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Constructions and Packings of Bull-Design

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Abstract

A bull-design of order n is a decomposition of the complete graph K_n into bulls. Such systems exist precisely when $n \equiv 0, 1 \pmod 5$. The necessary conditions of the existence of bull-designs of λK_n are the follows: $\lambda \not\equiv 0 \pmod 5$ and $n \equiv 0, 1 \pmod 5$, or $\lambda \equiv 0 \pmod 5$ and for all n. In this paper we showed the necessary conditions are also sufficient, and gave the minimum leave of the packing of the complete graph K_n.

Keywords: Decompositions, bull-designs, Packings

1 Introduction

A bull is a graph which is obtained by attaching two edges to two vertices of a triangle. In what follows we will denote the following bull by $(a, b, c; d, e)$ or $(a, c, b; e, d)$ (See Figure 1). A bull-design of order n, or bull-decomposition of order n, is a pair $(X, \mathcal{A})$, where X is the vertex set of the complete graph K_n and $\mathcal{A}$ is an edge-disjoint decomposition of K_n into copies of bulls. Following

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design terminology, we call these copies as blocks. Such designs exist precisely when $n \equiv 0, 1 \pmod 5$ [3]. One can even find that such designs are 2-colorable [3]. Similarly, a bull-decomposition of a graph G is a partition of edges of G into copies of bull.

Recently, some papers investigated the graph-decomposition for the special blocks. J. C. Bermond and J. Schonheim [1] considered this problem for the blocks with four vertices or less; E. J. Billington and D. L. Kreher[2], consider the intersection of G-designs; S. El-Zanati and C. A. Rodger [3] considered the blocking set of G-design for special blocks; C. M. Fu and W. C. Huang [4] considered the intersection for kite-designs.

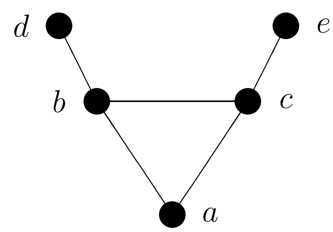


Figure 1. Bull $(a, b, c; d, e)$

A packing of the graph G with bulls (or a packing of G for brevity) is a set M of edge-disjoint bulls in G; the leave L of the packing M is the set of edges of G not belonging to any bulls in M. When the cardinality of the bull set M is as large as possible (or $|L|$ is as small as possible), the packing is called the maximum packing of G and the corresponding leave is called the minimum leave, denoted by $L(G)$. In the section 3, we will give the minimum leave of the packing of the complete graph K_n.

How about the bull-decomposition of the graph λK_n? The necessary conditions of the existence of such decompositions are the follows: $\lambda \not\equiv 0 \pmod 5$ and $n \equiv 0, 1 \pmod 5$, or $\lambda \equiv 0 \pmod 5$ and for all n. We will use the minimum leave to construct the bull-decomposition of the graph λK_n in section 4.

2 Some small examples of bull-design

Now, we will give some small examples of bull-design for the general constructions in the section 3.

Example 2.1 *The set $\mathcal{A}$ is a bull-decomposition of the graph $K_{5,5,5}$, where* $\mathcal{A} = \{(6,\ 1,\ 11; 9,\ 3),\ (12,\ 1,\ 7; 14,\ 11),\ (1,\ 8,\ 13; 2,\ 4),\ (6,\ 4,\ 14; 8,\ 2),\ (5,\ 8,$

12; 14, 6), (13, 2, 7; 6, 14), (9, 2, 15; 12, 6), (2, 10, 11; 1, 5), (7, 4, 15; 10, 1), (13, 5, 9; 7, 11), (13, 3, 6; 7, 5), (15, 3, 8; 14, 11), (12, 3, 10; 9, 15), (12, 4, 9; 11, 14), (14, 5, 10; 15, 13)} *and the partite sets of the graph* $K_{5,5,5}$ *are the set* $\{\{1,2,3,4,5\},\{6,7,8,9,10\},\{11,12,13,14,15\}\}$. $\diamond$

Example 2.2 *The set* $\mathcal{A}$ *is a bull-decomposition of the graph* $K_7 \setminus K_2$, *where* $\mathcal{A} = \{(2,\ 5,\ 6; 4,\ 1), (3,\ 6,\ 7; 4,\ 5), (5,\ 1,\ 3; 7,\ 4), (7,\ 2,\ 4; 3,\ 1)\}$. $\diamond$

