Parametric Optimization for the Lipschitz Function

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Abstract

We consider the parametric minimization problem with a Lipschitz objective function. We propose an approach for solving the original problem in a finite number of steps in order to obtain a solution with a given accuracy.

Keywords: parametric optimization, discretization, algorithm, Lipschitz function

1 Introduction

Parametric optimization problem plays an important role in optimization theory and applications. There are many parametric utility maximization problems in communication network. In these problems time is treated as a parameter which can take continuous or discrete values. Some parametric network utility maximization (NUM) problem with discrete time has been considered in [1] where duality are employed. They allow the flow utilities, link capacities and routing matrices to vary over time, and they introduced the concept of delivery contracts, which couple the flow rate across the time. We can write the problems of network utility maximization with delivery contact as follows the parametric minimization problem.

$$\max\{U(f, t) | f \in M_D(t)\}, \quad t \in \{1, 2, \dots, T\},$$

where

$$\begin{aligned} M_D(t) &= \{f \in R^n | R(t)f(t) \geq c(t), \\ C_j(t)f_j(t) &\geq q_j(t), j \in \{1, \dots, n\}, 0 \leq f(t) \leq f^{max}\}, \end{aligned}$$

$R(t)_{ij} \in R^{m \times n}$ is routing matrix, $c(t)$ is the vector of the link capacities, $f_j(t)$ denotes the flow rate at which flow j , $C_j(t)$, contract matrix $q_j(t)$ delivery contract at time t , and $U(f, t)$ is the total utility, over all flows and at time t and defined by $U(f, t) = \sum_{j=1}^n U_j(f_j(t), t)$ with the utility $U_j(f_j(t), t)$ derived by flow rate $f_j(t)$.

2 General Parametric Optimization Problem

We consider the following one-parametric optimization problem $P(t)$:

$$\min \{f(x, t) | x \in M(t)\}, \quad t \in [t_A, t_B], \quad (2.1)$$

where

$$M(t) = \{x \in R^n | h_i(x, t) = 0, i \in I, g_j(x, t) \leq 0, j \in J\},$$

with $I = \{1, \dots, m\}$, $J = \{1, \dots, s\}$, $m < n$, and $t_A < t_B$.

It is assumed that [4]:

(E1) There exists a continuous function $x : [t_A, t_B] \rightarrow R^n$ such that $x(t)$ is local minimizer for $P(t)$.

(E2) $x(t_A)$ is know.

(V1) There is neighborhood U of $\{(x(t), t) \in R^n \times [t_A, t_B]\}$ such that for all $(x, t) \in U$ the functions f , h_i , and g_j ($i \in I$, $j \in J$) are twice continuously differentiable with respect to their first variable x .

(V2) The LICO is satisfied at the $x(t)$ for each $t \in [t_A, t_B]$.

Assumptions (V1) and (V2) together imply that there exist functions $\lambda_i : [t_A, t_B] \rightarrow R^m$, $\mu_j \rightarrow R^s$, which are uniquely defined, so that $(x(t), \lambda(t), \mu(t))$ satisfies the KKT conditions. Additionally, we need the following second order sufficient condition:

(V3) $D_x^2 L(x(t))|_{T_{x(t)}^+ M(t)}$ is positive defined $\forall t \in [t_A, t_B]$, where $z = (x, t)$, (in particular $z(t) = (x(t), t)$)

$$T_{x(t)}^+ M(t) = \bigcap_{i \in I} \text{Ker} D_x h_i(z(t)) \cap \bigcap_{j \in J_{z(t)}^+} \text{Ker} D_x g_j(z(t)),$$

$$J_{z(t)}^+ = \{j \in J | g_j(z(t)) = 0, \mu_j(t) > 0\}$$

and $L(z)$ denotes Lagrangian function at z , i.e.,

$$L(z) = f(z) + \sum_{i \in I} \lambda_i h_i(z) + \sum_{j \in J} \mu_j g_j(z).$$

With (E2) we have a starting point for the pathfollowing process. We determine a KKT point $(v, t) = (x, \lambda, \mu, t)$ of $P(t)$ by means of an algorithm of the following:

