Small Zeros of Quadratic Forms Mod P^m

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Abstract. Let $Q(\mathbf{x}) = Q(x_1, x_2, \dots, x_n)$ be a quadratic form over $\mathbb{Z}$, p be an odd prime, and $\Delta = ((-1)^{n/2} \det A_Q/p)$. A solution of the congruence $Q(\mathbf{x}) \equiv 0 \pmod{p^m}$ is said to be a primitive solution if $p \nmid x_i$ for some i . We prove that if this congruence has a primitive solution then it has a primitive solution with $\|\mathbf{x}\| \leq \max\{6^{1/n} p^{m[(1/2)+(1/n)]}, 2^{2(n+1)/(n-2)} 3^{2/(n-2)}\}$.

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1 Introduction

Let $Q(\mathbf{x}) = Q(x_1, x_2, \dots, x_n) = \sum_{1 \leq i \leq j \leq n} a_{ij} x_i x_j$ be a quadratic form with integer coefficients and p be an odd prime. Set $\|\mathbf{x}\| = \max |x_i|$. When n is even we let $\Delta_p(Q) = ((-1)^{n/2} \det A_Q/p)$ if $p \nmid \det A_Q$ and $\Delta_p(Q) = 0$ if $p \mid \det A_Q$, where $(\cdot/p)$ denotes the Legendre-Jacobi symbol and A_Q is the $n \times n$ defining matrix for $Q(\mathbf{x})$. $Q(\mathbf{x})$ is called nonsingular $\pmod{p}$ if $p \nmid \det A_Q$.

Consider the congruence

$$Q(\mathbf{x}) \equiv 0 \pmod{M}, \quad (1)$$

where M is a positive integer. The problem of finding a small solution of (1) means finding a nonzero integral solution $\mathbf{x}$ such that $\|\mathbf{x}\| \leq M^\delta$ for some positive constant $\delta < 1$. The constant δ may depend on n , but not on M . Our interest in this paper is in finding a small primitive solution of (1) in the case where $M = p^m$, $m \geq 2$, a solution $\mathbf{x}$ with $\gcd(x_1, \dots, x_n, M) = 1$. A primitive solution is sought to rule out solutions of the type $p\mathbf{y}$ where $\mathbf{y}$ satisfies $Q(\mathbf{y}) \equiv 0 \pmod{p}$. For the quadratic form $Q(x) = x_1^2 + \dots + x_n^2$, it is clear that any nonzero solution $\mathbf{x}$ of (1) must satisfy, $\max |x_i| \geq \frac{1}{\sqrt{n}} M^{1/2}$. Thus $\delta = 1/2$ is the best possible exponent for a small solution in general.

Schinzel, Schlickewi and Schmidt [15] proved that (1) has a nonzero solution with $\|\mathbf{x}\| < M^{(1/2)+1/2(n-1)}$ for $n \geq 3$. Thus for any $\varepsilon > 0$ we get a nonzero solution of (1) with $\|\mathbf{x}\| < M^{(1/2)+\varepsilon}$ provided n is sufficiently large. We note that the solution obtained by their method is not necessarily a primitive solution. Indeed, when $M = p^m$, $m \geq 2$ they use a solution of the type $p\mathbf{y}$ with $Q(\mathbf{y}) \equiv 0 \pmod{p}$.

Dealing with $M = p$, p an odd prime, Heath-Brown [13] obtained a nonzero solution of (1) with $\|\mathbf{x}\| \ll p^{1/2} \log p$ for $n \geq 4$. His result was an improvement on the result of [15] in this case. Wang Yuan [16],[17] and [18] generalized Heath-Brown's work to all finite fields. Cochrane, in a sequence of papers [1], [2] and [3] improved this to $\|\mathbf{x}\| < \max\{2^{19}p^{1/2}, 2^{22}10^6\}$. An improvement on Cochrane's result is given by Hakami [7, Theorem 1.3] and [10, Theorem 1] who obtained $\|\mathbf{x}\| < \min\{p^{2/3}, 2^{19}p^{1/2}\}$.

