Inference Methods for Stochastic Volatility Models

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Abstract

In the present paper we consider estimation procedures for stationary Stochastic Volatility models, making inferences about the latent volatility of the process. We show that a sequence of generalized least squares regressions enables us to determine the estimates. Finally, we make inferences iteratively by using the Kalman Filter algorithm.

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1 Introduction

Stochastic Volatility (SV) models have been received a growing interest in time series analysis since they find many financial applications as, for example, option pricing, asset allocation and risk management. For a comprehensive discussion on SV models see, for example, Taylor [3] and Tsyplakov [4]. Let us consider the basic stochastic volatility model given by

$$(1.1) \quad \begin{aligned} y_t &= \exp \left\{ \frac{1}{2} h_t(\theta) \right\} u_t \\ h_t(\theta) &= \mu + \rho h_{t-1}(\theta) + v_t \end{aligned}$$

where the error terms $u_t \sim IIN(0, 1)$ and $v_t \sim IIN(0, \sigma_v^2)$ are assumed to be independent of one other, and the parameter vector $\theta = (\mu \ \rho \ \sigma_v^2)$ is in a compact parameter set $\Theta \subset (0, +\infty)^3$. To ensure stationarity, we always set $|\rho| < 1$. In this case we have

$$(1.2) \quad h_t(\theta) = \mu(1 - \rho)^{-1} + \sum_{i=0}^{\infty} \rho^i v_{t-i}$$

hence $E(h_t) = \mu(1 - \rho)^{-1}$ and $\text{Var}(h_t) = \sigma_v^2(1 - \rho^2)^{-1}$. Squaring (1.1) and taking logs, we get the state-space representation

$$(1.3) \quad \begin{aligned} x_t &= \alpha + h_t(\theta) + e_t \\ h_t(\theta) &= \mu + \rho h_{t-1}(\theta) + v_t \end{aligned}$$

where $x_t = \log y_t^2$, $\alpha = E[\log u_t^2]$ is a real constant ($\simeq -1.27$) and $e_t = \log u_t^2 - \alpha$ is a martingale difference but not normal, i.e., $e_t \sim IID(0, \sigma_e^2)$, where $\sigma_e^2 = \pi^2/2$ (here $\log$ denotes the natural logarithm). See Breidt and Carriquiry [1]. Following Kim-Nelson [2], Chp.3, we present two alternative ways of making inferences about h_t (volatility) conditional on information available up to time t (given θ). In Section 2 we show that a sequence of generalized least squares (GLS) regressions enables us to determine $h_{t|t} = E(h_t|\Psi_t)$, where Ψ_t denotes the information set up to time t . In Section 3 we make inferences about h_t by employing the Kalman Filter algorithm.

2 Generalized Least Squares Estimation

As usual, we approximate the SV model in (1.3) by a Gaussian state-space model. From (1.3) we get

$$(2.1) \quad h_t = \mu(1 - \rho)^{-1}(1 - \rho^{t-1}) + \rho^{t-1}h_1 + \sum_{i=0}^{t-2} \rho^i v_{t-i}$$

for $t \geq 2$. Thus

$$\begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_{t-1} \\ h_t \end{bmatrix} = \begin{bmatrix} \rho^{-t+1} \\ \rho^{-t+2} \\ \vdots \\ \rho^{-1} \\ 1 \end{bmatrix} h_t - \begin{bmatrix} \rho^{-1}v_2 + \rho^{-2}v_3 + \dots + \rho^{-t+2}v_{t-1} + \rho^{-t+1}v_t \\ \rho^{-1}v_3 + \dots + \rho^{-t+3}v_{t-1} + \rho^{-t+2}v_t \\ \vdots \\ \rho^{-1}v_t \\ 0 \end{bmatrix} - \mu \begin{bmatrix} \frac{\rho^{-t+1}-1}{1-\rho} \\ \frac{\rho^{-t+2}-1}{1-\rho} \\ \vdots \\ \rho^{-1} \\ 0 \end{bmatrix}$$

