a_n \geq f_m a_m,\]

for \(n \geq m \geq l[4]\). Clearly, if \((a_n)\) is a quasi-\(f\)-power increasing sequence, then the \((a_n f_n)\) is a quasi-increasing sequence.

Let \(\sum a_n\) be an infinite series with sequence of partial sums \(\{s_n\}\). Let \((p_n)\) be a sequence of positive numbers such that

\[P_n = \sum_{v=0}^{n} p_v \rightarrow \infty, \quad \text{as } n \rightarrow \infty.\]

Then the sequence-to-sequence transformation

\[(1.4) \quad T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v, \quad P_n \neq 0,\]
defines the \( \left( \bar{N}, p_n \right) \)-mean of the sequence \( (s_n) \) generated by the sequence of coefficients \( \{p_n\} \).

The series \( \sum a_n \) is said to be summable \( \left| \bar{N}, p_n \right|_{k}, k \geq 1 \), if

\[
\sum_{n=1}^{\infty} \left( \frac{p_n}{p_{n+1}} \right)^{k-1} |T_n - T_{n-1}|^k < \infty .
\]

The series \( \sum a_n \) is said to be summable \( \left| \bar{N}, p_n ; \delta \right|_{k}, k \geq 1, \delta \geq 0 \), if

\[
\sum_{n=1}^{\infty} \left( \frac{p_n}{p_{n+1}} \right)^{\delta + k-1} |T_n - T_{n-1}|^k < \infty .
\]

2. Known Theorems

Dealing with quasi-\( \beta \)-power increasing sequence Bor and Debnath \[2\] have established the following theorem:

**2.1. Theorem:**

Let \( (X_n) \) be a quasi-\( \beta \)-power increasing sequence for 0 < \( \beta \) < 1 and \( (\lambda_n) \) be a real sequence.

If the conditions

\[
\sum_{n=1}^{m} \frac{p_n}{n} = O(P_m),
\]

\[
\lambda_n X_n = O(1),
\]

\[
\sum_{n=1}^{m} \left| \frac{p_n}{n} \right|^k = O(X_m),
\]

\[
\sum_{n=1}^{m} \frac{p_n}{P_n} \left| \frac{f_n}{P_n} \right|^k = O(X_m)
\]

and
\[
(2.1.5) \quad \sum_{n=1}^{m} nX_n \Delta^2 \lambda_n < \infty
\]
are satisfied, where \( t_n \) is the \((C,1)\) mean of the sequence \((na_n)\). Then the series \( \sum a_n \lambda_n \) is summable \( \left[ N, p_n \right]_k, k \geq 1 \).

Subsequently Leindler [3] established a similar result reducing certain condition of Bor. He established:

2.2. Theorem:

Let the sequence \((X_n)\) be a quasi-\(\beta\)-power increasing sequence for \(0 < \beta < 1\), and the real sequence \((\lambda_n)\) satisfies the conditions

\[
(2.2.1) \quad \sum_{n=1}^{m} \lambda_n = O(m)
\]
and

\[
(2.2.2) \quad \sum_{n=1}^{m} |\Delta \lambda_n | = O(m).
\]

Furthermore, suppose the conditions (2.1.3), (2.1.4) and

\[
(2.2.3) \quad \sum_{n=1}^{m} nX_n (\beta) |\Delta \lambda_n | < \infty,
\]
hold, where \( X_n (\beta) = \max(n^\beta X_n, \log n) \). Then the series \( \sum a_n \lambda_n \) is summable \( \left[ N, p_n \right]_k, k \geq 1 \).

