(k, n; f) – Arcs of Type (1, n) in

PG(2, q), with q ≤ 8

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Abstract

In this paper we discussed the existence of (k, n; f) – arcs of type (1, n) in the projective plane of order q ≤ 8; with \( \text{Im}(f) = \{0, 1, \omega\} \) and \( \omega \in \{2, 3, ..., n - 1\} \), that different from G. Raguso and L. Rella (k, n; f) – arcs [12] and we deduced that there are no such (k, n; f) – arcs of type (1, n) with \( \text{Im}(f) = \{0, 1, \omega\} \) in PG(2, q) for all \( \omega \in \{2, 3, ..., n - 1\} \), q ≤ 7 and for all n satisfy \( (n - 1)|q \). But when q = 8, we have only (19, 9; f) – arc of type (1, 9), with \( \text{Im}(f) = \{0, 1, 4\} \), such that the points of weight 4 form an oval and the points of weight 1 are the points of some 0–secant of this oval.
Keywords: (k, n; f) – arcs, weighted arcs, PG(2, q), weighting line.

Introduction

In 1978, A. Barlotti [2] presented the notion of a $(k, n; \{w_i\})$ – set of kind $s$. The $(k, n; \{w_i\})$ – set of kind 2 in a projective plane, also called $(k, n; \{w_i\})$ – arcs, where studied by M. Barnabei [3]. The $(k, n; f)$ – arcs of type $(n - 2, n)$ in a finite projective plane was developed by E. D’Agostini [5] in 1979. B. J. Wilson [13] gives studying to the $(k, n; f)$ – arcs of type $(n - 3, n)$ in a finite projective plane and continue in this study the authors F. K. Hameed [7] and M. Y. Abass [1]. Also the $(k, n; f)$ – arcs of type $(n - 5, n)$ was developed by R. D. Mahmood [11] and completed by F. K. Hameed and others [9]. The notion of $(k, n; f)$ – arcs of type $(1, n)$ introduced by G. Raguso and L. Rella [12]. Also, F. K. Hameed [8] generalize the results of the monoidal arcs in PG(2, q).

In this paper we investigated the $(k, n; f)$ – arcs of type $(1, n)$ and its properties in the finite projective plane of order $q \leq 8$.

1. Preliminaries

We will denote by $\text{PG}(2, q)$ the projective desarguesian plane of order $q = p^h$, by $\mathcal{P}$ the set of all points of the plane and by $\mathcal{R}$ the set of all lines of the plane. Then $\text{PG}(2, q)$ have $q^2 + q + 1$ points and $q^2 + q + 1$ lines. On each line lie $q + 1$ points and through every point there pass $q + 1$ lines.

Definition 1.1. [10]

$(k, n)$ – arc in $\text{PG}(2, q)$ is a set of $k$ points no $n + 1$ of which are collinear, where $n \geq 2$. Write simply $k$ – arc for $(k, 2)$ – arc.

Definition 1.2. [10]

A line $\ell$ in $\text{PG}(2, q)$ is an $i$ – secant of a $(k, n)$ – arc $\mathcal{K}$ if $|\ell \cap \mathcal{K}| = i$. Let $\tau_i$ denote the total number of $i$ – secants to $\mathcal{K}$ in $\text{PG}(2, q)$, then the type of $\mathcal{K}$ is defined by $(\tau_0, \tau_1, ..., \tau_n)$. 
Lemma 1.1. [10]
For a $(k, n) - \text{arc} \ K$, the following equations hold:

1. $\sum_{i=0}^{n} \tau_i = q^2 + q + 1$ ;
2. $\sum_{i=1}^{n} i \tau_i = k(q + 1)$ ;
3. $\sum_{i=2}^{n} \frac{i(i-1)}{2} \tau_i = \frac{k(k-1)}{2}$.

For any function $f$ from $\mathcal{P}$ to the set of the natural numbers $\mathbb{N}$ we will say that $f(P)$ is the weight of the point $P$. From such $f$ we may define a function $F$ from $\mathcal{R}$ to $\mathbb{N}$ in the following way:

$$F(r) = \sum_{P \in r} f(P)$$

and we will say that $F(r)$ is the weight of the line $r$. Moreover, if $F(r) = j$ we will also say that $r$ is a "$-\text{weighting}"$ line.