Define:

$$\mathbf{a}_t = (\rho^{-t+1} \quad \rho^{-t+2} \quad \dots \quad \rho^{-1} \quad 1)'$$

and

$$B_t = \begin{bmatrix} \rho^{-1} & \rho^{-2} & \dots & \rho^{-t+2} & \rho^{-t+1} \\ 0 & \rho^{-1} & \dots & \rho^{-t+3} & \rho^{-t+2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \rho^{-1} \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix} \quad C_t = \begin{bmatrix} \alpha - \mu(1 - \rho)^{-1}(\rho^{-t+1} - 1) \\ \alpha - \mu(1 - \rho)^{-1}(\rho^{-t+2} - 1) \\ \vdots \\ \alpha - \mu\rho^{-1} \\ \alpha \end{bmatrix}$$

Using the measurement equation in (1.3) and the above matrix relation, we have

$$(2.2) \quad \mathbf{x}_t = C_t + \mathbf{a}_t h_t + \boldsymbol{\epsilon}_t$$

where $\mathbf{x}_t = (x_1 \ x_2 \ \dots \ x_t)'$, $\boldsymbol{\epsilon}_t = -B_t(v_2 \ \dots \ v_t)' + \mathbf{e}_t$ and $\mathbf{e}_t = (e_1 \ e_2 \ \dots \ e_t)'$. Then we have

$$(2.3) \quad E(\boldsymbol{\epsilon}_t \boldsymbol{\epsilon}_t') = \sigma_e^2 I_t + \sigma_v^2 B_t B_t' = \Omega_t.$$

One could apply GLS to model (2.2) for $t = 2, \dots, T$. Then we get

$$(2.4) \quad h_{t|t} = (\mathbf{a}_t' \Omega_t^{-1} \mathbf{a}_t)^{-1} \mathbf{a}_t' \Omega_t^{-1} (\mathbf{x}_t - C_t)$$

hence

$$(2.5) \quad h_{t|t} = h_t + (\mathbf{a}_t' \Omega_t^{-1} \mathbf{a}_t)^{-1} \mathbf{a}_t' \Omega_t^{-1} \boldsymbol{\epsilon}_t.$$

Then we have

$$E(h_{t|t}) = E(h_t) = \mu(1 - \rho)^{-1}$$

and

$$P_{t|t} = E[h_t - h_{t|t}]^2 = (\mathbf{a}_t' \Omega_t^{-1} \mathbf{a}_t)^{-1}.$$

Define

$$\mathbf{b}_t = B_t' \mathbf{a}_t \|\mathbf{a}_t\|^{-2} = \frac{1}{1 - \rho^{-2t}} \begin{bmatrix} \rho^{-t}(1 - \rho^{-2}) \\ \rho^{-t+1}(1 - \rho^{-4}) \\ \vdots \\ \rho^{-2}(1 - \rho^{-2t+2}) \end{bmatrix}.$$

Theorem 2.1 *With the above notation, we have*

$$P_{t|t} = \sigma_e^2 \|\mathbf{a}_t\|^{-2} + \sigma_v^2 \|\mathbf{b}_t\|^2$$

where

$$\|\mathbf{a}_t\|^2 = \frac{1 - \rho^{-2t}}{1 - \rho^{-2}} \quad \|\mathbf{b}_t\|^2 = \frac{\rho^{-4t+2} - 1}{\rho^2(1 - \rho^2)(1 - \rho^{-2t})^2} - \frac{2t - 1}{\rho^{2t+2}(1 - \rho^{-2t})^2}.$$

Proof. We apply the Sherman-Morrison-Woodbury (SMW) formula, i.e., if A and C are invertible matrices, then