Example 2.3 *Let* $\mathcal{A}_1 = \{(3,\ 6,\ 7; 4,\ 5), (5,\ 2,\ 8; 6,\ 3), (2,\ 4,\ 7; 8,\ 1), (3,\ 4,\ 5; 1,\ 6), (6,\ 1,\ 8; 5,\ 7)\}$, $\mathcal{A}_2 = \{(3,\ 6,\ 7; 2,\ 8), (5,\ 1,\ 3; 7,\ 8), (8,\ 4,\ 5; 1,\ 7), (1,\ 6,\ 8; 5,\ 2), (7,\ 2,\ 4; 5,\ 6)\}$, $\mathcal{A}_3 = \{(1,\ 5,\ 6; 2,\ 8), (8,\ 3,\ 4; 6,\ 5), (4,\ 2,\ 6; 3,\ 7), (3,\ 5,\ 7; 8,\ 2), (1,\ 7,\ 8; 4,\ 2)\}$, $\mathcal{A}_4 = \{((5,\ 6,\ 1; 3,\ 8), (4,\ 3,\ 8; 5,\ 7), (2,\ 6,\ 7; 4,\ 3), (4,\ 1,\ 7; 3,\ 5), (5,\ 2,\ 8; 4,\ 6)\}$ *and* $\mathcal{A}_5 = \{(7,\ 2,\ 6; 4,\ 8), (7,\ 4,\ 1; 8,\ 5), (8,\ 5,\ 2; 4,\ 3), (1,\ 3,\ 6; 5,\ 4), (3,\ 7,\ 8; 5,\ 1)\}$. *Then,* $\mathcal{A}_i$ *is a bull-decomposition of the graph* $K_8 \setminus G_i$, *where* $E(G_1) = \{\{1,2\},\{1,3\},\{2,3\}\}$, $E(G_2) = \{\{1,2\},\{2,3\},\{3,4\}\}$, $E(G_3) = \{\{1,2\},\{1,3\},\{1,4\}\}$, $E(G_4) = \{\{1,2\},\{2,3\},\{4,5\}\}$ *and* $E(G_5) = \{\{1,2\},\{3,4\},\{5,6\}\}$. $\diamond$

Example 2.4 *The set* $\mathcal{A}$ *is a bull-decomposition of the graph* $K_9 \setminus K_4$, *where* $\mathcal{A} = \{(1,\ 5,\ 6; 3,\ 4), (1,\ 7,\ 9; 5,\ 6), (2,\ 6,\ 7; 3,\ 4), (4,\ 8,\ 9; 6,\ 5), (7,\ 3,\ 8; 9,\ 1), (8,\ 2,\ 5; 9,\ 4)\}$. *Now,* $\mathcal{A}'$ *comes from* $\mathcal{A}$ *with* $(2,\ 6,\ 7; 3,\ 4) \cup \{\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$ *replaced by* $\{(1,3,4;6,2),(6,2,7;3,4)\}$. *Then,* $\mathcal{A}'$ *is a bull-decomposition of the graph* $K_9 \setminus K_2$. $\diamond$

Example 2.5 *The set* $\mathcal{A}$ *is a bull-decomposition of the graph* $K_{12} \setminus K_2$, *where* $\mathcal{A} = \{(2,\ 3,\ 6; 4,\ 9), (3,\ 7,\ 9; 6,\ 10), (2,\ 8,\ 10; 7,\ 11), (4,\ 8,\ 11; 6,\ 9), (5,\ 8,\ 12; 9,\ 1), (1,\ 5,\ 10; 6,\ 12), (8,\ 3,\ 1; 5,\ 11), (4,\ 9,\ 12; 5,\ 2), (2,\ 7,\ 11; 5,\ 6), (4,\ 7,\ 10; 1,\ 6), (4,\ 6,\ 1; 12,\ 9), (4,\ 5,\ 2; 11,\ 9), (11,\ 3,\ 12; 10,\ 7)\}$. $\diamond$