(A) Start with v^i , let v^{i+1} be a KKT point of the problem $P(v^i, t)$, $i = 1, 2, \dots$ where $P(\bar{v}, t)$:

$$\begin{aligned} \min \{ & \varphi(x, \bar{v}, t) | h_i(\bar{x}, t) + D_x h_i(\bar{x}, t)(x - \bar{x}) \leq 0, i \in I, \\ & g_j(\bar{x}, t) + D_x g_j(\bar{x}, t)(x - \bar{x}) \leq 0, j \in J. \} \end{aligned} \quad (2.2)$$

with

$$\varphi(x, \bar{v}, t) : R^n \times R^{n+m+s} \times [t_A, t_B] \rightarrow R$$

There are several choices of functions φ . For example, φ can be as follows.

$$\begin{aligned} \varphi(x, \bar{v}, t) = f(x, t) + \sum_{i \in I} \bar{\lambda}_i (h_i(x, t) - h_i(\bar{x}, t) - D_x h_i(\bar{x}, t)(x - \bar{x})) + \\ \sum_{j \in J} \bar{\mu}_j (g_j(x, t) - g_j(\bar{x}, t) - D_x g_j(\bar{x}, t)(x - \bar{x})). \end{aligned}$$

The pathfollowing algorithm for solving problem (2.1) is given [4].

Now we explain the pathfollowing process outlined above. Assume that a suitable partition $[t_A, t_B]$ is given:

$$t_A = t_0 \leq \dots \leq t_i \leq t_{i+1} \leq \dots \leq t_N \quad (2.3)$$

and the point $v(t_A)$, compute v_k , $k = 1, \dots, N$, as a KKT point of

$$P(v_{k-1}, t_k) \quad (2.4)$$

with $P(\cdot, \cdot)$ from (2.2).

Lemma 2.1 *Assume (E1), (V1) and (V2). Then the function $v : [t_A, t_B] \rightarrow R^{n+m+s}$ defined by the KKT points of $P(t)$ is continuous.*

3 Parametric Minimization with Lipschitz Utility Function

We consider the parametric minimization problem:

$$\min f(x, t), \quad t \in [t_A, t_B]$$

subject to constraint:

$$Ax \leq b,$$

where $f : R^n \times [t_A, t_B] \rightarrow R$ is the Lipschitz function with respect to x , A is a matrix of $(m \times n)$, $b \in R^m$. Denote by $M = \{x \in R^n | Ax \leq b\}$.

Definition 3.1 (Lipschitz continuity in time). *The function $f(x, t)$ is said to be locally Lipschitz continuous in t if for some $\Delta > 0$ there exists $\alpha \geq 0$ such that*

$$|f(x, t) - f(x, \hat{t})| \leq \alpha |t - \hat{t}|$$

$t, \hat{t} \in [t_A, t_B]$ with $|t - \hat{t}| \leq \Delta$, $x \in M(t)$. The constant α is called the Lipschitz modulus.

Lemma 3.2. Assume that the function $f(x, t)$ is the locally Lipschitz continuous in t with modulus α for each $x \in M$ then there always exists a discretization

$$t_A = t_0 < t_1 < \dots < t_i < t_{i+1} < \dots < t_N = t_B$$

Proof: We discretize $[t_A, t_B]$ in the following way

$$t_A = t_0 < t_1 < \dots < t_i < t_{i+1} < \dots < t_N = t_B$$

such that $0 < t_k - t_{k-1} \leq \Delta$, $k = 1, \dots, N$ with $\Delta = \frac{t_B - t_A}{N}$.

It is clear, for any $t \in [t_A, t_B]$, there is $k \in 1, \dots, N$ such that $t \in [t_{k-1}, t_k]$. Consequently,

$$|t - t_{k-1}| \leq \Delta, \quad k = 1, \dots, N$$

Due to condition, there exists $\alpha > 0$ such that

$$|f(x(\hat{t}), \hat{t}) - f(x(t), t)| < \alpha |t - \hat{t}|, \quad \forall t, \hat{t} \in [t_A, t_B].$$

Now we choose N as follows:

$$\varepsilon = \alpha \frac{t_A - t_B}{N}.$$

Now we take any $t \in [t_A, t_B]$ and compute

$$|f(x^*(t), t) - f(x^*(t_k), t_k)| \leq \alpha |t - t_k| \leq \alpha \frac{t_A - t_B}{N} = \varepsilon$$

which proves the lemma.