Hakami [8, Theorem 1] and [9, Theorem 1] respectively, generalized Cochrane's method to find a primitive solution of (1) with $\|\mathbf{x}\| \ll p$ for $n \geq 4$ when $M = p^2$ and with $\|\mathbf{x}\| \ll p^{3/2}$, for $n \geq 6$ when $M = p^3$ where $Q(\mathbf{y})$ is nonsingular $\pmod{p}$. Final Cochrane and Hakami [6, Theorem 1] proved that $\|x\| \leq p$ for $n > 1$.

For $M = pq$ a product of two distinct primes, we find Heath-Brown [11] obtained $\|\mathbf{x}\| \ll M^{(1/2)+\varepsilon}$, for $n > 4$ and $\varepsilon > 0$. Later Cochrane [5] sharpened this result to $\|\mathbf{x}\| \ll M^{1/2}$, for $n > 4$.

Our purpose in this paper is to generalize $\pmod{p}$ methods for obtaining a small primitive solution of

$$Q(\mathbf{x}) \equiv 0 \pmod{p^m}. \quad (2)$$

and more generally of obtaining a primitive solution in a box $\mathcal{B}$ with $|\mathcal{B}|$ sufficiently large.

Our main result is the following theorem.

Theorem 1.1 *For any quadratic form $Q(\mathbf{x})$ with n even, $n \geq 4$ and any odd prime power p^m , $m \geq 2$, there exists a primitive solution of (2) with*

$$\|\mathbf{x}\| \leq \max\{6^{1/n}p^{m[(1/2)+(1/n)]}, 2^{2(n+1)/(n-2)}3^{2/(n-2)}\}.$$

Theorem 1.1 is an immediate consequence of Corollary 3.1. To prove Theorem 1.1 we shall use finite Fourier series over $\mathbb{Z}_{p^m}$, the ring of integers $\pmod{p^m}$.

2 Basic Identities and Lemmas

Henceforth we shall assume that n is even, p is an odd prime, and that $Q(\mathbf{x})$ is a nonsingular quadratic form $\pmod{p}$. Let $e_{p^m}(\alpha) = e^{2\pi i \alpha / p^m}$. Let $V_{p^m} =$

$V_{p^m}(Q)$ be the set of zeros of Q contained in $\mathbb{Z}_{p^m}^n$ and let $Q^*(\mathbf{y})$ be the quadratic form associated with inverse of the matrix for $Q \pmod{p}$. For $\mathbf{y} \in \mathbb{Z}_{p^m}^n$ set

$$\phi(V_{p^m}, \mathbf{y}) = \begin{cases} \sum_{\mathbf{x} \in V} e_{p^m}(\mathbf{x} \cdot \mathbf{y}) & \text{for } \mathbf{y} \neq \mathbf{0} \\ |V_{p^m}| - p^{m(n-1)} & \text{for } \mathbf{y} = \mathbf{0} \end{cases}$$

We abbreviate complete sums over $\mathbb{Z}_{p^m}^n$ and $\mathbb{Z}_p^n$ in the manner

$$\sum_{\mathbf{x}} = \sum_{\mathbf{x} \pmod{p^m}} = \sum_{x_1=1}^{p^m} \cdots \sum_{x_n=1}^{p^m}, \quad \sum_{\mathbf{x} \pmod{p}} = \sum_{x_1=1}^p \cdots \sum_{x_n=1}^p.$$

The following lemma gives us a formula for $\phi(V, \mathbf{y})$.

