Buffon-Laplace Type Problems for Three Regular Lattices and ”Body Test” a Parallelogram

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Abstract

In this paper we consider three regular lattices with cells, respectively, rectangle, parallelogram and trapezium and we compute the probability that the parallelogram intersects a side of the lattice.

Keywords: Geometric Probability, stochastic geometry, random sets, random convex sets and integral geometry

1 Rectangle cell

Let $\mathcal{R}_1 (l, m)$ be the lattice with fundamental cell $C_0^{(1)}$ an rectangle of side $l, m$.

The ”body test” is a parallelogram $p$ of side $a, b$ and angle $\beta$ with $\frac{\pi}{3} \leq \beta \leq \frac{\pi}{2}$, $\sqrt{a^2 + b^2 + 2ab \cos \beta} < \frac{1}{2} \min(l, m)$, $\text{areap} = ab \sin \beta$. 

![fig.1](image-url)
We want to compute the probability that parallelogram $p$ intersects a side of the lattice $R_1$, therefore the probability $P_{int}^{(1)}$ that $p$ intersects a side of the fundamental cell $C_0^{(1)}$.

The position of the parallelogram $p$ is determined by his barycentre and the angle $\varphi$ represented in the figure 2.

To compute the probability $P_{int}^{(1)}$ we consider the limit positions of "body test" for a fixed value of $\varphi$. We denote with $\hat{C}_0^{(1)}$ the figure determined from these positions, we have the figure

\[
\text{area} \hat{C}_0^{(1)}(\varphi) = \text{area}C_0^{(1)} - 4 \text{area}p - 2 [\text{area}a_1 + \text{area}a_2 + \ldots + \text{area}a_6].
\]

The figure 2 give us

\[
\overline{A_1A_2A_3} = \pi - \beta, \quad \overline{AA_2A_1} = \beta - \varphi, \quad \overline{AA_1A_2} = \frac{\pi}{2} - \beta + \varphi.
\]

Therefore

\[
|AA_1| = a \sin (\beta - \varphi), \quad |AA_2| = a \cos (\beta - \varphi),
\]

hence

\[
\text{area}a_1(\varphi) = \frac{a^2}{4} \sin 2(\beta - \varphi).
\]

From the formulas (2) follow

\[
\varphi \leq \beta, \quad \varphi \geq \beta - \frac{\pi}{2},
\]
Buffon-Laplace type problems therefore, as $\beta \leq \frac{\pi}{2}$, we have $\varphi \geq 0$.

Hence
\[
\varphi \in [0, \beta].
\] (5)

In the same way from the figure 2 we obtain
\[
|BB_1| = b \sin \varphi, \quad |BB_2| = b \cos \varphi,
\] (6)

therefore
\[
areaa_3(\varphi) = \frac{b^2}{4} \sin 2\varphi.
\] (7)

Now we consider the figure

![Figure 3](image)

fig.3

We have
\[
\overrightarrow{B_1B_4B_6} = \overrightarrow{B_4B_1B_2} = \beta,
\]
\[
\overrightarrow{B_1B_6B_4} = \overrightarrow{BB_1B_2} = \frac{\pi}{2} - \varphi,
\]
\[
\overrightarrow{B_6B_1B_4} = \frac{\pi}{2} - \beta + \varphi.
\] (8)

With these values, the triangle $B_1B_4B_6$ give us
\[
\frac{|B_1B_6|}{\sin \beta} = \frac{a}{\cos \varphi} = \frac{|B_4B_6|}{\cos (\beta - \varphi)},
\]

therefore
\[
|B_1B_6| = \frac{a \sin \beta}{\cos \varphi}, \quad |B_4B_6| = \frac{a \cos (\beta - \varphi)}{\cos \varphi}
\] (9)

and, consequently,
\[
areaa_4(\varphi) = \frac{a^2 \sin \beta \cos (\beta - \varphi)}{2 \cos \varphi}.
\] (10)

From the figure 2 we have
\[
\Delta O_1A_4A_7 = \Delta O_2B_4B_7,
\]
therefore
\[ |A_4A_7| = |B_4B_7|. \]

Hence
\[ |A_1A_7| = b - |A_4A_7|, \]
\[ |B_6B_7| = |B_4B_7| + |B_4B_6| = |A_4A_7| \cdot \frac{a \cos (\beta - \varphi)}{\cos \varphi}. \]

From here and considering that \(|A_1A_7| = |B_6B_7|\), we obtain
\[ |A_4A_7| = \frac{1}{2} \left[ b - \frac{a \cos (\beta - \varphi)}{\cos \varphi} \right], \]
therefore
\[ |A_1A_7| = \frac{1}{2} \left[ b + \frac{a \cos (\beta - \varphi)}{\cos \varphi} \right]. \quad (11) \]

From the figure

and with the (11) we have
\[ h_2 = |A_1A_7| \sin B_6A_1A_7 = \]
\[ \frac{1}{2} \left[ b \cos \varphi + a \cos (\beta - \varphi) \right]. \]