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}.$$

Setting $A = \sigma_e^2 I_t$, $B = B_t$, $C = \sigma_v^2 I_{t-1}$ and $D = B_t'$, from (2.3) we get

$$\Omega_t^{-1} = \sigma_e^{-2} I_t - \sigma_e^{-2} B_t (\sigma_v^{-2} I_{t-1} + \sigma_e^{-2} B_t' B_t)^{-1} \sigma_e^{-2} B_t'$$

hence

$$\mathbf{a}_t' \Omega_t^{-1} \mathbf{a}_t = \sigma_e^{-2} \mathbf{a}_t' \mathbf{a}_t - \sigma_e^{-2} \mathbf{a}_t' B_t (\sigma_v^{-2} I_{t-1} + \sigma_e^{-2} B_t' B_t)^{-1} \sigma_e^{-2} B_t' \mathbf{a}_t.$$

Apply again the SMW formula with $A = \sigma_e^{-2} \mathbf{a}_t' \mathbf{a}_t$, $B = -\sigma_e^{-2} \mathbf{a}_t' B_t$, $C = (\sigma_v^{-2} I_{t-1} + \sigma_e^{-2} B_t' B_t)^{-1}$ and $D = \sigma_e^{-2} B_t' \mathbf{a}_t$. Then we have

$$\begin{aligned} P_{t|t} &= (\mathbf{a}_t' \Omega_t^{-1} \mathbf{a}_t)^{-1} = \sigma_e^2 (\mathbf{a}_t' \mathbf{a}_t)^{-1} - (\mathbf{a}_t' \mathbf{a}_t)^{-1} \mathbf{a}_t' B_t \\ &\quad (\sigma_v^{-2} I_{t-1} + \sigma_e^{-2} B_t' B_t - \sigma_e^{-2} B_t' \mathbf{a}_t (\mathbf{a}_t' \mathbf{a}_t)^{-1} \mathbf{a}_t' B_t)^{-1} B_t' \mathbf{a}_t (\mathbf{a}_t' \mathbf{a}_t)^{-1} \\ &= \sigma_e^2 \|\mathbf{a}_t\|^{-2} + (\mathbf{a}_t' \mathbf{a}_t)^{-1} \mathbf{a}_t' B_t (\sigma_v^{-2} I_{t-1})^{-1} B_t' \mathbf{a}_t (\mathbf{a}_t' \mathbf{a}_t)^{-1} \\ &= \sigma_e^2 \|\mathbf{a}_t\|^{-2} + \sigma_v^2 \|\mathbf{b}_t\|^2 \end{aligned}$$

which gives the result of the statement.

Theorem 2.2

$$P_\infty = \lim_{T \rightarrow +\infty} \frac{1}{T} \sum_{t=1}^T P_{t|t} = \sigma_v^2 (1 - \rho^2)^{-1} = \text{var}(h_t).$$

Proof. To compute the partial sums of the series we took advantage of the software "Mathematica". We have (recall that $|\rho| < 1$)

$$\sum_{t=1}^T \|\mathbf{a}_t\|^{-2} = \sum_{t=1}^T \frac{1 - \rho^{-2}}{1 - \rho^{-2t}} = \frac{(\rho^2 - 1) \psi_{\rho^2}^{(0)}(T+1) + (1 - \rho^2) \psi_{\rho^2}^{(0)}(1)}{\rho^2 \log(\rho^2)}$$

where $\psi_q^{(0)}(z) = \psi_q(z) = \partial \log \Gamma_q(z) / \partial z$ denotes the q -digamma function. Since

$$(2.6) \quad \lim_{T \rightarrow +\infty} \frac{1}{T} \psi_{\rho^2}^{(0)}(T+1) = 0$$

we get

$$(2.7) \quad \lim_{T \rightarrow +\infty} \frac{1}{T} \sum_{t=1}^T \|\mathbf{a}_t\|^{-2} = 0.$$