Lemma 2.1 *Let n be an even positive integer. For $\mathbf{y} \in \mathbb{Z}^n$, put $\mathbf{y}' = p^{-j}\mathbf{y}$ in case $p \mid \mathbf{y}$. Then*

$$\phi(V, \mathbf{y}) = p^{(mn/2)-m} \sum_{\substack{j=0 \\ p^j \mid y_i \text{ for all } i}}^{m-1} \delta_j p^{jn/2} \omega_j(\mathbf{y}'),$$

where

$$\delta_j = \begin{cases} 1 & \text{if } m-j \text{ is even,} \\ \Delta & \text{if } m-j \text{ is odd,} \end{cases} \quad (3)$$

and

$$\omega_j(\mathbf{y}') = \begin{cases} p^{m-j} - p^{m-j-1}, & p^{m-j} \mid Q^*(\mathbf{y}'), \\ -p^{m-j-1}, & p^{m-j-1} \parallel Q^*(\mathbf{y}'), \\ 0, & p^{m-j-1} \nmid Q^*(\mathbf{y}'). \end{cases}$$

The proof of Lemma 2.1 is given (with some work) in Carlitz [12], and in complete detail in [11, Theorem 1].

Let $\alpha(\mathbf{x})$ be a complex valued function defined on $\mathbb{Z}_{p^m}^n$ with Fourier expansion $\alpha(\mathbf{x}) = \sum_{\mathbf{y}} a(\mathbf{y}) e_{p^m}(\mathbf{x} \cdot \mathbf{y})$ where $a(\mathbf{y}) = p^{-mn} \sum_{\mathbf{x}} \alpha(\mathbf{x}) e_{p^m}(-\mathbf{x} \cdot \mathbf{y})$. Then by definition of $\phi(V, \mathbf{y})$,

$$\begin{aligned}
\sum_{\mathbf{x} \in V_{p^m}} \alpha(\mathbf{x}) &= a(\mathbf{0}) |V_{p^m}| + \sum_{\mathbf{y} \neq \mathbf{0}} a(\mathbf{y}) \sum_{\mathbf{x} \in V_{p^m}} e_{p^m}(\mathbf{y} \cdot \mathbf{x}) \\
&= a(\mathbf{0}) |V_{p^m}| + \sum_{\mathbf{y} \neq \mathbf{0}} a(\mathbf{y}) \phi(V, \mathbf{y}) \\
&= a(\mathbf{0}) [\phi(V, \mathbf{0}) + p^{m(n-1)}] + \sum_{\mathbf{y} \neq \mathbf{0}} a(\mathbf{y}) \phi(V, \mathbf{y}) \\
&= a(\mathbf{0}) p^{m(n-1)} + \sum_{\mathbf{y}} a(\mathbf{y}) \phi(V, \mathbf{y}) \\
&= p^{-mn} \sum_{\mathbf{x}} \alpha(\mathbf{x}) p^{m(n-1)} + \sum_{\mathbf{y}} a(\mathbf{y}) \phi(V, \mathbf{y}).
\end{aligned}$$

Thus by Lemma 2.1,

$$\begin{aligned}
\sum_{\mathbf{x} \in V_{p^m}} \alpha(\mathbf{x}) &= p^{-m} \sum_{\mathbf{x}} \alpha(\mathbf{x}) + p^{(mn/2)-m} \sum_{\mathbf{y}} a(\mathbf{y}) \sum_{\substack{j=0 \\ p^j | y_i \text{ for all } i}}^{m-1} \delta_j p^{jn/2} \omega_j(\mathbf{y}') \\
&= p^{-m} \sum_{\mathbf{x}} \alpha(\mathbf{x}) + p^{(mn/2)-m} \sum_{j=0}^{m-1} \delta_j p^{jn/2} \sum_{\substack{\mathbf{y} \\ p^j | y_i \text{ for all } i}} a(\mathbf{y}) \omega_j(\mathbf{y}').
\end{aligned}$$