Then, considering of the relations (3), (9) and (6) we can write
\[ |A_1B_6| = m - a \sin (\beta - \varphi) - \]
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\[ b \sin \varphi - \frac{a \sin \beta}{\cos \varphi}. \]

Therefore

\[ \text{area}_{a_2}(\varphi) = |A_1B_6| \cdot h_2, \]

hence

\[ \text{area}_{a_2}(\varphi) = \frac{1}{2} [b \cos \varphi + a \cos (\beta - \varphi)] [m - b \sin \varphi - a \sin (\beta - \varphi)] - \frac{ab \sin \beta}{2} - \frac{a^2 \sin \beta \cos (\beta - \varphi)}{2 \cos \varphi}. \] (12)

The formulas (4), (7), (10), and (12) give us

\[ \text{area}_{a_1}(\varphi) + \text{area}_{a_2}(\varphi) + \text{area}_{a_3}(\varphi) + \]

\[ \text{area}_{a_4}(\varphi) = \frac{m}{2} [b \cos \varphi + a \cos (\beta - \varphi)] - ab \sin \beta. \] (13)

Now we consider the figure

We have

\[ \overline{A_2A_3A_6} = \pi - \beta, \]
\[ \overline{A_2A_6A_3} = \overline{AA_2A_1} = \beta - \varphi \]

and the triangle \( A_2A_3A_6 \) give us

\[ \frac{|A_2A_6|}{\sin \beta} = \frac{b}{\sin (\beta - \varphi)} = \frac{|A_3A_6|}{\sin \varphi} \]

therefore

\[ |A_2A_6| = \frac{b \sin \beta}{\sin (\beta - \varphi)}, \]
\[ |A_3A_6| = \frac{b \sin \varphi}{\sin (\beta - \varphi)}. \] (14)
Hence

\[ \text{area}_{a_6} = \frac{b^2 \sin \beta \sin \varphi}{2 \sin (\beta - \varphi)}. \]  

(15)

From the figure 2 we have

\[ \Delta O_1 B_3 B_5 = \Delta O_2 C_3 C_5, \]

therefore

\[ |B_3 B_5| = |C_3 C_5|. \]

Then, because of \(|B_2 B_5| = |C_5 C_6|\) and considering of the relation (14) we have

\[ |B_3 B_5| = \frac{1}{2} \left[ a - \frac{b \sin \varphi}{\sin (\beta - \varphi)} \right] \]

and

\[ |B_2 B_5| = \frac{1}{2} \left[ a + \frac{b \sin \varphi}{\sin (\beta - \varphi)} \right]. \]

Considering the figure

we have

\[ h_5 = |B_2 B_5| \sin (\beta - \varphi) = \]

\[ \frac{1}{2} \left[ a \sin (\beta - \varphi) + b \sin \varphi \right]. \]

Moreover, with the relation (6), (14) and (3), we have

\[ |B_2 B_6| = l - b \cos \varphi - \]

\[ a \cos (\beta - \varphi) - \frac{b \sin \beta}{\sin (\beta - \varphi)}. \]

Therefore

\[ \text{area}_{a_5} (\varphi) = \frac{l}{2} \left[ a \sin (\beta - \varphi) + b \sin \varphi \right] - \]

\[ \frac{a^2}{4} \sin 2(\beta - \varphi) - \frac{b^2}{4} \sin 2\varphi - \]
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\[
\frac{b^2 \sin \beta \sin \varphi}{2 \sin (\beta - \varphi)} - ab \sin \beta. \tag{16}
\]

The formulas (15) and (16) give us

\[
\text{area}a_5 (\varphi) + \text{area}a_6 (\varphi) = \\
\frac{l}{2} [a \sin (\beta - \varphi) + b \sin \varphi] - \\
\frac{a^2}{4} \sin 2 (\beta - \varphi) - \frac{b^2}{4} \sin 2 \varphi - ab \sin \beta. \tag{17}
\]

Replacing in (1) the expressions (13) and (17) we obtain

\[
\text{area} \hat{C}_0 (1) = \\
lm - m [b \cos \varphi + a \cos (\varphi - \alpha)] - \\
l [b \sin \varphi - a \sin (\varphi - \beta)] + \\
\frac{a^2}{2} \sin 2 (\varphi - \beta) + \frac{b^2}{2} \sin 2 \varphi. \tag{18}
\]

Denoting with \(M^{(1)}\), the set of the "body test" that have the barycentre in the cell \(C_0^{(1)}\) and with \(N_1\) the set of the "body test" completely contained in \(C_0^{(1)}\), we have [4]:

\[
P^{(1)}_{\text{int}} = 1 - \frac{\mu(N_1)}{\mu(M_1)}, \tag{19}
\]

where \(\mu\) is the Lebesgue measure in Euclidean plane.

To compute the measures \(\mu(M_1)\) and \(\mu(N_1)\) we use the Poincaré kinematic measure [3]:

\[
dK = d\varphi \wedge dx \wedge dy,
\]

where \(x, y\) are the coordinate of the barycentre of the parallelogram \(p\) and \(\varphi\) is the angle already defined.