Example 2.6 *The set* $\mathcal{A}$ *is a bull-decomposition of the graph* $K_{13} \setminus K_8$, *where* $\mathcal{A} = \{(1,\ 9,\ 10; 6,\ 7), (1,\ 11,\ 12; 8,\ 7), (2,\ 10,\ 11; 8,\ 5), (13,\ 2,\ 12; 9,\ 5), (11,\ 3,\ 9; 13,\ 7), (3,\ 10,\ 12; 6,\ 8), (10,\ 4,\ 13; 11,\ 1), (4,\ 9,\ 12; 8,\ 6), (9,\ 5,\ 13; 10,\ 7), (6,\ 11,\ 13; 7,\ 8)\}$. *A bull-decomposition of the graph* $K_{13} \setminus G$ *can be obtained by embedding a bull-decomposition of the graph* $K_8 \setminus G$ *to a bull-decomposition of the graph* $K_{13} \setminus K_8$, *where* G *is any subgraph of* K_{13} *with three edges.* $\diamond$

Example 2.7 *The set* $\mathcal{A}$ *is a bull-decomposition of the graph* $K_{14} \setminus K_4$, *where* $\mathcal{A} = \{(1,\ 14,\ 5; 2,\ 13), (7,\ 9,\ 1; 5,\ 6), (1,\ 10,\ 12; 5,\ 2), (5,\ 12,\ 8; 13,\ 9), (9,\ 3,\ 11; 12,\ 1), (10,\ 2,\ 8; 5,\ 1), (14,\ 8,\ 13; 6,\ 1), (9,\ 2,\ 13; 11,\ 10), (13,\ 3,\ 6; 8,\ 10),$

(7, 5, 11; 3, 12), (9, 4, 10; 12, 11), (7, 14, 3; 11, 10), (2, 6, 7; 5, 10), (12, 9, 14; 6, 10), (14, 4, 6; 5, 12), (4, 8, 11; 7, 6), (4, 7, 13; 12, 11)}. *Now,* $\mathcal{A}'$ *comes from* $\mathcal{A}$ *with* (9, 4, 10; 12, 11) $\cup \{\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$ *replaced by* $\{(9,4,10;2,11),(1,3,4;2,12)\}$. *Then,* $\mathcal{A}'$ *is a bull-decomposition of the graph* $K_{14} \setminus K_2$. ⋄

Example 2.8 *The set* $\mathcal{A}$ *is a bull-decomposition of the graph* $K_{22} \setminus K_{12}$*, where* $\mathcal{A}$ = {(1, 13, 14; 4, 12), (1, 15, 16; 5, 6), (1, 17, 18; 11, 12), (1, 19, 20; 3, 4), (1, 21, 22; 4, 5), (2, 14, 15; 11, 12), (2, 16, 17; 5, 6), (2, 18, 19; 11, 12), (2, 20, 21; 12, 11), (3, 13, 15; 2, 22), (16, 3, 14; 22, 6), (15, 4, 17; 22, 12), (18, 4, 16; 19, 11), (17, 5, 19; 14, 22), (5, 18, 20; 6, 7), (19, 6, 21; 13, 3), (6, 20, 22; 14, 2), (17, 7, 14; 21, 4), (7, 15, 18; 6, 22), (16, 8, 19; 13, 7), (20, 8, 17; 14, 21), (18, 8, 21; 15, 12), (17, 9, 13; 21, 11), (18, 9, 14; 22, 21), (9, 15, 19; 11, 13), (9, 16, 20; 12, 11), (10, 13, 18; 20, 3), (10, 14, 19; 22, 11), (10, 15, 20; 21, 3), (10, 16, 21; 22, 5), (10, 17, 22; 3, 8), (12, 13, 22; 5, 11), (16, 7, 13; 22, 21)}. *A bull-decomposition of the graph* $K_{22} \setminus K_2$ *can be obtained by embedding a bull-decomposition of the graph* $K_{12} \setminus K_2$ *to a bull-decomposition of the graph* $K_{22} \setminus K_{12}$. ⋄