Definition 1.3. [6]
A $(k, n; f) - \text{arc} \ K$ in $PG(2, q)$ is a function $f : \mathcal{P} \rightarrow \mathbb{N}$ such that $k = |\text{support of } f (\text{the points of non-zero weight})|$ and $n = \max F$.

Let us remark that an ordinary $(k, n) - \text{arc}$ is a $(k, n; f) - \text{arc}$ with $\text{Im}(f) = \{0, 1\}$.

Let us use the following notation:

$\omega = \max f$, $W = \sum_{P \in \mathcal{P}} f(P)$ and $W$ will be called the weight of $K$.

$L_i = f^{-1}(i)$ and $l_i = |L_i|$, $i = 0, 1, ..., \omega$.

$[M]$ indicates the set of all lines through the point $M$.

Definition 1.5. [8]
A $(k, n; f) - \text{arc} \ K$ is called monoidal if $\text{Im} f = \{0, 1, \omega\}$ and $l_\omega = 1$.

Definition 1.6. [6]
The characters of a $(k, n; f) - \text{arc} \ K$ are the integers $t_j = |F^{-1}(j)|$ for $j = 0, 1, ..., n$.

Definition 1.7. [6]
The type of a $(k, n; f) - \text{arc} \ K$ is the set of $\text{Im} F$. To write explicitly the type of $K$ we can use the sequence $(n_1, ..., n_\rho)$ where $n_\lambda \in \text{Im} F$, $\lambda = 1, ..., \rho$ and $n_1 < n_2 < \cdots < n_\rho = n$.

It is well known from [6] that:

$$k = \sum_{i=1}^{\omega} l_i.$$  \hspace{1cm} (1.1)
\[ W = \sum_{i=1}^{\omega} i \cdot l_i . \]  
(1.2)

\[ \sum_{r \in [M]} F(r) = W + qf(M) . \]  
(1.3)

\[ |\text{Im} F| \geq 2 . \]  
(1.4)

A useful result, mentioned in [6], is the following:

If there exists a point \( P \) of a \((k, n; f)\)-arc \( K \) such that every line through it is a \( n \)-weighting line, then

\[ P \in L_\omega \]  
(1.5)

If \( M \in L_\omega \) and \( u \in [M] \), then \( F(u) = n \).

Hence

\[ W \leq (n - \omega)q + n . \]  
(1.6)

An arc with weight such that the equality holds, is called maximal. Of course, a maximal arc is also such that through a point of maximal weight there pass only \( n \)-weighting lines.

Finally we shall recall [6] the following relations concerning the characters of a \((k, n; f)\)-arc \( K \):

\[ \sum_{j=0}^{n} t_j = q^2 + q + 1 \]  
(1.7)

\[ \sum_{j=1}^{n} j t_j = (q + 1)W \]  
(1.8)

\[ \sum_{j=2}^{n} \binom{j}{2} t_j = \binom{W}{2} + q \sum_{i=2}^{\omega} \binom{i}{2} l_i . \]  
(1.9)

2. \((k, n; f)\)-arcs of type \((m, n)\)

From now on, \( K \) shall denote a \((k, n; f)\)-arc of type \((m, n)\), where \(|\text{Im} f| \geq 3\). Let firstly state the following:

**Lemma 2.1.**[7] The weight \( W \) of a \((k, n; f)\)-arc of type \((m, n)\) satisfies:

\[ m(q + 1) \leq W \leq (n - \omega)q + n . \]

We call arcs for which the values in lemma (2.1) are attained, maximal and minimal \((k, n; f)\)-arcs of type \((m, n)\) respectively.

**Theorem 2.1.**[7]

Let \( K \) be a \((k, n; f)\)-arc of type \((m, n)\), \( m > 0 \) and let \( \nu_m^s \) and \( \nu_n^s \) respectively the number of lines of weight \( m \) and the number of lines of weight \( n \) passing through a point of weight \( s \). Then

\[ (n - m)\nu_m^s = (n - s)(q + 1) - (W - s) ; \]

\[ (n - m)\nu_n^s = (W - s) - (m - s)(q + 1) . \]
Theorem 2.2.[7]
A necessary conditions for the existence of a \((k,n;f)\) – arc \(K\) of type \((m,n)\), \(m > 0\) are that:
\((1)\) \(q \equiv 0 \mod (n-m)\); \((2)\) \(\omega \leq n - m\); \((3)\) \(m \leq n - 2\).