The formula (5) give us

\[
\mu(M_1) = \int_0^{\beta} d\varphi \iint_{\{(x,y) \in C_0^{(1)}\}} dx dy = \\
\int_0^{\beta} \left[ \text{area} C_0^{(1)} \right] d\varphi = \beta \text{area} C_0^{(1)} = \beta lm \tag{20}
\]

and, considering of the (18) ,
\[ \mu (N_1) = \int_0^\beta d\varphi \int \int \{ (x,y) \in \tilde{C}_0^{(1)}(\varphi) \} \, dx \, dy = \]
\[ \int_0^\beta \left[ \text{area} \tilde{C}_0^{(1)} \right] d\varphi = \beta lm - \left\{ \left\lfloor m \sin \beta + l(1 - \cos \beta) \right\rfloor (a + b) - \frac{1 - \cos 2\beta}{4} (a^2 + b^2) \right\}. \quad (21) \]

The formulas (19), (20) and (21) give us

\[ P^{(1)}_{\text{int}} = \frac{(1 - \cos \beta) l + m \sin \beta}{\beta lm} (a + b) - \frac{\sin^2 \beta}{\beta lm} (a^2 + b^2). \quad (22) \]

For \( \beta = \frac{\pi}{2} \), the parallelogram \( p \) becomes a rectangle of side \( a \) and \( b \) and the probability \( P^{(1)}_{\text{int}} \) becomes

\[ P = 2 \frac{(l + m) (a + b) - (a^2 + b^2)}{(a + b)}, \]

formula already found in a previous paper [1].

If \( l \to \infty \), the cell \( C_0^{(1)} \) becomes a line of wide \( m \) and the probability \( P^{(1)}_{\text{int}} \) becomes

\[ P' = \frac{1 - \cos \beta}{\beta} (a + b), \]

that represents an extension of probability of Buffon.

## 2 Parallelogram cell

Let \( \mathcal{R}_2 (l, m, d) \) be the regular lattice with fundamental cell \( C_0^{(2)} \) a parallelogram of side \( l, m \) and angle \( \alpha \) with \( \frac{\pi}{3} \leq \alpha \leq \frac{\pi}{2} \).

The ”body test” is the same parallelogram \( p \) of the section 1 with \( \beta < \alpha \).

With the notations of the section 1 we have the figure
and the formula
\[
\text{area} \widehat{C}^{(2)}_0(\varphi) = \text{area} C^{(2)}_0 - 4 \text{area} p - 2 [\text{area} b_1 + \text{area} b_2 + \ldots + \text{area} b_{64}] .
\] (23)

From the figure

we have

\[
A_1 A A_2 = \pi - \alpha, \quad \widehat{A A_2 A}_1 = \beta - \varphi,
\]
\[
\widehat{A A_1 A}_2 = \varphi + \alpha - \beta .
\] (24)

With these values from the triangle $AA_1 A_2$ we have

\[
|BB_1| = \frac{a \sin \varphi}{\sin \alpha},
\]
\[
|BB_2| = \frac{a \sin (\varphi + \alpha)}{\sin \alpha}
\]

Therefore

\[
\text{area} b_1(\varphi) = \frac{a^2 \sin (\beta - \varphi) \sin (\varphi + \alpha - \beta)}{2 \sin \alpha} .
\] (25)
Moreover from the relations (24) follow that
\[ \beta - \varphi \geq 0, \quad \varphi + \alpha - \beta \geq 0, \]
therefore, because of \( \beta < \alpha \),
\[ \varphi \in [0, \beta]. \quad \text{(26)} \]

Now we consider the figure

\[ |BB_1| = \frac{b \sin \varphi}{\sin \alpha}, \quad |BB_2| = \frac{b \sin (\varphi + \alpha)}{\sin \alpha}, \quad \text{(27)} \]
and consequently
\[ area_{aB_3}(\varphi) = \frac{b^2 \sin \varphi \sin (\varphi + \alpha)}{2 \sin \alpha}. \quad \text{(28)} \]

From the figure
we have
\[ \overrightarrow{B_5B_4B_1} = \beta, \]
\[ \overrightarrow{B_5B_1B_4} = \overrightarrow{AA_1A_2} = \varphi + \alpha - \beta, \]
\[ \overrightarrow{B_1B_5B_4} = \pi - \alpha - \varphi. \] (29)

With these values the triangle \( B_1B_4B_5 \) give us
\[ |B_1B_5| = \frac{a \sin \beta}{\sin (\varphi + \alpha)}, \]
\[ |B_4B_5| = \frac{a \sin (\varphi + \alpha - \beta)}{\sin (\varphi + \alpha)}. \] (30)

Hence
\[ \text{arc} a_4(\varphi) = \frac{a^2 \sin \beta \sin (\varphi + \alpha - \beta)}{2 \sin (\varphi + \alpha)}. \] (31)