But by noticing that the inner sum $\sum_{\mathbf{y}} a(p^j \mathbf{y}') \omega_j(\mathbf{y}')$ can be written

$$\begin{aligned}
\sum_{\substack{\mathbf{y} \\ p^j | y_i \text{ for all } i}} a(\mathbf{y}) \omega_j(\mathbf{y}') &= \sum_{\mathbf{y}' \pmod{p^{m-j}}} a(p^j \mathbf{y}') \omega_j(\mathbf{y}') \\
&= \sum_{\substack{\mathbf{y}' \\ p^{m-j} | Q^*(\mathbf{y}')}} a(p^j \mathbf{y}') (p^{m-j} - p^{m-j-1}) - \sum_{\substack{\mathbf{y}' \\ p^{m-j-1} || Q^*(\mathbf{y}')}} a(p^j \mathbf{y}') p^{m-j-1} \\
&= \sum_{\substack{\mathbf{y}' \\ p^{m-j} | Q^*(\mathbf{y}')}} a(p^j \mathbf{y}') p^{m-j} - p^{m-j-1} \sum_{\substack{\mathbf{y}' \\ p^{m-j-1} | Q^*(\mathbf{y}')}} a(p^j \mathbf{y}'),
\end{aligned}$$

we obtain,

Lemma 2.2 (The fundamental identity) *For any complex valued $\alpha(\mathbf{x})$*

as given above

$$\begin{aligned} \sum_{\mathbf{x} \in V_{p^m}} \alpha(\mathbf{x}) &= p^{-m} \sum_{\mathbf{x} \pmod{p^m}} \alpha(\mathbf{x}) \\ &+ p^{mn/2} \sum_{j=0}^{m-1} p^{jn/2} \delta_j \left(p^{-j} \sum_{\substack{y'_i=1 \\ p^{m-j}|Q^*(\mathbf{y}')}}^{p^{m-j}} a(p^j \mathbf{y}') - p^{-j-1} \sum_{\substack{y'_i=1 \\ p^{m-j-1}|Q^*(\mathbf{y}')}}^{p^{m-j}} a(p^j \mathbf{y}') \right), \end{aligned}$$

where δ_j as we defined in (3): $\delta_j = 1$ when $m - j$ is even and $\delta_j = \Delta$ if $m - j$ is odd.

3 Proof of Theorem 1.1 when $\Delta = +1$

Let $\mathcal{B}$ be the box of points in $\mathbb{Z}^n$ given by

$$\mathcal{B} = \{\mathbf{x} \in \mathbb{Z}^n \mid a_i \leq x_i < a_i + B_i, 1 \leq i \leq n\}, \quad (4)$$

where $a_i, B_i \in \mathbb{Z}$ and $1 \leq B_i \leq p^m$, $1 \leq i \leq n$. Then $|\mathcal{B}| = \prod_{i=1}^n B_i$, the cardinality of $\mathcal{B}$. View the box $\mathcal{B}$ in (4) as a subset of $\mathbb{Z}_{p^m}^n$ and let $\alpha(\mathbf{x}) = \chi_{\mathcal{B}} * \chi_{\mathcal{B}}$ with Fourier expansion $\alpha(\mathbf{y}) = \sum_{\mathbf{y}} a(\mathbf{y}) e_{p^m}(\mathbf{x} \cdot \mathbf{y})$. Thus for any $\mathbf{y} \in \mathbb{Z}_{p^m}^n$,

$$|a(\mathbf{y})| = p^{-mn} \prod_{i=1}^n \left| \frac{\sin^2 \pi B_i y_i / p^m}{\sin^2 \pi y_i / p^m} \right|,$$

where the term in the product is taken to be m_i if $y_i = 0$. In particular

$$a(\mathbf{y}) \leq p^{-mn} \prod_{i=1}^n \min \left\{ B_i^2, \left(\frac{p^m}{2y_i} \right)^2 \right\}.$$

Consider the congruence (2):

$$Q(\mathbf{x}) \equiv 0 \pmod{p^m}.$$

For later reference, we construct the following lemma which help us on estimating the sum $\sum_{y_i=1}^{p^{m-j}} a(p^j \mathbf{y})$.