3. \((k,n;f)\) – arcs of type \((1,n)\)

In this section we discussed \((k,n;f)\) – arcs of type \((1,n)\). Then we get the following properties as in [12]:
\((a)\) \(\text{Im} f \subseteq \{0, 1, \omega\}\);
\((b)\) The weight \(W\) of a \((k,n;f)\) – arc of type \((1,n)\) satisfies:
\[q + 1 \leq W \leq (n - \omega)(q + 1) + \omega = (n - \omega)q + n.\]
\((c)\) From theorem (2.1) we have:
\[
\begin{align*}
v_i^s &= \frac{q(n - s) - W + n}{n - 1} \\
v_n^s &= \frac{q(s - 1) + W - 1}{n - 1}
\end{align*}
\]
\((d)\) From theorem (2.2) we have: \(q \equiv 0 \mod (n - 1)\) and \(\omega \leq n - 1\).
\((e)\) From the equations (1.7) and (1.8), the characters of a \((k,n;f)\) – arc of type \((1,n)\) are given by:
\[
\begin{align*}
t_1 &= \frac{q + 1}{n - 1}\left(n \frac{q^2 + q + 1}{q + 1} - W\right) \\
t_n &= \frac{q + 1}{n - 1}\left(W - \frac{q^2 + q + 1}{q + 1}\right)
\end{align*}
\]
\((f)\) From the equation (1.9), the weight \(W\) of the plane must be a root of the following equation of degree two:
\[W^2 - W[n(q + 1) + 1] + n(q^2 + q + 1) + q\omega(\omega - 1)t_\omega = 0.\] (3.1)

Proposition 3.1.[12]
A necessary condition for the existence in a projective plane \(\pi\) of order \(q\) of a \((k,n;f)\) – arc of type \((1,n)\) with \(\omega \geq 2\) is that the total weight of the plane is \(W = (n - \omega)q + n\).

Also from [12], we have the following:
From (3.3) it follows that:

\( \omega | nq, \quad (\omega - 1)(\omega q - n + 1) \) and so \( (\omega - 1)(q - n + 1) \).

4. (k, n; f) – arcs of type (1, n) in PG(2, q), with q = 2, 3, 4, 5, 7 and 8

In this section we test if there exist a (k, n; f) – arc of type (1, n) in PG(2, q), when q = 2, 3, 4, 5, 7 and 8, that distinct from the arcs of G. Raguso and L. Rella [11]. Then we partition this section into the following cases:

4.1. (k, n; f) – arcs of type (1, n) in PG(2, 2)

From section (3), part (d) we have \( q \equiv 0 \mod (n - 1) \) and \( \omega \leq n - 1 \). Then \( n = 3 \) and \( \omega = 2 \). But this case leads to the arcs of G. Raguso and L. Rella as in the following theorem:

**Theorem 4.1.1.** [12]

The (k, n; f) – arcs of PG(2, q), of type (1, n) with \( \omega = n - 1, n \geq 3 \) are precisely the monoidal (k, n; f) – arcs having as points of weight 1 those of a line.

4.2. (k, n; f) – arcs of type (1, n) in PG(2, 3)

In this case we have \( n = 4 \) and \( \omega \leq 3 \). Since \( \omega > 1 \), then we can take \( \omega = 2 \) and for this value we have the following theorem:
Theorem 4.2.1.[12]
In a projective plane $\pi$ of odd order $q$ the $(k, n; f)$ – arcs of type $(1, n)$ with $n = q + 1$ and $\omega = 2$ are precisely those which have as points of weight 1 the points of an oval $C$ and as points of weight $\omega$ the interior points of $C$.
Also we can take $\omega = 3$ and also we obtain the G. Raguso and L. Rella arc as in the Theorem (4.1.1).