4 Numerical results

In numerical experiments we have considered the problem of the following types:

$$\min\{f(x, t) = \langle Cx, x \rangle - \langle d, x \rangle t \mid Ax \leq b\}, \quad t \in [t_A, t_B], \quad (4.1)$$

where C is a positive defined matrix of $(n \times n)$, $d \in R^n$, A is a matrix of $m \times n$ and $b \in R^m$.

At each discretization t_i , $i = 0, \dots, N$ problem (4.1) has been solved by the conditional gradient methods. In all problems ε accuracy was taken as $\varepsilon = 0.001$.

Problem 1. $t_A = 4$, $t_B = 8$, $N = 12$,

$$C = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \quad d = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

$$A = \begin{pmatrix} -1 & -2 & -2 \\ 1 & 2 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 72 \end{pmatrix}.$$

#	t_i	x_1	x_2	x_3
0	4	0.0001	0	1.9999
1	4.3333	0	0	2.1666
2	4.6667	0	0	2.3333
3	5	-0.0004	0	2.5003
4	5.3333	-0.0002	0	2.6669
5	5.6667	-0.0001	0	2.8335
6	6	0	0.0001	3.0002
7	6.3333	0	0	3.1666
8	6.6667	0	0	3.3333
9	7.0000	-0.0001	0	3.5000
10	7.3333	0	0	3.6667
11	7.6667	-0.0001	-0.0001	3.8334
12	8.0000	-0.0001	-0.0001	4.0001
Optimum				
-12.0000				
-14.0833				
-16.3333				
-18.7500				
-21.3333				
-24.0833				
-27.0000				
-30.0833				
-33.3333				
-36.7500				
-40.3333				
-44.0833				
-48.0000				

Problem 2. $t_A = 5$, $t_B = 13$, $N = 18$,

$$C = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad d = \begin{pmatrix} 5 \\ 6 \\ 7 \\ -1 \end{pmatrix},$$

$$A = \begin{pmatrix} 2 & -3 & 1 & 4 \\ -3 & 4 & 2 & 1 \\ 4 & 6 & 7 & -2 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 13 \\ 4 \end{pmatrix}.$$

#	t_i
0	5
1	5.4444
2	5.8889
3	6.3333
4	6.7778
5	7.2222
6	7.6667
7	8.1111
8	8.5556
9	9
10	9.4444
11	9.8889
12	10.3333
13	10.7778
14	11.2222
15	11.6667
16	12.1111
17	12.5556
18	13

x_1	x_2	x_3	x_4
0.9440	-0.2941	0.2805	-0.0127
0.9868	-0.2513	0.2262	0.0116
1.0297	-0.2085	0.1719	0.0358
1.0726	-0.1657	0.1177	0.0601
1.1154	-0.1228	0.0634	0.0843
1.1583	-0.0800	0.0092	0.1085
1.2011	-0.0372	-0.0451	0.1328
1.2440	0.0056	-0.0993	0.1570
1.2869	0.0484	-0.1536	0.1813
1.3297	0.0912	-0.2079	0.2055
1.3726	0.1340	-0.2621	0.2297
1.4154	0.1768	-0.3164	0.2540
1.4583	0.2196	-0.3706	0.2782
1.5012	0.2624	-0.4249	0.3025
1.5440	0.3052	-0.4791	0.3267
1.5869	0.3480	-0.5334	0.3509
1.6297	0.3908	-0.5877	0.3752
1.6728	0.4339	-0.6422	0.3996
1.7156	0.4767	-0.6964	0.4238

Optimum
-21.5628
-23.7694
-26.0059
-28.2721
-30.5682
-32.8941
-35.2498
-37.6354
-40.0507
-42.4959
-44.9709
-47.4758
-50.0104
-52.5749
-55.1692
-57.7933
-60.4472
-63.1310
-65.8446