Example 2.9 *The set* $\mathcal{A}$ *is a bull-decomposition of the graph* $K_{23} \setminus K_{13}$*, where* $\mathcal{A}$ = {(1, 14, 15; 11, 12), (1, 16, 17; 12, 23), (1, 18, 19; 11, 12), (1, 20, 21; 12, 13), (1, 22, 23; 12, 13), (2, 15, 16; 11, 23), (2, 17, 18; 11, 13), (2, 19, 20; 13, 14), (2, 21, 22; 14, 15), (3, 14, 19; 5, 7), (3, 15, 20; 6, 7), (3, 16, 21; 5, 6), (3, 17, 22; 8, 6), (3, 18, 23; 12, 5), (2, 14, 23; 13, 7), (4, 21, 23; 11, 12), (4, 20, 22; 11, 9), (19, 5, 21; 22, 12), (18, 5, 20; 15, 6), (19, 6, 17; 23, 13), (18, 6, 16; 14, 13), (7, 15, 17; 8, 14), (16, 7, 14; 22, 12), (20, 8, 23; 14, 11), (8, 19, 22; 11, 13), (18, 8, 21; 16, 15), (20, 9, 17; 14, 5), (19, 9, 16; 23, 11), (18, 9, 15; 21, 10), (19, 10, 23; 14, 15), (10, 18, 22; 7, 11), (10, 17, 21; 12, 7), (10, 16, 20; 22, 13), (19, 4, 15; 17, 13), (18, 4, 14; 16, 22)}. *A bull-decomposition of the graph* $K_{23} \setminus G$ *can be obtained by embedding a bull-decomposition of the graph* $K_{13} \setminus G$ *to a bull-decomposition of the graph* $K_{23} \setminus K_{13}$*, where* G *is any subgraph of* K_{13} *with three edges.* ⋄

Example 2.10 *The set* $\mathcal{A}$ *is a bull-decomposition of the graph* $K_{24} \setminus K_{14}$*, where* $\mathcal{A}$ = {(1, 15, 16; 2, 4), (1, 17, 18; 6, 8), (1, 19, 20; 9, 10), (1, 21, 22; 11, 12), (1, 23, 24; 13, 14), (2, 16, 17; 8, 9), (2, 18, 19; 10, 11), (2, 20, 21; 12, 13), (2, 22, 23; 14, 15), (3, 15, 17; 14, 13), (3, 16, 18; 13, 11), (3, 19, 21; 10, 8), (3, 20, 22; 11, 9), (4, 17, 19; 14, 12), (4, 18, 20; 14, 9), (4, 21, 23; 10, 5), (4, 22, 24; 11, 16), (5, 16, 19; 12, 14), (5, 15, 18; 12, 13), (5, 17, 20; 24, 14), (21, 5, 24; 22, 9), (21, 6, 18; 15, 9), (6, 19, 22; 13, 8), (20, 6, 23; 16, 10), (19, 7, 15; 24, 4), (20, 7, 16; 23, 11), (7, 17, 21; 8, 12), (7, 18, 22; 23, 17), (8, 19, 23; 24, 3), (20, 8, 24; 15, 2), (21, 14, 16; 23, 9), (20, 13, 15; 24, 11), (18, 12, 24; 23, 6), (23, 11, 17; 24, 12), (10, 16, 22; 23, 13), (21, 9, 15; 23, 22), (15, 10, 24; 17, 3)}. *A bull-decomposition of the graph $K_{24} \setminus K_4$ ($K_{24} \setminus K_2$) can be obtained by embedding a bull-decomposition of the graph $K_{14} \setminus K_4$ ($K_{14} \setminus K_2$) to a bull-decomposition of the graph $K_{24} \setminus K_{14}$.* $\diamond$

3 Packing of bull-design

Since a bull-decomposition of order n exists for $n \equiv 0, 1 \pmod 5$ [3], the minimum leaves are the empty set. For the others, we need some constructions.

Take $2n \geq 6$ and $X = \{1, 2, \ldots, 2n\}$. Let $\mathcal{G}$ be a partition of X in sets of size 2 if $2n \equiv 0, 2 \pmod 6$ and of size 2 and 4 with one set of size 4 if $2n \equiv 4 \pmod 6$. The sets in $\mathcal{G}$ are called holes. Let $(X, \mathcal{G}, \mathcal{B})$ be a group divisible design (GDD) with groups $\mathcal{G}$ and blocks $\mathcal{B}$ of size 3 (see [5]).

$(10n + 2)$-Construction

Let $V = \{\infty_1, \infty_2\} \cup (X \times \{1, 2, 3, 4, 5\})$ and define a collection $\mathcal{A}$ of bulls on V as follows:

1. If $\mathcal{G}$ contains a hole g of size 4, take a bull-decomposition of the graph $K_{22} \setminus K_2$ on $\{\infty_1, \infty_2\} \cup (g \times \{1, 2, 3, 4, 5\})$ and put these bulls in $\mathcal{A}$.