4.3. $(k, n; f)$ – arcs of type $(1, n)$ in $\text{PG}(2, 4)$
In this case we have $n = 3$ and $n = 5$, but $n = 3$ cannot be occur in $\text{PG}(2, 4)$ according to the following theorem:

Theorem 4.3.1.[12]
There do not exist $(k, 3; f)$ – arcs of type $(1, 3)$ in $\text{PG}(2, q), q \neq 2$.
Then remains $n = 5$ and this implies that $\omega \leq 4$.
$\Rightarrow \omega = 2 \Rightarrow \text{Im}(f) = \{0, 1, 2\}$; this case discussed in [12] as in the following theorem:

Theorem 4.3.2.[12]
In a projective plane $\pi$ of even order $q (\neq 2)$ the $(k, n; f)$ – arcs of type $(1, n)$ with $n = q + 1$ and $\omega = 2$ are precisely those which have as points of weight 1 the points of a line $r$ and as points of weight $\omega$ the points of $(\frac{q(q-1)}{2}, \frac{q}{2})$ – arc of type $(0, \frac{q}{2})$.
$\Rightarrow \omega = 3 \Rightarrow \text{Im}(f) = \{0, 1, 3\}$; this case impossible because $\omega \not \mid nq$, i.e. $3 \not \mid 5 * 4$.
$\Rightarrow \omega = 4 \Rightarrow \text{Im}(f) = \{0, 1, 4\}$; this case discussed in [12] as in the theorem (4.1.1).

4.4. $(k, n; f)$ – arcs of type $(1, n)$ in $\text{PG}(2, 5)$
In this case we have $n = 6$ and $\omega \leq 5$.
$\Rightarrow \omega = 2 \Rightarrow \text{Im}(f) = \{0, 1, 2\}$; this case discussed in [12] as in theorem (4.2.1).
$\Rightarrow \omega = 3 \Rightarrow \text{Im}(f) = \{0, 1, 3\}$; in this case we have the following:
The equations in (3.2), become:
\[
\begin{align*}
v_1^0 &= 3, & v_6^0 &= 3 \\
v_3^0 &= 2, & v_6^1 &= 4, & v_6^3 &= 6 \\
t_1 &= 12, & t_6 &= 19
\end{align*}
\]
(4.4.1)
Also the equations in (3.3) become:
\[ l_3 = 5 \Rightarrow l_1 = 6 \Rightarrow l_0 = 20 \quad (4.4.2) \]

Since \( n = 6 \) then there is no more than two points of weight 3 on a line. Then the points of weight 3 form 5-arc in PG(2, 5). This 5-arc must have \( \tau_1 + \tau_2 \leq t_6 \), but \( \tau_1 = 10, \tau_2 = 10 \Rightarrow \tau_1 + \tau_2 > t_6 = 19 \) and this contradiction. Then we have the following theorem:

**Theorem 4.4.1.** There is no \((11, 6; f)\) – arc of type \((1, 6)\) in PG(2, 5), when \( \text{Im}(f) = \{0, 1, 3\} \).

\[ \Rightarrow \omega = 4 \Rightarrow \text{Im}(f) = \{0, 1, 4\}; \text{this case impossible because } \omega \nmid nq, \text{ i.e. } 4 \nmid 6 \cdot 5. \]

\[ \Rightarrow \omega = 5 \Rightarrow \text{Im}(f) = \{0, 1, 5\}; \text{this case discussed in [12] as in the theorem (4.1.1).} \]

### 4.5. \((k, n; f)\) – arcs of type \((1, n)\) in PG(2, 7)

In this case we have \( n = 8 \) and \( \omega \leq 7 \).

\[ \Rightarrow \omega = 2 \Rightarrow \text{Im}(f) = \{0, 1, 2\}; \text{this case discussed in [12] as in theorem (4.2.1).} \]

\[ \Rightarrow \omega = 3 \Rightarrow \text{Im}(f) = \{0, 1, 3\}; \text{this case impossible because } \omega \nmid nq, \text{ i.e. } 3 \nmid 8 \cdot 7. \]

\[ \Rightarrow \omega = 4 \Rightarrow \text{Im}(f) = \{0, 1, 4\}; \text{in this case we have the following:} \]

The equations in (3.2), become:

\[
\begin{align*}
v_1^0 &= 4, \quad v_6^0 = 4 \\
v_1^1 &= 3, \quad v_6^1 = 5, \quad v_8^4 = 8
\end{align*}
\]

\[ t_1 = 24, \quad t_8 = 33 \quad (4.5.1) \]