Recently, extending the above results to quasi-\(f\)-power increasing sequence, Sulaiman [5] have established the following theorem:

2.3. Theorem:

Let \( f = (f_n) = (n^\beta \log^\gamma n) \), \( 0 \leq \beta < 1, \gamma \geq 0 \) be a sequence. Let \((X_n)\) be a quasi-\(f\)-power sequence and \((\lambda_n)\) a sequence of constants satisfying the conditions

\[
(2.3.1) \quad \lambda_n \to 0 \text{ as } n \to \infty,
\]
(2.3.2) \[ \sum_{n=1}^{\infty} nX_n |\Delta^2 \lambda_n| < \infty, \]

(2.3.3) \[ |\lambda_n| X_n = O(1), \]

(2.3.4) \[ \sum_{n=1}^{\infty} \frac{1}{nX_n^k} |t_n| = O(X_n) \]

and

(2.3.5) \[ \sum_{n=1}^{\infty} \frac{P_n}{P_{n-1}} \frac{1}{X_n^{k-1}} |t_n| = O(X_n), \]

where \( t_n \) is the \((C,1)\) mean of the sequence \((n \lambda_n)\). Then the series \( \sum a_n \lambda_n \) is summable \( \left\lfloor N, p_n \right\rfloor, k \geq 1 \).

In what follows in this paper we prove the following theorem.

3. Main Theorem

Let \( f = (f_n) = (n^\beta \log^r n) \) be a sequence and \((X_n)\) be a quasi-\( f \)-power sequence. Let \((\lambda_n)\) a sequence of constants such that

(3.1) \[ \lambda_n \to 0, \text{ as } n \to \infty, \]

(3.2) \[ \sum_{n=1}^{\infty} nX_n |\Delta^2 \lambda_n| < \infty, \]

(3.3) \[ |\lambda_n| X_n = O(1), \]

(3.4) \[ \sum_{n=v+1}^{m} \left(\frac{P_n}{P_{n-1}}\right)^{\delta k-1} \frac{1}{P_{n-1}} = O\left(\frac{P_m}{P_v}\right)^{\delta k-1}, \]

(3.5) \[ \sum_{n=v}^{\infty} \left(\frac{P_n}{P_{n-1}}\right)^{\delta k-1} \frac{|t_n|}{X_n^{k-1}} = O(X_m), \]

(3.6) \[ \sum_{n=1}^{\infty} \left(\frac{P_n}{P_{n-1}}\right)^{\delta k} \frac{|t_n|}{nX_n^{k-1}} = O(X_m). \]
Then the series $\sum a_n \lambda_n$ is summable $\left| N, p_n ; \delta \right|_k$, $k \geq 1, \delta \geq 0$.

In order to prove the theorem we require the following lemma.

4. Lemma:

Let $f = (f_n) = \left( n^{\beta} \log^\gamma n \right), 0 \leq \beta < 1, \gamma \geq 0$ be a sequence and $(X_n)$ be a quasi- $f$-power increasing sequence. Let $\left( \lambda_n \right)$ be a sequence of constants satisfying (3.1) and (3.2). Then

\begin{align*}
& (4.1) \quad n \sum_{n=1}^{\infty} |X_n| \Delta \lambda_n = O(1) \\
& (4.2) \quad \sum_{n=1}^{\infty} X_n |\Delta \lambda_n| < \infty.
\end{align*}

4.1. Proof of Lemma:

As $\Delta \lambda_n \to 0$ and $n^{\beta} \log^\gamma n X_n$ is non-decreasing, we have

\[ n X_n |\Delta \lambda_n| = n^{1-\beta} \log^{-\gamma} n n^{\beta} \log^\gamma n X_n \sum_{t=n}^{\infty} |\Delta \lambda_{v_t}| \]
\[ = O(1) n^{1-\beta} \log^{-\gamma} n \sum_{t=n}^{\infty} v^{\beta} \log^\gamma v X_v |\Delta \lambda_v| \]
\[ = O(1) \sum_{t=n}^{\infty} v^{1-\beta} \log^{-\gamma} v v^{\beta} \log^\gamma v X_v |\Delta \lambda_v| \]
\[ = O(1) \sum_{t=n}^{\infty} X_v |\Delta \lambda_v| = O(1). \]

This establishes (4.1).