Problem 3. $t_A = 1$, $t_B = 9$, $N = 23$,

$$C = \begin{pmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 5 & 4 & 3 & 2 \\ 3 & 4 & 5 & 4 & 3 \\ 2 & 3 & 4 & 5 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \quad d = \begin{pmatrix} 7 \\ -8 \\ 9 \\ 3 \\ 1 \end{pmatrix},$$

$$A = \begin{pmatrix} 2 & 7 & 5 & 6 & 4 \\ -3 & 1 & 2 & 4 & 5 \\ 4 & 2 & 3 & 1 & 6 \\ 2 & 1 & 3 & 4 & 7 \end{pmatrix}, \quad b = \begin{pmatrix} 13 \\ 9 \\ 7 \\ 8 \end{pmatrix}.$$

#	t_i	x_1	x_2	x_3	x_4	x_5
0	1	3.4621	-7.4917	5.2418	0.1858	-1.2960
1	1.3478	4.4902	-9.9535	6.9208	0.5869	-2.0672
2	1.6957	5.5070	-12.3478	8.6542	0.8407	-2.8559
3	2.0435	6.5164	-14.6979	10.4238	0.9971	-3.6564
4	2.3913	7.5259	-17.0479	12.1933	1.1535	-4.4568
5	2.7391	8.5353	-19.3979	13.9627	1.3100	-5.2572
6	3.087	9.5447	-21.7479	15.7322	1.4664	-6.0577
7	3.4348	10.5541	-24.0979	17.5017	1.6228	-6.8581
8	3.7826	11.5635	-26.4479	19.2711	1.7792	-7.6585
9	4.1304	12.5729	-28.7979	21.0406	1.9356	-8.4589
10	4.4783	13.5824	-31.1479	22.8101	2.0920	-9.2593
11	4.8261	14.5918	-33.4979	24.5795	2.2485	-10.0597
12	5.1739	15.6012	-35.8479	26.3490	2.4049	-10.8601
13	5.5217	16.6106	-38.1979	28.1185	2.5613	-11.6606
14	5.8696	17.6200	-40.5479	29.8879	2.7177	-12.4610
15	6.2174	18.6295	-42.8979	31.6574	2.8741	-13.2614
16	6.5652	19.6389	-45.2480	33.4269	3.0305	-14.0618
17	6.913	20.6483	-47.5980	35.1963	3.1870	-14.8622
18	7.2609	21.6577	-49.9480	36.9658	3.3434	-15.6626
19	7.6087	22.6671	-52.2980	38.7353	3.4998	-16.4630
20	7.9565	23.6766	-54.6480	40.5047	3.6562	-17.2635
21	8.3043	24.6860	-56.9980	42.2742	3.8126	-18.0639
22	8.6522	25.6954	-59.3480	44.0437	3.9691	-18.8643
23	9	26.7048	-61.6980	45.8131	4.1255	-19.6647
Optimum						
		-0.0677				
		-0.1205				
		-0.1880				
		-0.2699				
		-0.3663				
		-0.4771				
		-0.6023				
		-0.7420				
		-0.8960				
		-1.0645				
		-1.2474				
		-1.4447				
		-1.6565				
		-1.8826				
		-2.1232				
		-2.3782				
		-2.6476				
		-2.9315				
		-3.2298				
		-3.5424				
		-3.8695				
		-4.2111				
		-4.5670				
		-4.9374				

Problem 4. $t_A = 3$, $t_B = 7$, $N = 31$,

$$C = \begin{pmatrix} 6 & 5 & 4 & 3 & 2 & 1 \\ 5 & 6 & 5 & 4 & 3 & 2 \\ 4 & 5 & 6 & 5 & 4 & 3 \\ 3 & 4 & 5 & 6 & 5 & 4 \\ 2 & 3 & 4 & 5 & 6 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}, \quad d = \begin{pmatrix} 7 \\ -8 \\ 9 \\ 3 \\ 1 \\ 12 \end{pmatrix},$$

$$A = \begin{pmatrix} 2 & 7 & 5 & 6 & 4 & 1 \\ -3 & 1 & 2 & 4 & 5 & 7 \\ 4 & 2 & 3 & 1 & 6 & 8 \\ 2 & 1 & 3 & 4 & 7 & 5 \end{pmatrix}, \quad b = \begin{pmatrix} 13 \\ 9 \\ 7 \\ 8 \end{pmatrix}.$$