Also the equations in (3.3) become:

\[ l_4 = 7 \Rightarrow l_1 = 8 \Rightarrow l_0 = 42 \quad (4.5.2) \]

Since \( n = 8 \) then there is no more than two points of weight 4 on a line. Then the points of weight 4 form 7-arc in PG(2, 7). This 7-arc must have \( \tau_1 + \tau_2 \leq t_8 \), because 1-secants and 2-secants of the points of weight 4 are 8-weighting lines but \( \tau_1 = 14, \tau_2 = 21 \Rightarrow \tau_1 + \tau_2 > t_8 = 33 \) and this contradiction. Then we have the following theorem:

**Theorem 4.5.1.** There is no \((15, 8; f)\) – arc of type \((1, 8)\) in PG(2, 7), when \( \text{Im}(f) = \{0, 1, 4\} \).

\[ \Rightarrow \omega = 5 \Rightarrow \text{Im}(f) = \{0, 1, 5\}; \text{this case impossible because } \omega \nmid nq, \text{ i.e. } 5 \nmid 8 \cdot 7. \]
\( (k, n; f) \) – arcs of type \((1,n)\)

\[ \Rightarrow \omega = 6 \Rightarrow \text{Im}(f) = \{0,1,6\}; \text{ this case impossible because } \omega \nmid nq, \text{ i.e. } 6 \nmid 8 \ast 7. \]

\[ \Rightarrow \omega = 7 \Rightarrow \text{Im}(f) = \{0,1,7\}; \text{ this case discussed in [11] as in the theorem (4.1.1).} \]

4.6. \( (k, n; f) \) – arcs of type \((1,n)\) in PG(2, 8)

In this case we have \( n = 5 \) and \( n = 9 \). For \( n = 5 \) we have \( \omega \leq 4 \), then

\[ \Rightarrow \omega = 2 \Rightarrow \text{Im}(f) = \{0, 1, 2\}; \text{ in this case we have the following:} \]

The equations in (3.2), become:

\[
\begin{align*}
    v_1^0 &= 4, & v_8^0 &= 5 \\
    v_1^1 &= 2, & v_5^3 &= 7, & v_5^2 &= 9 \\
    t_1 &= 26, & t_5 &= 47
\end{align*}
\]

(4.6.1)

Also the equations in (3.3) become:

\[ l_2 = 8 \Rightarrow l_1 = 13 \Rightarrow l_0 = 52 \] (4.6.2)

Since \( n = 5 \) then there is no more than two points of weight 2 on a line. Then the points of weight 2 form 8 – arc in PG(2, 8). Every 1 – secant and 2 – secant of 8 – arc (the points of weight 2) are 5 – weighting lines. Let \( S = \{P_1, P_2, \ldots, P_8\} \) be the points of weight 2 and let \( O \) be an oval in PG(2, 8), then the points of \( O \setminus S = \{P_9, P_{10}\} \) are not of weight 2, then \( v_{5}^{f(P_9)} \) and \( v_{5}^{f(P_{10})} \) are 5 or 7 as in equation (4.6.1), but through them there pass exactly eight 1 – secants of \( S \) that is \( v_{5}^{f(P_9)} = v_{5}^{f(P_{10})} = 8 \) and this is contradiction.

Then we have the following theorem:

**Theorem 4.6.1.** There is no \((21, 5; f)\) – arc of type \((1,5)\) in PG(2, 8), when \( \text{Im}(f) = \{0, 1, 2\} \).

\[ \Rightarrow \omega = 3 \Rightarrow \text{Im}(f) = \{0,1,3\}; \text{ this case impossible because } \omega \nmid nq, \text{ i.e. } 3 \nmid 5 \ast 8. \]

\[ \Rightarrow \omega = 4 \Rightarrow \text{Im}(f) = \{0,1,4\}; \text{ this case impossible because } (\omega - 1) \nmid (q - n + 1) \text{ i.e. } 3 \nmid (8 - 5 + 1 = 4). \]