Next

\[ \sum_{n=1}^{\infty} X_n |\Delta \lambda_n| = \sum_{n=1}^{\infty} \left( \sum_{r=1}^{n} X_r \right) |\Delta \lambda_r| + \left( \sum_{r=1}^{n} X_r \right) |\Delta \lambda_r| \]
\[ = O(1) \sum_{n=1}^{\infty} \left( \sum_{r=1}^{n} r^{-\beta} \log^{-\gamma} r r^{\beta} \log^\gamma r X_r \right) |\Delta \lambda_r| \]
\[ + O(1) \left( \sum_{r=1}^{n} r^{-\beta} \log^{-\gamma} r r^{\beta} \log^\gamma r X_r \right) |\Delta \lambda_n| \]
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\begin{align*}
&= O(1) \sum_{n=1}^{m} \left( n^\beta \log n \right) X_n \Delta \lambda_v \sum_{r=1}^{n} r^{-\beta - \epsilon} \log^{-\gamma} r \, r^{\epsilon} \\
&+ O(1) m^\beta X_m \Delta \lambda_m \left| \log n \sum_{r=1}^{m} r^{-\beta - \epsilon} \log^{-\gamma} r \, r^{\epsilon} \right|, \epsilon < 1 - \beta.
\end{align*}

\begin{align*}
&= O(1) \sum_{n=1}^{m} \left( n^\beta \log n \right) X_n \Delta \lambda_v \sum_{r=1}^{n} r^{-\beta - \epsilon} \log^{-\gamma} n^{\epsilon} \\
&+ O(1) m^\beta X_m \Delta \lambda_m \left| \log m \sum_{r=1}^{m} r^{-\beta - \epsilon} \log^{-\gamma} r \, r^{\epsilon} \right|
\end{align*}

\begin{align*}
&= O(1) \sum_{n=1}^{m} n^{\beta + \epsilon} X_n \left| \Delta \lambda_v \left( \int_1^n u^{-\beta - \epsilon} \, du \right) \right| + O(1) m^{\beta + \epsilon} X_m \left| \Delta \lambda_m \left( \int_1^m u^{-\beta - \epsilon} \, du \right) \right|
\end{align*}

\begin{align*}
&= O(1) \sum_{n=1}^{m} n X_n \Delta \lambda_v \left| \Delta \lambda_v \right| + O(1) m X_m \Delta \lambda_m \\
&= O(1).
\end{align*}

This establishes (4.2).

5. Proof of Theorem:

Let \( (T_n) \) be the sequence of \( (N, p_n) \) mean of the series \( \sum_{n=1}^{\infty} a_n \lambda_n \), then

\begin{align*}
T_n &= \frac{1}{p_n} \sum_{v=0}^{n} p_v a_v \lambda_v \\
&= \frac{1}{p_n} \sum_{v=0}^{n} (p_n - p_{v-1}) a_v \lambda_v.
\end{align*}

Hence for \( n \geq 1 \)

\begin{align*}
T_n - T_{n-1} &= \frac{p_n}{p_n p_{n-1}} \sum_{v=0}^{n} p_{v-1} a_v \lambda_v \\
&= \frac{p_n}{p_n p_{n-1}} \sum_{v=0}^{n} v a_v \left( \frac{1}{p_v - p_{v-1}} \lambda_v \right) \\
&= \frac{(n+1) p_n}{n} t_n \lambda_n + \frac{p_n}{p_n p_{n-1}} \sum_{v=0}^{n-1} p_{v+1} \lambda_v \frac{v+1}{v} + \frac{p_n}{p_n p_{n-1}} \sum_{v=0}^{n-1} p_v t_v \lambda_v \frac{v+1}{v} - \Delta \lambda_v \\
&= \frac{p_n}{p_n p_{n-1}} \sum_{v=0}^{n-1} p_v t_v \lambda_{v+1} \frac{v+1}{v}.
\end{align*}
= T_{a_1} + T_{a_2} + T_{a_3} + T_{a_4} (say).