From the figure 7 we obtain
\[ |A_4A_6| = |B_4B_6|, \quad |A_1A_6| = |B_5B_6|, \]
therefore, considering of the second relation (31),
\[ |A_4A_6| = \frac{1}{2} \left[ b - \frac{a \sin (\varphi + \alpha - \beta)}{\sin (\varphi + \alpha)} \right] \]
and, consequently,
\[ |A_1A_6| = \frac{1}{2} \left[ b + \frac{a \sin (\varphi + \alpha - \beta)}{\sin (\varphi + \alpha)} \right]. \] (32)

Considering the figure
and the formulas (30) and (33) we obtain
\[ \overline{A_6A_1B_5} = \overline{B_4B_5B_1} = \pi - (\varphi + \alpha), \]
\[ h_2 = |A_1A_6| \sin \overline{A_6A_1B_5} = \]
\[ \frac{1}{2} [b \sin (\varphi + \alpha) + a \sin (\varphi + \alpha - \beta)]. \]

Then, with the formulas (25), (28) and (31), we have
\[ |A_1A_5| = m - |AA_1| - |B_1B_5| - |BB_1| = \]
\[ m - \frac{a \sin (\beta - \varphi) + b \sin \varphi}{\sin \alpha} - \frac{a \sin \beta}{\sin (\varphi + \alpha)}. \]

Therefore
\[ \text{areab}_2(\varphi) = \frac{1}{2} \left[ m - \frac{a \sin (\beta - \varphi) + b \sin \varphi}{\sin \alpha} \right]. \]
\[ [b \sin (\varphi + \alpha) + a \sin (\varphi + \alpha - \beta)] - \]
\[ \frac{ab \sin \beta}{2} - \frac{a^2 \sin \beta \sin (\varphi + \alpha - \beta)}{2 \sin (\varphi + \alpha)}. \] (33)

The relations (26), (29), (32) and (34) give us
\[ \text{areab}_1(\varphi) + \text{areab}_2(\varphi) + \text{areab}_3(\varphi) + \]
\[ \text{areab}_4(\varphi) = \frac{m}{2} [b \sin (\varphi + \alpha) + \]
\[ a \sin (\varphi + \alpha - \beta)] - ab \sin \beta. \] (34)

The figure
\[ \text{fig.12} \]
give us
\[ \overline{B_2B_3B_8} = \pi - \beta, \quad \overline{B_3B_2B_8} = \beta - \varphi \]
and
\[ |B_3B_8| = \frac{a \sin (\beta - \varphi)}{\sin \varphi}, \]
|B_2B_8| = \frac{a \sin \beta}{\sin \varphi}. \quad (35)

Therefore

\text{area}_{ab_6}(\varphi) = \frac{a^2 \sin \beta \sin (\beta - \varphi)}{2 \sin \varphi}. \quad (36)

From the figure 7 follow that

|B_3B_7| = |C_3C_5|, \quad |B_7B_8| = |C_2C_5|

and considering of the first relation (36) we obtain

|C_2C_5| = \frac{1}{2} \left[ \frac{a \sin (\beta - \varphi)}{\sin \varphi} \right].

Considering of the figure

we have

h_5 = |C_2C_5| \sin \varphi = \frac{1}{2} \left[ b \sin \varphi + a \sin (\beta - \varphi) \right]

and, with the formulas (25), (28) and (33) we can write

|B_8C_2| = l - |BB_2| - |B_2B_8| - |CC_2| = 

l - a \frac{\sin (\varphi + \alpha - \beta) + b \sin (\varphi + \alpha)}{\sin \varphi} - a \frac{\sin \beta}{\sin \varphi}.

Hence

\text{area}_{ab_5} = \frac{1}{2} \left[ l - a \frac{\sin (\varphi + \alpha - \beta) + b \sin (\varphi + \alpha)}{\sin \varphi} \right].

[b \sin \varphi + a \sin (\beta - \varphi)] - \frac{ab \sin \beta}{2} - \frac{a^2 \sin \beta \sin (\beta - \varphi)}{2 \sin \varphi}. \quad (37)

The relations (37) and (38) give us

\text{area}_{ab_5} + \text{area}_{ab_6} = \frac{l}{2} [b \sin \varphi + a \sin (\beta - \varphi)] -
\[
\frac{1}{2 \sin \alpha} \left[ \frac{a^2 \sin (\varphi + \alpha - \beta) + b \sin (\varphi + \alpha)}{\sin \varphi} \right] - ab \sin \beta. \quad (38)
\]

Replacing in the (23) the expressions (35) and (39) we obtain
\[
\text{area} \hat{C}_0^{(2)} (\varphi) = lm \sin \alpha - \left\{ l \left[ b \sin \varphi + a \sin (\beta - \varphi) \right] + m \left[ b \sin (\varphi + \alpha) + a \sin (\varphi + \alpha - \beta) \right] - \frac{a^2 [\cos (2\varphi + \alpha - 2\beta) - \cos \alpha] + b^2 [\cos \alpha - \cos (2\varphi + \alpha)]}{2 \sin \varphi} \right\}. \quad (39)
\]