2. For each hole g of size 2, take a bull-decomposition of the graph $K_{12} \setminus K_2$ on $\{\infty_1, \infty_2\} \cup (g \times \{1, 2, 3, 4, 5\})$ and put these bulls in $\mathcal{A}$.

3. For each block $\{a, b, c\}$ of the GDD, take a bull-decomposition of $K_{5,5,5}$ on $\{(a, j) \mid 1 \leq j \leq 5\} \cup \{(b, j) \mid 1 \leq j \leq 5\} \cup \{(c, j) \mid 1 \leq j \leq 5\}$ and put these bulls in $\mathcal{A}$.

$(10n + 3)$-Construction

Let $V = \{\infty_1, \infty_2, \infty_3\} \cup (X \times \{1, 2, 3, 4, 5\})$ and define a collection $\mathcal{A}$ of bulls on V as follows:

1. If $\mathcal{G}$ contains a hole g of size 4, take a bull-decomposition of the graph $K_{23} \setminus K_3$ on $\{\infty_1, \infty_2, \infty_3\} \cup (g \times \{1,2,3,4,5\})$ and put these bulls in $\mathcal{A}$.

2. Take a hole g' of size 2 and a bull-decomposition of the graph $K_{13} \setminus G$ on $\{\infty_1, \infty_2, \infty_3\} \cup (g' \times \{1,2,3,4,5\})$ and put these bulls in $\mathcal{A}$, where the graph G can be taken the spanning subgraph of K_{13} with 3 edges as in examples 2.10 - 2.14.

3. For each hole g of size 2, $g \neq g'$, take a bull-decomposition of the graph $K_{13} \setminus K_3$ on $\{\infty_1, \infty_2, \infty_3\} \cup (g \times \{1,2,3,4,5\})$ and put these bulls in $\mathcal{A}$.

4. For each block $\{a, b, c\}$ of the GDD, take a bull-decomposition of $K_{5,5,5}$ on $\{(a,j) \mid 1 \leq j \leq 5\} \cup \{(b,j) \mid 1 \leq j \leq 5\} \cup \{(c,j) \mid 1 \leq j \leq 5\}$ and put these bulls in $\mathcal{A}$.

$(10n + 4)$-Construction

Let $V = \{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (X \times \{1,2,3,4,5\})$ and define a collection $\mathcal{A}$ of bulls on V as follows:

1. If $\mathcal{G}$ contains a hole g of size 4, take a bull-decomposition of the graph $K_{24} \setminus K_4$ on $\{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (g \times \{1,2,3,4,5\})$ and put these bulls in $\mathcal{A}$.

2. Take a hole g' of size 2 and a bull-decomposition of the graph $K_{14} \setminus K_2$ on $\{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (g' \times \{1,2,3,4,5\})$ and put these bulls in $\mathcal{A}$.

3. For each hole g of size 2, $g \neq g'$, take a bull-decomposition of the graph $K_{14} \setminus K_4$ on $\{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (g \times \{1,2,3,4,5\})$ and put these bulls in $\mathcal{A}$.

4. For each block $\{a, b, c\}$ of the GDD, take a bull-decomposition of $K_{5,5,5}$ on $\{(a,j) \mid 1 \leq j \leq 5\} \cup \{(b,j) \mid 1 \leq j \leq 5\} \cup \{(c,j) \mid 1 \leq j \leq 5\}$ and put these bulls in $\mathcal{A}$.

Take $n \geq 1$ and $Y = \{1, 2, \ldots, 2n+1\}$. Let $(Y, \bullet)$ be an idempotent commutative quasigroup of order $2n+1$ [5].

$(10n+7)$-Construction

Let $V = \{\infty_1, \infty_2\} \cup (Y \times \{1,2,3,4,5\})$ and define a collection $\mathcal{A}$ of bulls on V as follows:

1. Take each point $y \in Y$ and a bull-decomposition of the graph $K_7 \setminus K_2$ on $\{\infty_1, \infty_2\} \cup (y \times \{1,2,3,4,5\})$ and put these bulls in $\mathcal{A}$.