In order to prove the theorem, using Minkowski’s inequality it is enough to show that

\[ \sum_{n=1}^{\infty} \left( \frac{P_n}{P_n} \right)^{\delta + k - 1} \left| T_{n,j} \right| < \infty, \quad j = 1, 2, 3, 4. \]

Applying Holders inequality, we have

\[ \sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{\delta + k - 1} \left| T_{n,j} \right|^k = \sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{\delta + k - 1} \left| \frac{n + 1}{n} \frac{P_n}{P_n} \delta \lambda_n \right|^k \]

\[ = O(1) \sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{\delta - 1} \frac{|\lambda_n|}{X_n^{k-1}} \left| \lambda_n \right|^k \]

\[ = O(1) \sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{\delta - 1} \frac{|\lambda_n|}{X_n^{k-1}} \left| \lambda_n \right| \]

\[ = O(1) \sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{\delta - 1} \frac{|\lambda_n|}{X_n^{k-1}} \left| \lambda_n \right| \]

\[ = O(1). \]

Next

\[ \sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{\delta + k - 1} \left| T_{n,j} \right|^k = \sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{\delta + k - 1} \left| \frac{n + 1}{n} \frac{P_n}{P_n} \delta \lambda_n \right|^k \]

\[ = O(1) \sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{\delta - 1} \frac{1}{P_n} \sum_{v=1}^{n-1} \left| \frac{P_n}{P_n} \right|^k \left| \lambda_n \right|^k \left( \sum_{v=1}^{n-1} \frac{P_n}{P_n} \right)^{\delta - 1} \]

\[ = O(1) \sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{\delta - 1} \left| \frac{P_n}{P_n} \right| \left| \lambda_n \right|^k \left( \sum_{v=1}^{n-1} \frac{P_n}{P_n} \right)^{\delta - 1} \]

\[ = O(1), \text{ as in the case of } T_{n1}. \]

Next

\[ \sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{\delta + k - 1} \left| T_{n,j} \right|^k = \sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{\delta + k - 1} \left| \frac{n + 1}{n} \frac{P_n}{P_n} \delta \lambda_n \right|^k \]

\[ = O(1), \text{ as in the case of } T_{n1}. \]
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\[
O(1) \sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{\alpha k - 1} \frac{1}{P_{n-1}^{\alpha k}} \sum_{v=1}^{\alpha k - 1} \frac{1}{X_v^{k-1}} |\Delta \lambda_v| \left( \sum_{i=1}^{n-1} X_i |\Delta \lambda_i| \right)^{k-1}
\]

\[
= O(1) \sum_{n=1}^{m} P_n \frac{1}{X_v^{k-1}} |\Delta \lambda_v| \sum_{n=1}^{m+1} \left( \frac{P_n}{P_n} \right)^{\alpha k - 1} \frac{1}{P_{n-1}^{\alpha k}}.
\]

\[
= O(1) \sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{\alpha k} \frac{1}{X_v^{k-1}} |\Delta \lambda_v| \sum_{n=1}^{m+1} \left( \frac{P_n}{P_n} \right)^{\alpha k - 1} \frac{1}{P_{n-1}^{\alpha k}}.
\]

\[
= O(1) \sum_{n=1}^{m} \left( \sum_{v=1}^{m+1} \frac{P_v}{P_v} \right)^{\alpha k} \frac{1}{X_v^{k-1}} |\Delta \lambda_v| \left( \sum_{v=1}^{m+1} \frac{P_v}{P_v} \right)^{\alpha k - 1} \frac{1}{P_{n-1}^{\alpha k}}.
\]

\[
= O(1) \sum_{n=1}^{m} \left( \sum_{v=1}^{m+1} \frac{P_v}{P_v} \right)^{\alpha k} \frac{1}{X_v^{k-1}} |\Delta \lambda_v| \sum_{v=1}^{m+1} \frac{P_v}{P_v} \frac{1}{P_{n-1}^{\alpha k}}.
\]