With the notations of the previous point we have
\[
P_{\text{int}}^{(2)} = 1 - \frac{\mu (N_2)}{\mu (M_2)}, \quad (40)
\]

where for the formula (27),
\[
\mu (M_2) = \int_0^\beta d\varphi \int \int_{\{(x,y) \in C_0^{(2)}\}} dxdy =
\]
\[
\int_0^\beta \left[ \text{area} C_0^{(2)} \right] d\varphi = \beta \text{area} C_0^{(2)} = \beta \sin \alpha \cdot lm \quad (41)
\]

and, considering of the (40),
\[
\mu (N_2) = \int_0^\beta d\varphi \int \int_{\{(x,y) \in \hat{C}_0^{(2)}(\varphi)\}} dxdy =
\]
\[
\int_0^\beta \left[ \text{area} \hat{C}_0^{(2)} (\varphi) \right] d\varphi = \beta \sin \alpha \cdot lm - (1 - \cos \beta) (a + b) l + m \left\{ a \left[ \cos \alpha - \cos (\alpha - \beta) \right] + b \left[ \cos (\alpha + \beta) - \cos \alpha \right] \right\} + \frac{1}{4 \sin \alpha} \left\{ a^2 \left[ \sin \alpha - \sin (\alpha - 2\beta) - 2\beta \cos \alpha \right] + b^2 \left[ 2\beta \cos \alpha - \sin (\alpha + 2\beta) + \sin \alpha \right] \right\}. \quad (42)
\]

The formulas (41), (42) and (43) give us
\[
P_{\text{int}}^{(2)} = \frac{1}{\beta \sin \alpha \cdot lm} ((1 - \cos \beta) (a + b) l -
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\[ m \{ b \cos (\alpha + \beta) - \cos \alpha \} + a \{ \cos \alpha - \cos (\alpha - \beta) \} \] -
\[ \frac{1}{4 \sin \alpha} \left\{ a^2 \left[ \sin \alpha - \sin (\alpha - 2\beta) - 2\beta \cos \alpha \right] + b^2 \left[ \sin \alpha - \sin (\alpha + 2\beta) + 2\beta \cos \alpha \right] \right\}. \] (43)

For \( \alpha = \frac{\pi}{2} \), the cell \( C_{0}^{(2)} \) becomes a rectangle of side \( l, m \), therefore it is the same of the cell \( C_{0}^{(1)} \) and we have
\[ P_{\text{int}}^{(2)} = P_{\text{int}}^{(1)}. \]

In the same way for \( \beta = \frac{\pi}{2} \), the "boby test" \( p \) becomes a rectangle of side \( a, b \) and the probability \( P_{\text{int}}^{(2)} \) is written
\[ P = \frac{2}{\pi \sin \alpha \cdot lm} \left\{ (a + b) l - [a \cos \alpha - \sin \alpha - b \cos \alpha + \sin \alpha] m - \frac{1}{4 \sin \alpha} [a^2 (2 \sin \alpha - \pi \cos \alpha) + b^2 (2 \sin \alpha + \pi \cos \alpha)] \right\}. \]

3 Trapezium cell

Let \( R_3 (l, m, \alpha) \) a lattice with the fundamental cell \( C_{0}^{(3)} \) the trapezium represented in the figure

![fig.14](image-url)
with $l \leq m$, $\frac{\pi}{3} \leq \alpha \leq \frac{\pi}{2}$.

The "body test" is the same parallelogram $p$ of the previous point with $b \leq l$, $b \leq \alpha$.

We have

$$areaC_0^{(3)} = (l + m \cos \alpha) m \sin \alpha.$$  \hfill (44)

As in the previous points we want to compute the probability $P_{int}^{(3)}$ that the "body test" $p$ intersects a side of the lattice $R_3$.

Using the notations of the previous points we have the figure

![Figure 15](image15.png)

and the formula

$$areaC_0^{(3)}(\varphi) = areaC_0^{(3)} - 4b \sin \beta - [area_{d_1} + area_{d_2} + \ldots + area_{d_{12}}].$$  \hfill (45)

From the figure

![Figure 16](image16.png)
follow
\[ \overline{A_1A_4} = \pi - \alpha, \quad \overline{AA_1A_4} = \alpha - \varphi \]
and the triangle \( AA_1A_4 \) give us
\[ |AA_1| = \frac{a \sin \varphi}{\sin \alpha}, \quad |AA_4| = \frac{a \sin (\alpha - \varphi)}{\sin \alpha}, \]
therefore
\[ area_{a1}(\varphi) = \frac{a^2 \sin \varphi \sin (\alpha - \varphi)}{2 \sin \alpha}. \] (47)

The figure

\[ BB_2B_1 = \beta - \varphi, \quad BB_1B_2 = \pi - \alpha - \beta + \varphi \] (48)
and
\[ |BB_1| = \frac{b \sin (\beta - \varphi)}{\sin \alpha}, \quad |BB_1| = \frac{b \sin (\alpha + \beta - \varphi)}{\sin \alpha}, \]
therefore
\[ area_{a2}(\varphi) = \frac{b^2 \sin (\beta - \varphi) \sin (\alpha + \beta - \varphi)}{2 \sin \alpha}. \] (50)
Moreover from the first formula (48) we have
\[ \varphi \leq \beta. \] (51)