#	t_i	x_1	x_2	x_3	x_4	x_5	x_6	Optimum
0	3.0000	8.8036	-20.3800	13.6301	5.4467	-13.3698	5.8034	-0.6468
1	3.1290	9.1632	-21.2411	14.2009	5.7170	-13.9603	6.0339	-0.7021
2	3.2581	9.5227	-22.1022	14.7717	5.9872	-14.5507	6.2645	-0.7597
3	3.3871	9.8823	-22.9634	15.3425	6.2575	-15.1412	6.4950	-0.8196
4	3.5161	10.2419	-23.8245	15.9133	6.5278	-15.7317	6.7255	-0.8817
5	3.6452	10.6014	-24.6856	16.4841	6.7980	-16.3222	6.9561	-0.9460
6	3.7742	10.9610	-25.5467	17.0549	7.0683	-16.9127	7.1866	-1.0127
7	3.9032	11.3205	-26.4079	17.6257	7.3386	-17.5032	7.4171	-1.0815
8	4.0323	11.6801	-27.2690	18.1965	7.6088	-18.0937	7.6477	-1.1527
9	4.1613	12.0397	-28.1301	18.7673	7.8791	-18.6842	7.8782	-1.2261
10	4.2903	12.3992	-28.9913	19.3381	8.1494	-19.2747	8.1087	-1.3018
11	4.4194	12.7588	-29.8524	19.9089	8.4196	-19.8652	8.3393	-1.3797
12	4.5484	13.1183	-30.7135	20.4797	8.6899	-20.4557	8.5698	-1.4599
13	4.6774	13.4779	-31.5746	21.0505	8.9602	-21.0461	8.8004	-1.5424
14	4.8065	13.8375	-32.4358	21.6213	9.2304	-21.6366	9.0309	-1.6271
15	4.9355	14.1970	-33.2969	22.1921	9.5007	-22.2271	9.2614	-1.7141
16	5.0645	14.5566	-34.1580	22.7629	9.7710	-22.8176	9.4920	-1.8033
17	5.1935	14.9161	-35.0192	23.3337	10.0412	-23.4081	9.7225	-1.8948
18	5.3226	15.2757	-35.8803	23.9045	10.3115	-23.9986	9.9530	-1.9886
19	5.4516	15.6353	-36.7414	24.4753	10.5818	-24.5891	10.1836	-2.0846
20	5.5806	15.9948	-37.6025	25.0461	10.8520	-25.1796	10.4141	-2.1829
21	5.7097	16.3544	-38.4637	25.6170	11.1223	-25.7701	10.6446	-2.2835
22	5.8387	16.7139	-39.3248	26.1878	11.3926	-26.3606	10.8752	-2.3863
23	5.9677	17.0735	-40.1859	26.7586	11.6629	-26.9511	11.1057	-2.4914
24	6.0968	17.4331	-41.0471	27.3294	11.9331	-27.5415	11.3362	-2.5987
25	6.2258	17.7926	-41.9082	27.9002	12.2034	-28.1320	11.5668	-2.7083
26	6.3548	18.1522	-42.7693	28.4710	12.4737	-28.7225	11.7973	-2.8202
27	6.4839	18.5118	-43.6305	29.0418	12.7439	-29.3130	12.0278	-2.9343
28	6.6129	18.8713	-44.4916	29.6126	13.0142	-29.9035	12.2584	-3.0507
29	6.7419	19.2309	-45.3527	30.1834	13.2845	-30.4940	12.4889	-3.1694
30	6.8710	19.5904	-46.2139	30.7542	13.5547	-31.0845	12.7194	-3.2903
31	7.0000	19.9500	-47.0750	31.3250	13.8250	-31.6750	12.9500	-3.4135

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