2. For each $a, b \in Y$, $1 \leq a < b \leq 2n+1$, take bulls $\{((a \bullet b, 2), (a, 1), (b, 1); (b, 4), (a, 4)) + (0, i) \mid 0 \leq i \leq 4\}$ and put these bulls in $\mathcal{A}$.

$(10n+8)$-Construction

Let $V = \{\infty_1, \infty_2, \infty_3\} \cup (Y \times \{1,2,3,4,5\})$ and define a collection $\mathcal{A}$ of bulls on V as follows:

1. Take one point $y' \in Y$ and a bull-decomposition of the graph $K_8 \setminus G$ on $\{\infty_1, \infty_2, \infty_3\} \cup (y' \times \{1,2,3,4,5\})$ and put these bulls in $\mathcal{A}$, where where the graph G can be taken the spanning subgraph of K_8 with 3 edges as in examples 2.3 - 2.7.

2. Take each point $y \in Y$, $y \neq y'$, and a bull-decomposition of the graph $K_8 \setminus K_3$ on $\{\infty_1, \infty_2, \infty_3\} \cup (y \times \{1,2,3,4,5\})$ and put these bulls in $\mathcal{A}$.

3. For each $a, b \in Y$, $1 \leq a < b \leq 2n+1$, take bulls $\{((a \bullet b, 2), (a, 1), (b, 1); (b, 4), (a, 4)) + (0, i) \mid 0 \leq i \leq 4\}$ and put these bulls in $\mathcal{A}$.

$(10n+9)$-Construction

Let $V = \{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (Y \times \{1,2,3,4,5\})$ and define a collection $\mathcal{A}$ of bulls on V as follows:

1. Take one point $y' \in Y$ and a bull-decomposition of the graph $K_9 \setminus K_2$ on $\{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (y' \times \{1,2,3,4,5\})$ and put these bulls in $\mathcal{A}$.

2. Take each point $y \in Y$, $y \neq y'$, and a bull-decomposition of the graph $K_9 \setminus K_4$ on $\{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (y \times \{1,2,3,4,5\})$ and put these bulls in $\mathcal{A}$.

3. For each $a, b \in Y$, $1 \leq a < b \leq 2n+1$, take bulls $\{((a \bullet b, 2), (a, 1), (b, 1); (b, 4), (a, 4)) + (0, i) \mid 0 \leq i \leq 4\}$ and put these bulls in $\mathcal{A}$.

From the above constructions and small examples in section 2, we can obtain the maximum packing $L(K_n)$ of bull-design. (see Figure 2)

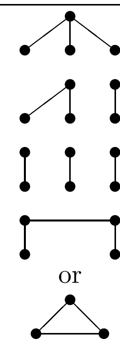
n	$n \equiv 0$ or $1 (\mathrm{mod}\ 5)$	$n \equiv 2$ or $4 (\mathrm{mod}\ 5)$	$n \equiv 3 (\mathrm{mod}\ 5)$
$L(K_n)$	$\emptyset$		

Figure 2: Leaves of maximum packing.

4 Construction of the bull-design of λK_n

Now, we will give the bull-decomposition of the graph λK_n by the minimum leave in section 3. The necessary conditions of the existence of such decompositions are the follows: $\lambda \not\equiv 0 \pmod 5$ and $n \equiv 0, 1 \pmod 5$, or $\lambda \equiv 0 \pmod 5$ and for all n. Since a bull-decomposition of order n exists for $n \equiv 0, 1$ (mod 5) [3], the existence results for the bull-decomposition of λK_n are just to show the existence for $n \not\equiv 0, 1 \pmod 5$ and $\lambda = 5$.

For $n = 10k + 2$, $10k + 4$, $10k + 7$ or $10k + 9$, the minimum leave of the packing of K_n with bulls is one edge. Combining the 5 edges, we obtain a bull-decomposition of the graph $5K_n$. For $n = 10k + 3$ or $10k + 8$, we take the minimum leave of the packing of K_n with bulls by the vertex-disjoint three edges. Combining the five leaves of this type, we obtain a bull-decomposition of the graph $5K_n$. From those methods, we obtain the following Main Theorem.

Main Theorem An bull-designs of λK_n exists if and only if $\lambda \not\equiv 0 \pmod 5$ and $n \equiv 0, 1 \pmod 5$, or $\lambda \equiv 0 \pmod 5$ and for all n.

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