Considering the figure

\[ BB_2B_1 = \beta - \varphi, \quad BB_1B_2 = \pi - \alpha - \beta + \varphi \] (48)
and the second formula (48) we can write

\[ \begin{align*}
\overline{B_1B_4B_7} &= \beta, \\
\overline{B_4B_1B_7} &= \alpha - \varphi, \\
\overline{B_1B_7B_4} &= \pi - \alpha - \beta + \varphi.
\end{align*} \]

(52)

Because of this from the triangle \( B_1B_4B_7 \) we obtain

\[ \begin{align*}
|B_1B_7| &= \frac{a \sin \beta}{\sin (\alpha + \beta - \varphi)}, \\
|B_4B_7| &= \frac{a \sin (\alpha - \varphi)}{\sin (\alpha + \beta - \varphi)}.
\end{align*} \]

(53)

and consequently

\[ \text{area}_d (\varphi) = \frac{a^2 \sin \beta \sin (\alpha - \varphi)}{2 \sin (\alpha + \beta - \varphi)}. \]

(54)

The figure 15 give us

\[ |A_1A_6| = |B_4B_6|, \quad |A_1A_6| = |B_6B_7|. \]

From here and with the relation (53) we obtain

\[ \begin{align*}
|A_2A_6| &= \frac{1}{2} \left[ b - \frac{a \sin (\alpha - \varphi)}{\sin (\alpha + \beta - \varphi)} \right], \\
|A_1A_6| &= \frac{1}{2} \left[ b + \frac{a \sin (\alpha - \varphi)}{\sin (\alpha + \beta - \varphi)} \right].
\end{align*} \]

(55)

Now we consider the figure
and the last relation (52) we have

\[ h_2 = \frac{1}{2} [b \sin (\alpha + \beta - \varphi) + a \sin (\alpha - \varphi)] . \]

Then, considering of the formulas (46), (49) and (53), we can write

\[ |A_1B_7| = m - |AA_1| - |B_1B_7| - |BB_1| = \]

\[ m - \frac{a \sin \varphi + b \sin (\beta - \varphi)}{\sin \alpha} - \frac{a \sin \beta}{\sin (\alpha + \beta - \varphi)}. \]

Therefore

\[ \text{aread}_2 (\varphi) = \frac{1}{2} \left[ m - \frac{a \sin \varphi + b \sin (\beta - \varphi)}{\sin \alpha} \right]. \]

\[ b \sin (\alpha + \beta - \varphi) + a \sin (\alpha - \varphi) - \frac{ab \sin \beta}{2} - \]

\[ \frac{a^2 \sin \beta \sin (\alpha - \varphi)}{2 \sin (\alpha + \beta - \varphi)}. \]  

(56)

The figure

\[ \]  

\[ \text{aread}_5 (\varphi) = \frac{a \sin \beta \sin \varphi}{2 \sin (\beta - \varphi)}. \]

(59)

From the figure

\[ \]
follow
\[ \overline{C C_1 C_2} = \pi - (\varphi + \alpha) \]
and the triangle \( C C_1 C_2 \) give us
\[ |C C_1| = \frac{a \sin \varphi}{\sin \alpha}, \quad |C C_2| = \frac{a \sin (\varphi + \alpha)}{\sin \alpha}, \]
therefore
\[ \text{area}_7(\varphi) = \frac{a^2 \sin \varphi \sin (\varphi + \alpha)}{2 \sin \alpha}. \]  
(61)

From the figure 15 follow
\[ |B_3 B_5| = |C_3 C_6|, \quad |B_5 B_8| = |C_2 C_6|. \]

From here we obtain
\[ |C_2 C_6| = \frac{1}{2} \left[ b + \frac{a \sin \varphi}{\sin (\beta - \varphi)} \right]. \]

Then the figure
\[ \overline{C_6 C_2 B_8} = \beta - \varphi, \quad h_6 = \frac{1}{2} [b \sin (\beta - \varphi) + a \sin \varphi]. \]
At the end from the formulas (49), (57) and (59) we obtain
\[ |B_8C_2| = l + 2m \cos \alpha - |BB_2| - \]
\[ |B_2B_8| - |CC_2| = l + 2m \cos \alpha - \]
\[ \frac{a \sin (\alpha + \varphi) + b \sin (\alpha + \beta - \varphi)}{\sin \alpha} - \frac{a \sin \beta}{\sin (\beta - \varphi)}. \]

Hence
\[
\text{aread}_6 (\varphi) = \frac{1}{2} (l + m \cos \alpha) [b \sin (\beta - \varphi) + a \sin \varphi] - \]
\[ \frac{ab \sin \beta}{2} - \frac{a^2 \sin \beta \sin \varphi}{2 \sin (\beta - \varphi)} \]
\[ \frac{[a \sin \varphi + b \sin (\beta - \varphi)] [a \sin (\varphi + \alpha) + b \sin (\alpha + \beta - \varphi)]}{2 \sin \alpha}. \quad (62) \]