For \( n = 9 \) we have \( \omega \leq 8 \), then

\[ \Rightarrow \omega = 2 \Rightarrow \text{Im}(f) = \{0,1,2\}; \text{ this case discussed in [11] as in the theorem (4.3.2).} \]

\[ \Rightarrow \omega = 3 \Rightarrow \text{Im}(f) = \{0, 1, 3\}; \text{ in this case we have the following:} \]
The equations in (3.2), become:
\[
\begin{align*}
v^0_1 &= 3 \\
v^0_2 &= 6 \\
v^1_1 &= 2 \\
v^3_2 &= 7 \\
v^3_3 &= 9 \\
t_1 &= 18 \\
t_9 &= 55
\end{align*}
\]
(4.6.3)

Also the equations in (3.3) become:
\[
l_3 = 16 \Rightarrow l_1 = 9 \Rightarrow l_0 = 48
\]
(4.6.4)

Since \( n = 9 \) then there is no more than three points of weight \( 3 \) on a line. Then the points of weight \( 3 \) form \((16, 3) – \text{arc}\) in PG(2, 8). This is contradiction because the maximum size of \((u, 3) – \text{arc}\) in PG(2, 8) is \( u = 15 \), as in [4].

**Theorem 4.6.2.** There is no \((25, 9; f) – \text{arc}\) of type \((1, 9)\) in PG(2, 8), when \( \text{Im}(f) = \{0, 1, 3\} \).

\[\Rightarrow \omega = 4 \Rightarrow \text{Im}(f) = \{0, 1, 4\} \] in this case we have the following:

The equations in (3.2), become:
\[
\begin{align*}
v^0_1 &= 4 \\
v^0_2 &= 5 \\
v^1_1 &= 3 \\
v^3_2 &= 6 \\
v^3_3 &= 9 \\
t_1 &= 27 \\
t_9 &= 46
\end{align*}
\]
(4.6.5)

Also the equations in (3.3) become:
\[
l_4 = 10 \Rightarrow l_1 = 9 \Rightarrow l_0 = 54
\]
(4.6.6)

Since \( n = 9 \) then there is no more than two points of weight \( 4 \) on a line. Then the points of weight \( 4 \) form 10 – \text{arc} (oval) in PG(2, 8). The oval in PG(2, 8) has no 1 – secant. It has only 2 – secants and 0 – secants and every 2 – secant is a 9 – weighting line, then we have 45 9 – weighting lines having two points of weight \( 4 \) but from (4.6.5) we have 46 9 – weighting lines \( (t_9 = 46) \), then we have a 9 – weighting line on which there is no points of weight \( 4 \), then the points of weight 1 are collinear on this line which is a 0 – secant of the oval.

**Theorem 4.6.3.** There is \((19, 9; f) – \text{arc}\) of type \((1, 9)\) in PG(2, 8), when \( \text{Im}(f) = \{0, 1, 4\} \), in which the points of weight \( 4 \) form an oval and the points of weight \( 1 \) are the points of some 0 – secant of this oval.

\[\Rightarrow \omega = 5 \Rightarrow \text{Im}(f) = \{0, 1, 5\} \] this case impossible because \( \omega \nmid nq \), i.e. \( 5 \nmid 9 \times 8 \).
\( \Rightarrow \omega = 6 \Rightarrow \text{Im}(f) = \{0,1,6\} \); in this case we have the following:

The equations in (3.2), become:

\[
\begin{align*}
\nu_1^0 &= 6, & \nu_0^0 &= 3 \\
\nu_1^1 &= 5, & \nu_0^1 &= 4, & \nu_0^6 &= 9 \\
\tau_1 &= 45, & \tau_0 &= 28 \\
\end{align*}
\]

(4.6.7)

Also the equations in (3.3) become:

\[
l_6 = 4 \Rightarrow l_1 = 9 \Rightarrow l_0 = 60
\]

(4.6.8)

Since \( n = 9 \) then there is no two points of weight 6 on a line. This is contradiction with the axiom of the projective geometry.

**Theorem 4.6.4.** There is no \((13, 9; f) – \text{arc of type } (1, 9)\) in PG(2, 8), when \(\text{Im}(f) = \{0, 1, 6\}\).

\( \Rightarrow \omega = 7 \Rightarrow \text{Im}(f) = \{0,1,7\} \); this case impossible because \( \omega \nmid nq \), i.e. \( 7 \nmid 9 \ast 8 \).

\( \Rightarrow \omega = 8 \Rightarrow \text{Im}(f) = \{0,1,8\} \); this case discussed in [12] as in the theorem (4.1.1).

**References**


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