Lemma 3.1 *Let $\mathcal{B}$ be any box of type (4) and $\alpha(\mathbf{x}) = \chi_{\mathcal{B}} * \chi_{\mathcal{B}}(\mathbf{x})$. Then we have*

$$\sum_{y_i=1}^{p^{m-j}} a(p^j \mathbf{y}) \leq \frac{|\mathcal{B}|}{p^{jn}} \prod_{B_i \geq p^{m-j}} \frac{2B_i}{p^{m-j}}.$$

Proof. First,

$$\begin{aligned}
\sum_{y_i=1}^{p^{m-j}} a(p^j \mathbf{y}) &= \sum_{\mathbf{y}} \sum_{x_i=1}^{p^m} \frac{1}{p^m} \alpha(\mathbf{x}) e_{p^m}(-\mathbf{x} \cdot p^j \mathbf{y}) \\
&= \sum_{x_i=1}^{p^m} \frac{1}{p^{mn}} \alpha(\mathbf{x}) \sum_{y_i} e_{p^m}(-\mathbf{x} \cdot p^j \mathbf{y}) \\
&= \sum_{x_i=1}^{p^m} \frac{1}{p^{mn}} \alpha(\mathbf{x}) \sum_{y_i=1}^{p^{m-j}} e_{p^{m-j}}(-\mathbf{x} \cdot \mathbf{y}) \\
&= \sum_{\substack{x_i=1 \\ \mathbf{x} \equiv \mathbf{0} \pmod{p^{m-j}}}}^{p^m} p^{-mn} \alpha(\mathbf{x}) p^{(m-j)n} \\
&= p^{-jn} \sum_{\substack{x_i=1 \\ \mathbf{x} \equiv \mathbf{0} \pmod{p^{m-j}}}}^{p^m} \alpha(\mathbf{x}) \\
&= p^{-jn} \sum_{\mathbf{u} \in \mathcal{B}} \sum_{\substack{\mathbf{v} \in \mathcal{B} \\ \mathbf{u} + \mathbf{v} \equiv \mathbf{0} \pmod{p^{m-j}}}} 1 \\
&\leq p^{-jn} \prod_{i=1}^n B_i \left(\left\lceil \frac{B_i}{p^{m-j}} \right\rceil + 1 \right). \tag{5}
\end{aligned}$$

To verify the last inequality in (5) we count the number of solutions of the congruence

$$\mathbf{u} + \mathbf{v} \equiv \mathbf{0} \pmod{p^{m-j}},$$

with $\mathbf{u}, \mathbf{v} \in \mathcal{B}$. In fact for each choice of $\mathbf{v}$, there are at most $\prod_{i=1}^n ([B_i/p^{m-j}] + 1)$ choices $\mathbf{u}$. Thus the total number of solutions is less than or equal to $\prod_{i=1}^n ([B_i/p^{m-j}] + 1)$. Hence it follows from (5),

$$\sum_{y_i=1}^{p^2} a(p\mathbf{y}) \leq p^{-jn} \prod_{i=1}^n B_i \left(\left\lceil \frac{B_i}{p^{m-j}} \right\rceil + 1 \right). \tag{6}$$

We may split the product in (6) such that

$$\prod_{i=1}^n B_i \left(\left\lceil \frac{B_i}{p^{m-j}} \right\rceil + 1 \right) = \prod_{B_i < p^{m-j}} B_i \prod_{B_i \geq p^{m-j}} B_i \left(\frac{2B_i}{p^{m-j}} \right).$$

Then by this equality we therefore have

$$\sum_{y_i=1}^{p^{m-j}} a(p^j \mathbf{y}) \leq \frac{|\mathcal{B}|}{p^{jn}} \prod_{B_i \geq p^{m-j}} \frac{2B_i}{p^{m-j}}.$$

Our proof is complete. $\square$

Now we can proceed to estimate the error term in the fundamental identity which is given by

$$Error = p^{mn/2} \sum_{j=0}^{m-1} p^{jn/2} \delta_j \left(p^{-j} \sum_{\substack{y'_i=1 \\ p^{m-j}|Q^*(\mathbf{y}')}}^{p^{m-j}} a(p^j \mathbf{y}') - p^{-j-1} \sum_{\substack{y'_i=1 \\ p^{m-j-1}|Q^*(\mathbf{y}')}}^{p^{m-j}} a(p^j \mathbf{y}') \right).$$