Now we consider the figure

From here follow
\[ C_1C_4C_7 = \beta, \quad C_4C_1C_7 = \alpha - \beta + \varphi, \]
\[ C_1C_7C_4 = \pi - \alpha - \varphi. \quad (63) \]

With these values the triangle \(C_1C_4C_7\) give us
\[ |C_1C_7| = \frac{b \sin \beta}{\sin (\alpha + \varphi)}, \]
\[ |C_4C_7| = \frac{\sin (\alpha - \beta + \varphi)}{\sin (\alpha + \varphi)}, \quad (64) \]

therefore
\[
\text{aread}_8 (\varphi) = \frac{b^2 \sin \beta \sin (\alpha - \beta + \varphi)}{2 \sin (\varphi + \alpha)}. \quad (65) \]

From the figure
and from the formula (48) follow

\[ DD_1 D_4 = \pi - \alpha, \quad DD_1 D_4 = BB_2 B_1 = \beta - \varphi, \]

and then, from the triangle \( DD_1 D_4 \) we have

\[ |DD_4| = \frac{b \sin (\beta - \varphi)}{\sin \alpha}, \]
\[ |DD_1| = \frac{b \sin (\alpha - \beta + \varphi)}{\sin \alpha}. \] (67)

Hence

\[ \text{aread}_{10} (\varphi) = \frac{b^2 \sin (\beta - \varphi) \sin (\alpha - \beta + \varphi)}{2 \sin \alpha}. \] (68)

Moreover from the third relation (65) we have \( \varphi \geq \beta - \alpha \) and, because of \( \beta \leq \alpha \), follow

\[ \varphi \geq 0. \]

This relation with the (51) give us

\[ \varphi \in [0, \beta]. \] (69)

From the figure 15 we obtain

\[ |C_4C_7| = |D_3D_6|, \quad |C_5C_7| = |D_4D_6|. \]

From these relations and from the (63) we have

\[ |C_5C_7| = \frac{1}{2} \left[ a + \frac{b \sin (\alpha - \beta + \varphi)}{\sin (\varphi + \alpha)} \right]. \]

Now we consider the figure
From the third relation (62) follow $C_5C_7C_4 = \varphi + \alpha$, therefore

$$h_9 = |C_5C_7| \sin C_5C_7C_4 =$$

$$\frac{1}{2} [a \sin (\varphi + \alpha) + b \sin (\alpha - \beta + \varphi)].$$

On the other hand (59), (63) and (66) give us

$$|C_7D_4| = m - |CC_1| - |C_1C_7| - |DD_4| =$$

$$m - \frac{a \sin \varphi + b \sin (\beta - \varphi)}{\sin \alpha} - \frac{b \sin \beta}{\sin (\varphi + \alpha)}.$$

So

$$\text{area}_9 (\varphi) = \frac{m}{2} [a \sin (\varphi + \alpha) + b \sin (\alpha - \beta + \varphi)] -$$

$$\frac{1}{2 \sin \alpha} [a \sin (\varphi + \alpha) + b \sin (\alpha - \beta + \varphi)].$$

$$[a \sin \varphi + b \sin (\beta - \varphi)] - \frac{ab \sin \beta}{2} -$$

$$\frac{b^2 \sin \beta \sin (\alpha - \beta + \varphi)}{2 \sin (\varphi + \alpha)}.$$

(70)

From the figure

$$\text{fig.26}$$
and from the second formula (65) we have
\[
\overrightarrow{D_7D_2D_1} = \pi - \beta, \quad \overrightarrow{D_1D_7D_2} = \overrightarrow{DD_1D_4} = \beta - \varphi, \quad (71)
\]
and the triangle \( D_1D_2D_7 \) give us
\[
|D_1D_7| = \frac{a \sin \beta}{\sin (\beta - \varphi)}, \quad |D_2D_7| = \frac{a \sin \varphi}{\sin (\beta - \varphi)}. \quad (72)
\]
Therefore
\[
\text{aread}_{11} (\varphi) = \frac{a^2 \sin \beta \sin \varphi}{2 \sin (\beta - \varphi)}. \quad (73)
\]
From the figure 15, as in previous cases we have
\[
|A_4A_5| = \frac{1}{2} \left[ b + \frac{a \sin \varphi}{\sin (\beta - \varphi)} \right].
\]
The figure
\[
\text{fig.27}
\]
give us
\[
h_2 = |A_4A_5| \sin \overrightarrow{A_5A_4D_7} = \\
\frac{1}{2} \left[ b \sin (\beta - \varphi) + a \sin \varphi \right].
\]
In the same way from the formulas (46), (66) and (71) follow that
\[
|A_4D_7| = l - |AA_4| - |D_1D_7| - |DD_1| = \\
l - \frac{a \sin (\alpha - \varphi) + b \sin (\alpha - \beta + \varphi)}{\sin \alpha} - \frac{a \sin \beta}{\sin (\beta - \varphi)}.
\]
Hence
\[
\text{aread}_{12} (\varphi) = \frac{l}{2} \left[ a \sin \varphi + b \sin (\beta - \varphi) \right] -
\]
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\[
\frac{[a \sin (\alpha - \varphi) + b \sin (\alpha - \beta + \varphi)] [a \sin \varphi + b \sin (\beta - \varphi)]}{2 \sin \alpha} - \frac{ab \sin \beta}{2} + \frac{a^2 \sin \beta \sin \varphi}{2 \sin (\beta - \varphi)}. \tag{74}
\]