Furthermore, we have

$$\sum_{t=1}^T \frac{\rho^{-4t+2} - 1}{\rho^2 (1 - \rho^2) (1 - \rho^{-2t})^2} = \frac{\psi_{\rho^2}^{(1)}(T+1)}{\rho^2 \log^2(\rho^2)} + \frac{(\rho^2 + 1) \psi_{\rho^2}^{(0)}(T+1)}{\rho^2 (\rho^2 - 1) \log(\rho^2)} + \frac{T}{1 - \rho^2} + c$$

where c is the numerical constant

$$c = -\frac{\psi_{\rho^2}^{(1)}(1)}{\rho^2 \log^2(\rho^2)} - \frac{(\rho^2 + 1) \psi_{\rho^2}^{(0)}(1)}{\rho^2 (\rho^2 - 1) \log(\rho^2)}$$

and $\psi_q^{(1)}(z)$ denotes the first derivative of the q -digamma function. Now (2.6) and

$$(2.8) \quad \lim_{T \rightarrow +\infty} \frac{1}{T} \psi_{\rho^2}^{(1)}(T+1) = 0$$

imply

$$(2.9) \quad \lim_{T \rightarrow +\infty} \frac{1}{T} \sum_{t=1}^T \frac{\rho^{-4t+2} - 1}{\rho^2(1 - \rho^2)(1 - \rho^{-2t})^2} = \frac{1}{1 - \rho^2}.$$

It remains to consider the series

$$\sum_{t=1}^T \frac{2t-1}{\rho^{2t+2}(1 - \rho^{-2t})^2} = \sum_{t=1}^T \frac{2t-1}{\rho^2(\rho^t - \rho^{-t})^2}.$$

We take the first Taylor expansion around $\rho_0 \in (0, 1)$ of the function $(\rho^t - \rho^{-t})^2$ and use the following sequence of inequalities

$$\begin{aligned} 0 < \frac{2t-1}{(\rho^t - \rho^{-t})^2} &\sim \frac{2t-1}{(\rho_0^t - \rho_0^{-t})^2 + 2t(\rho_0^t - \rho_0^{-t})(\rho_0^{t-1} + \rho_0^{-t-1})(\rho - \rho_0)} \\ &< \frac{1}{(\rho_0^t - \rho_0^{-t})(\rho_0^{t-1} + \rho_0^{-t-1})(\rho - \rho_0)} \end{aligned}$$

for $0 < \rho < \rho_0$. But we have

$$\begin{aligned} \sum_{t=1}^T \frac{1}{(\rho^t - \rho^{-t})(\rho^{t-1} + \rho^{-t-1})} &= \frac{\rho\psi_{\rho}^{(0)}(T+1) - \rho\psi_{\rho}^{(0)}(T - \frac{i\pi}{2\log(\rho)} + 1)}{4\log(\rho)} \\ &+ \frac{-\rho\psi_{\rho}^{(0)}(T + \frac{i\pi}{2\log(\rho)} + 1) + \rho\psi_{\rho}^{(0)}(T - \frac{i\pi}{\log(\rho)} + 1)}{4\log(\rho)} + d \end{aligned}$$

where d is the numerical constant

$$d = \frac{-\rho\psi_{\rho}^{(0)}(1) + \rho\psi_{\rho}^{(0)}(1 - \frac{i\pi}{2\log(\rho)}) + \rho\psi_{\rho}^{(0)}(1 + \frac{i\pi}{2\log(\rho)}) - \rho\psi_{\rho}^{(0)}(1 - \frac{i\pi}{2\log(\rho)})}{4\log(\rho)}.$$

Then we have

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \sum_{t=1}^T \frac{1}{(\rho^t - \rho^{-t})(\rho^{t-1} + \rho^{-t-1})} = 0$$

as

$$\begin{aligned} \lim_{T \rightarrow +\infty} \frac{1}{T} \psi_{\rho}^{(0)}(T+1) &= \lim_{T \rightarrow +\infty} \frac{1}{T} \psi_{\rho}^{(0)}(T \pm \frac{i\pi}{2\log(\rho)} + 1) \\ &= \lim_{T \rightarrow +\infty} \frac{1}{T} \psi_{\rho}^{(0)}(T - \frac{i\pi}{\log(\rho)} + 1) = 0. \end{aligned}$$

This implies

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \sum_{t=1}^T \frac{2t-1}{\rho^2(\rho^t - \rho^{-t})^2} = 0$$

hence

$$(2.10) \quad \lim_{T \rightarrow +\infty} \frac{1}{T} \sum_{t=1}^T \|\mathbf{b}_t\|^2 = \frac{1}{1 - \rho^2}.$$

Finally, from (2.7) and (2.10), we get the result of the statement.