\[
= O(1) \sum_{n=1}^{m} \frac{P_n}{X_v^{k-1}} \frac{1}{X_v^{k-1}} |\Delta \lambda_v| \sum_{n=1}^{m+1} \frac{P_v}{P_v} \frac{1}{P_{n-1}^{\alpha k}}.
\]

\[
= O(1) \sum_{n=1}^{m} \frac{P_n}{X_v^{k-1}} \frac{1}{X_v^{k-1}} |\Delta \lambda_v| \sum_{v=1}^{m+1} \frac{P_v}{P_v} \frac{1}{P_{n-1}^{\alpha k}}.
\]

\[
= O(1) \sum_{n=1}^{m} \frac{P_n}{X_v^{k-1}} |\Delta \lambda_v| \left( \sum_{v=1}^{m+1} \frac{P_v}{P_v} \right)^{\alpha k - 1} \frac{1}{P_{n-1}^{\alpha k}}.
\]

\[
= O(1) \sum_{n=1}^{m} \frac{P_n}{X_v^{k-1}} |\Delta \lambda_v| \left( \sum_{v=1}^{m+1} \frac{P_v}{P_v} \right)^{\alpha k - 1} \frac{1}{P_{n-1}^{\alpha k}}.
\]

\[
= O(1) \sum_{n=1}^{m} \left( \sum_{v=1}^{m+1} \frac{P_v}{P_v} \right)^{\alpha k} \frac{1}{X_v^{k-1}} |\Delta \lambda_v| \left( \sum_{v=1}^{m+1} \frac{P_v}{P_v} \right)^{\alpha k - 1} \frac{1}{P_{n-1}^{\alpha k}}.
\]

\[
= O(1) \sum_{n=1}^{m} \left( \sum_{v=1}^{m+1} \frac{P_v}{P_v} \right)^{\alpha k} \frac{1}{X_v^{k-1}} |\Delta \lambda_v| \left( \sum_{v=1}^{m+1} \frac{P_v}{P_v} \right)^{\alpha k - 1} \frac{1}{P_{n-1}^{\alpha k}}.
\]

\[
= O(1) \sum_{n=1}^{m} \frac{P_n}{X_v^{k-1}} \frac{1}{X_v^{k-1}} |\Delta \lambda_v| \sum_{v=1}^{m+1} \frac{P_v}{P_v} \frac{1}{P_{n-1}^{\alpha k}}.
\]

\[
= O(1) \sum_{n=1}^{m} \frac{P_n}{X_v^{k-1}} \frac{1}{X_v^{k-1}} |\Delta \lambda_v| \sum_{v=1}^{m+1} \frac{P_v}{P_v} \frac{1}{P_{n-1}^{\alpha k}}.
\]

\[
= O(1) \sum_{n=1}^{m} \frac{P_n}{X_v^{k-1}} \frac{1}{X_v^{k-1}} |\Delta \lambda_v| \sum_{v=1}^{m+1} \frac{P_v}{P_v} \frac{1}{P_{n-1}^{\alpha k}}.
\]

\[
= O(1) \sum_{n=1}^{m} \frac{P_n}{X_v^{k-1}} \frac{1}{X_v^{k-1}} |\Delta \lambda_v| \sum_{v=1}^{m+1} \frac{P_v}{P_v} \frac{1}{P_{n-1}^{\alpha k}}.
\]

\[
= O(1) \sum_{n=1}^{m} \frac{P_n}{X_v^{k-1}} \frac{1}{X_v^{k-1}} |\Delta \lambda_v| \sum_{v=1}^{m+1} \frac{P_v}{P_v} \frac{1}{P_{n-1}^{\alpha k}}.
\]
\[ = O(1) \sum_{r=1}^{n} X_{r} |\Delta x_{r}| + O(1) X_{n} |x_{n}| \]

This completes the proof of the theorem.

References


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