Replacing in the formula (45) the relations (47), (50), (54), (55), (58), (60), (61), (64), (67), (69), (72) and (73) we obtain

\[
\text{area} \hat{C}_0^{(3)} = \text{area} C_0^{(3)} - l \left[ a \sin \varphi + b \sin (\varphi - \alpha) \right] - \\
\left[ a \sin (\varphi + \alpha) + b \sin (\alpha + \beta - \varphi) \right] + \\
\frac{a^2}{2} \sin 2\varphi + \frac{b^2}{2} \sin 2(\beta - \varphi) + \\
\frac{ab}{\sin \alpha} \left[ \left( \sin \alpha \cos \beta - \frac{\sin \beta \cos \alpha}{2} \right) \sin 2\varphi + \\
\frac{\cos \alpha \cos \beta}{2} \cos 2\varphi - \frac{\cos \alpha \cos \beta}{2} \right]. \tag{75}
\]

With the notations of the previous points we have

\[
P_{\text{int}}^{(3)} = 1 - \frac{\mu(N_3)}{\mu(M_3)}. \tag{76}
\]

Considering the relations (68) and (74) we can write

\[
\mu(M_3) = \int_0^\beta d\varphi \int \int \{ (x,y) \in C_0^{(3)} \} \ dx \ dy = \\
\int_0^\beta \left[ \text{area} C_0^{(3)} \right] d\varphi = \beta \text{area} C_0^{(3)} \tag{77}
\]

\[
\mu(N_3) = \int_0^\beta d\varphi \int \int \{ (x,y) \in \hat{C}_0^{(3)}(\varphi) \} \ dx \ dy = \\
\int_0^\beta \left[ \text{area} \hat{C}_0^{(3)}(\varphi) \right] d\varphi = \beta \text{area} C_0^{(3)} - \\
\left\{ \sin \frac{\beta}{2} \cdot l \left[ a \sin \frac{\beta}{2} + b \sin \left( \alpha - \frac{\beta}{2} \right) \right] \right\} -
\]
\[2 \sin \frac{\beta}{2} \sin \left( \alpha + \frac{\beta}{2} \right) m (a + b) + \frac{\sin^2 \beta}{2} \left( a^2 + b^2 \right) + \]
\[
\frac{ab}{\sin \alpha} \left[ \sin \beta \cos \left( \alpha + 2\beta \right) - \beta \cos \alpha \cos \beta \right]. \quad (78)
\]

The formulas (44), (75), (76) and (77) give us

\[
P^{(3)}_{\text{int}} = \frac{1}{\beta (l + m \cos \varphi) m \sin \alpha}.
\]

\[
\left(2 \sin \frac{\beta}{2} \left\{ a \sin \frac{\beta}{2} \left[ a \sin \frac{\beta}{2} + b \sin \left( \alpha - \frac{\beta}{2} \right) \right] \right\} \right) l -
\]
\[
2 \sin \frac{\beta}{2} \sin \frac{\beta}{2} \sin \left( \alpha + \frac{\beta}{2} \right) (a + b) m +
\]
\[
\frac{\sin^2 \beta}{2} \left( a^2 + b^2 \right) + \frac{ab}{\sin \alpha} \left[ \sin \beta \cos \left( \alpha + 2\beta \right) - \beta \cos \alpha \cos \beta \right]. \quad (79)
\]

If \( \alpha = \frac{\pi}{2} \), the cell \( C_{0}^{(3)} \) becomes a rectangle of side \( l, m \) and the probability \( P^{(3)}_{\text{int}} \) becomes

\[
P^{(3)}_{\text{int}} = \frac{1}{\beta lm} \left[ 2 \sin \frac{\beta}{2} \left( a \sin \frac{\beta}{2} + b \cos \frac{\beta}{2} \right) \right] l -
\]
\[
\sin \beta (a + b) m - \frac{\sin^2 \beta}{2} \left( a^2 + b^2 \right) \right] - ab \sin \beta \sin 2\beta \right].
\]

In this way we have

\[
P^{(3)}_{\text{int}} = P^{(1)}_{\text{int}}.
\]

At the end, if \( \beta = \frac{\pi}{2} \), the "body test" becomes a rectangle, the probability (78) is written

\[
P^{(3)}_{\text{int}} = \frac{2}{\pi (l + m \cos \alpha) m \sin \alpha} \left\{ [a +
\]
\[
b (\sin \alpha - \cos \alpha)] l - (\sin \alpha + \cos \alpha) (a + b) m +
\]
\[
\frac{a^2 + b^2}{2} - ab \cot \alpha \right\},
\]

probability already found in a previous paper [2].
References


[2] M. Pettineo, Un problema di tipo Laplace per un reticolo trapezoidale con il ”corpo test” rettangolo,...


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