3 Estimation by the Kalman Filter

Since Model (1.3) is in linear state-space form, predicted filtered and smoothed values of h_t can be computed recursively via the Kalman Filter algorithm. See Kim-Nelson [2], Sec.3.1.2. Define $h_{t|\tau} = E[h_t|\Psi_\tau]$, $P_{t|\tau} = E[h_t - h_{t|\tau}]^2$, $x_{t|\tau} = E[x_t|\Psi_\tau]$, $\eta_{t|\tau} = x_t - x_{t|\tau}$ and $f_{t|\tau} = E[\eta_{t|\tau}]^2$ for $1 \leq \tau \leq T$. For the one-step-ahead prediction, we have

$$\begin{aligned} h_{t|t-1} &= \mu + \rho h_{t-1|t-1} & P_{t|t-1} &= \rho^2 P_{t-1|t-1} + \sigma_v^2 \\ \eta_{t|t-1} &= x_t - x_{t|t-1} = x_t - \alpha - h_{t|t-1} = (h_t - h_{t|t-1}) + e_t \\ f_{t|t-1} &= P_{t|t-1} + \sigma_e^2. \end{aligned}$$

The initial states of the recursion are

$$h_{0|0} = \mu(1 - \rho)^{-1} \quad \text{and} \quad P_{0|0} = \sigma_v^2(1 - \rho^2)^{-1}.$$

The updating is given by

$$\begin{aligned} h_{t|t} &= h_{t|t-1} + K_t \eta_{t|t-1} = h_{t|t-1} + K_t(x_t - \alpha - h_{t|t-1}) \\ P_{t|t} &= P_{t|t-1} - K_t P_{t|t-1} = P_{t|t-1} - P_{t|t-1}^2 f_{t|t-1}^{-1} \end{aligned}$$

where

$$K_t = P_{t|t-1} f_{t|t-1}^{-1} = P_{t|t-1} (P_{t|t-1} + \sigma_e^2)^{-1}$$

is the Kalman gain. From these recursions, one can construct the (quasi) Gaussian log-likelihood

$$\ell(\theta|\Psi_T) = -\frac{T}{2} \log(2\pi) - \frac{1}{2} \sum_{t=1}^T \log f_{t|t-1} - \frac{1}{2} \sum_{t=1}^T \eta_{t|t-1}^2 f_{t|t-1}^{-1}.$$

The smoothed estimates and their variances are given by

$$\begin{aligned} h_{t|T} &= h_{t|t} + P_t^*(h_{t+1|T} - \mu - \rho h_{t|t}) \\ P_{t|T} &= P_{t|t} + [P_t^*]^2 (P_{t+1|T} - P_{t+1|t}) \end{aligned}$$

where

$$P_t^* = \rho P_{t|t} P_{t+1|t}^{-1}.$$

Solving the difference equations in $h_{t|t}$ and $P_{t|t}$ gives the estimates. More precisely, we have

$$\begin{aligned} P_{t|t} &= \sigma_v^2 \rho^{2t} (1 - \rho^2)^{-1} \prod_{i=0}^{t-1} (1 - K_{t-i}) + \sigma_v^2 \sum_{k=0}^{t-1} \rho^{2k} \prod_{j=0}^k (1 - K_{t-j}) \\ h_{t|t} &= \mu(1 - \rho)^{-1} + \sum_{i=0}^{\infty} \rho^i K_{t-i} \eta_{t-i|t-i-1} \\ \eta_{t|t-1} &= \alpha_t(L) v_t + \beta_t(L) e_t \end{aligned}$$

where $\alpha_t(L) = 1 + \sum_{r=1}^t \rho^r \prod_{s=1}^r (1 - K_{t-s}) L^r$ and $\beta_t(L) = \alpha_t(L)(1 - \rho L)$ (here L denotes the lag operator).

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