By Lemma 3.1 we get

$$|Error| \leq p^{mn/2} \sum_{j=0}^{m-1} p^{jn/2} \left| \sum_{y_i=1}^{p^{m-j}} \varphi(\mathbf{y}) a(p^j \mathbf{y}') \right| \quad (7)$$

where

$$\varphi(\mathbf{y}) = \begin{cases} p^j - p^{-j-1} & \text{if } p^{m-j} | Q^*(\mathbf{y}), \\ -p^{-j-1} & \text{if } p^{m-j-1} \nmid Q^*(\mathbf{y}). \end{cases}$$

Continuing from (7),

$$\begin{aligned} |Error| &\leq p^{mn/2} \sum_{j=0}^{m-1} p^{(jn/2)-j} \frac{|\mathcal{B}|}{p^{jn}} \prod_{B_i \geq p^{m-j}} \frac{2B_i}{p^{m-j}} \\ &= p^{mn/2} |\mathcal{B}| \sum_{j=0}^{m-1} \frac{1}{p^{(jn/2)+j}} \prod_{B_i \geq p^{m-j}} \frac{2B_i}{p^{m-j}}. \end{aligned} \quad (8)$$

We now restrict our attention to the case where $\mathcal{B}$ is cube, that is, $B_i = B$, $1 \leq i \leq n$. Say $p^{j_0} \leq B < p^{j_0+1}$ for some $j_0 \in \mathbb{N}$, $1 \leq j_0 < m$. Then

$$B \geq p^{m-j} \iff m-j \leq j_0 \iff j \geq m-j_0.$$

Continuing from (8), we get

$$\begin{aligned} |Error| &\leq p^{mn/2} |\mathcal{B}| \sum_{j=m-j_0}^{m-1} \frac{2^n |\mathcal{B}|}{p^{(jn/2)+j} p^{nm-nj}} + p^{mn/2} |\mathcal{B}| \sum_{j=0}^{m-j_0-1} \frac{1}{p^{(jn/2)+j}} \\ &= \frac{2^n |\mathcal{B}|^2}{p^{mn/2}} \sum_{j=m-j_0}^{m-1} p^{(n/2)j-j} + p^{mn/2} |\mathcal{B}| \sum_{j=0}^{m-j_0-1} \frac{1}{p^{(jn/2)+j}} \\ &< \frac{2^n |\mathcal{B}|^2}{p^{mn/2}} p^{[(n/2)-1](m-1)} \cdot 2 + p^{mn/2} |\mathcal{B}| \cdot 2 \end{aligned}$$

$$\begin{aligned}
&= 2^{n+1} |\mathcal{B}|^2 p^{-m-(n/2)+1} + 2p^{mn/2} |\mathcal{B}| \\
&= \frac{2^{n+1} |\mathcal{B}|^2}{p^{m+(n/2)-1}} + 2p^{mn/2} |\mathcal{B}|.
\end{aligned}$$

Proposition 3.1 Suppose that $m \geq 2$, $n \geq 4$, n even. Then for any cube $\mathcal{B}$ centered at the origin,

$$\sum_{\mathbf{x} \in V} \alpha(\mathbf{x}) \geq \frac{|\mathcal{B}|^2}{p^m} - |\text{Error}|,$$

where

$$|\text{Error}| \leq \underbrace{\frac{2^{n+1} |\mathcal{B}|^2}{p^{m+(n/2)-1}}}_{\text{Error 1}} + \underbrace{2p^{mn/2} |\mathcal{B}|}_{\text{Error 2}}.$$

We compare *Error 1* and *Error 2* in Proposition 3.1 to the main term $|\mathcal{B}|^2/p^m$. In order to make the left-hand side positive, we make each error term less than $1/3$ of the main term. For the error term *Error 1*, we need

$$\begin{aligned}
\frac{2^{n+1} |\mathcal{B}|^2}{p^{m+(n/2)-1}} &< \frac{1}{3} \frac{|\mathcal{B}|^2}{p^m} \iff p^{(n/2)-1} > 3 \cdot 2^{n+1} \\
&\iff p > 2^{2(n+1)/(n-2)} \cdot 3^{2/(n-2)}. \tag{9}
\end{aligned}$$

For the error term *Error 2*,

$$\begin{aligned}
2p^{mn/2} |\mathcal{B}| &< \frac{1}{3} \frac{|\mathcal{B}|^2}{p^m} \iff |\mathcal{B}| > 3 \cdot 2p^{(mn/2)+m} \\
&\iff B > 6^{1/n} p^{(m/2)+(m/n)}. \tag{10}
\end{aligned}$$

Collecting together the two criteria (9) and (10), we obtain

Theorem 3.2 Suppose that $m \geq 2$, $n \geq 4$, n even. Then for any cube $\mathcal{B}$ centered at the origin, if (9), (10) hold, then

$$\sum_{\mathbf{x} \in V} \alpha(\mathbf{x}) \geq \frac{1}{3} \frac{|\mathcal{B}|^2}{p^m}.$$

In particular

$$|V \cap (\mathcal{B} + \mathcal{B})| \geq \frac{|\mathcal{B}|}{3p^m}.$$

It remains to prove under the hypothesis of Theorem 3.2, the existence of primitive solutions of the congruence (1). Recall $\mathbf{x}$ is called primitive if $\gcd(x_1, \dots, x_n, p) = 1$. We shall write $p|\mathbf{x}$ for imprimitive points. Thus we have to prove

$$\sum_{\mathbf{x} \in V} \alpha(\mathbf{x}) > \sum_{\substack{\mathbf{x} \in V \\ p|\mathbf{x}}} \alpha(\mathbf{x}).$$

Corollary 3.1 Suppose $m \geq 2$, $n > m$, n even, $p > 2^{2(n+1)/(n-2)} 3^{2/(n-2)}$ and $\mathcal{B}$ is a cube with $B > 6^{1/n} p^{(m/2)+(m/n)}$. Then $\mathcal{B} + \mathcal{B}$ contains a primitive solution of (2).

Proof. As in the proof of Lemma 3.1 with $j = m - 1$, we have

$$\begin{aligned} \sum_{\substack{\mathbf{x} \in V \\ p|\mathbf{x}}} \alpha(\mathbf{x}) &= \sum_{\substack{p|x_i, \\ 1 \leq i \leq n}} \alpha(\mathbf{x}) < \sum_{\substack{\mathbf{u} \in \mathcal{B} \\ \mathbf{u} + \mathbf{v} \equiv 0 \pmod{p}}} \sum_{\mathbf{v} \in \mathcal{B}} 1 \\ &\leq \prod_{i=1}^n B_i \left(\left\lfloor \frac{B_i}{p} \right\rfloor + 1 \right) \\ &= B^n \left(\left\lfloor \frac{B}{p} \right\rfloor + 1 \right)^n \\ &< \frac{B^{2n}}{p^n} \left(1 + \frac{p}{B} \right)^n \\ &\leq \frac{B^{2n}}{p^n} (1 + \varepsilon)^n, \end{aligned}$$

where $\varepsilon < p^{-[(m/2)+(m/n)]+1}$. However according to Theorem 3.2 we have,

$$\sum_{\mathbf{x} \in V} \alpha(\mathbf{x}) \geq \frac{|\mathcal{B}|^2}{3p^m}.$$

Hence it follows that

$$\sum_{\substack{\mathbf{x} \in V \\ p \nmid \mathbf{x}}} \alpha(\mathbf{x}) \geq \frac{|\mathcal{B}|^2}{3p^m} - \sum_{\substack{\mathbf{x} \in V \\ p|\mathbf{x}}} \alpha(\mathbf{x}) \geq \frac{|\mathcal{B}|^2}{3p^m} - \frac{B^{2n}}{p^n} (1 + \varepsilon)^n > 0,$$

by our hypotheses on the size of p . This completes the proof. $\square$

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