On relations between right alternative and nuclear square AG-groupoids

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Abstract

We find some interesting relations between right alternative and left nuclear square AG-groupoids such as (1) A right alternative AG-groupoid S is left nuclear square, (2) For a right alternative AG-groupoid S the conditions (i) S is right nuclear square, (ii) S is middle nuclear square, (iii) S is nuclear square, are equivalent, (3) a right alternative AG^{**}-groupoid S is nuclear square, and (4) a cancellative left nuclear square AG-groupoid S is T^1 -AG-groupoid.

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1 Introduction

An AG-groupoid - i.e. a groupoid satisfying the identity $(xy)z = (zy)x$ called left invertive law is a generalization of commutative semigroups and is considered one of the most interesting structure among the non-associative structures. Recently some new classes of AG-groupoid have been discovered in [1, 2] and further study of each of these classes has been declared as an interesting future work. The concepts of nuclear square property and alternativity have been brought from quasigroups and loops to AG-groupoids [3]. C-loops are nuclear square [4] and many of the properties of C-loops hold due to their this property

[5]. In fact (left, middle, right) nuclear square properties make calculations too much easy. In this note we find some relations between right alternative AG-groupoids and (left, middle, right) nuclear square AG-groupoids and we find that the class of right alternative AG^{**}-groupoid S is nuclear square. Indeed this is the first class of AG-groupoids with this useful property. This opens a new chapter of research to find such other classes of AG-groupoids. The results, we have obtained, have been listed in the abstract. Besides that we prove that a cancellative left nuclear square AG-groupoid S is T^1 -AG-groupoid. For more details about T^1 -AG-groupoids we suggest [2, 8] and for general study about AG-groupoids and AG-groups we refer the reader to [3, 9, 10]. Moreover, to motivate the readers and researchers of the field, in the end of this article we give a conjecture that a cancellative left nuclear square T^1 -AG-groupoid is an AG-group.

2 Preliminaries

An AG-groupoid S is called left nuclear square if $a^2(bc) = (a^2b)c$, middle nuclear square if $a(b^2c) = (ab^2)c$ and right nuclear square if $a(bc^2) = (ab)c^2$, $\forall a, b, c \in S$ [3]. An AG-groupoid S is called right alternative AG-groupoid if it satisfies $b \cdot aa = ba \cdot a$, $\forall a, b \in S$. An AG-groupoid S is called T^1 -AG-groupoid if $ab = cd \Rightarrow ba = dc$ $\forall a, b, c, d \in S$. An element a of S is called left [resp. right] cancellative if $ax = ay \Rightarrow x = y$, $\forall x, y$ in S [resp. $xa = ya \Rightarrow x = y$]. An element a of S is called cancellative if it is both left and right cancellative. S is called left cancellative (right cancellative, cancellative) if every element of S is left cancellative (right cancellative, cancellative). AG^{**}-groupoid if it satisfies the identity $a(bc) = b(ac)$. An AG-groupoid with left identity and inverses is called AG-group. It can be easily verified that an AG-groupoid always satisfies the medial law: $(ab)(cd) = (ac)(bd)$.

3 Relations between right alternative and (left, middle, right) nuclear square AG-groupoids

We start by the following important lemma.

Lemma 3.1. *A right alternative AG-groupoid S is left nuclear square.*

Proof. Let S be a right alternative AG-groupoid, and let $a, b, c \in S$. Then
Hence S is left nuclear square. $\square$

A right alternative AG-groupoid S is not necessarily right nuclear square or middle nuclear square. The following is a counter example for both.

$$\begin{aligned}
a^2 \cdot bc &= (bc \cdot a)a && \text{by left invertive law} \\
&= (bc)(aa) && \text{by right alternative property} \\
&= (ba)(ca) && \text{by medial law} \\
&= (ca \cdot a)b && \text{by left invertive law} \\
&= (c \cdot aa)b && \text{by right alternative property} \\
&= (b \cdot aa)c && \text{by left invertive law} \\
&= (ba \cdot a)c && \text{by right alternative property} \\
&= (a^2b)c && \text{by left invertive law} \\
\Rightarrow a^2 \cdot bc &= a^2b \cdot c.
\end{aligned}$$

Example 1. A right alternative AG-groupoid of order 4.

$\cdot$	1	2	3	4
1	2	3	4	3
2	3	2	3	2
3	2	3	2	3
4	3	2	3	2

However, we have the following relation among them.

Theorem 3.2. Let S be a right alternative AG-groupoid. Then the following conditions are equivalent.

- (i) S is right nuclear square,
- (ii) S is middle nuclear square,
- (iii) S is nuclear square.

Proof. (i) $\Rightarrow$ (iii). Suppose (i) holds. Let $a, b, c \in S$. Then

$$\begin{aligned}
ab^2 \cdot c &= cb^2 \cdot a && \text{by left invertive law} \\
&= (cb \cdot b)a && \text{by right alternative property} \\
&= (b^2c)a && \text{by left invertive law} \\
&= (ac)b^2 && \text{by left invertive law} \\
&= a(cb^2) && \text{by right nuclear square property} \\
&= a(cb \cdot b) && \text{by right alternative property} \\
&= a(b^2c) && \text{by left invertive law} \\
\Rightarrow ab^2 \cdot c &= a \cdot b^2c.
\end{aligned}$$

Thus S is also middle nuclear square. But by Lemma 3.1, S is left nuclear square as well. Hence S is nuclear square, which is (iii).

(iii) $\Rightarrow$ (ii). Obvious.

Finally, we show that $(ii) \Rightarrow (i)$.

Suppose (ii) holds. Let $a, b, c \in S$. Then

$$\begin{aligned}
 ab \cdot c^2 &= c^2b \cdot a && \text{by left invertive law} \\
 &= (bc \cdot c)a && \text{by left invertive law} \\
 &= (b \cdot cc)a && \text{by right alternative property} \\
 &= (a \cdot cc)b && \text{by left invertive law} \\
 &= a \cdot c^2b && \text{by middle nuclear square property} \\
 &= a(bc \cdot c) && \text{by left invertive law} \\
 &= a(bc^2) && \text{by right alternative property} \\
 \Rightarrow ab \cdot c^2 &= a \cdot bc^2.
 \end{aligned}$$

which proves (i) . Hence the theorem is proved. $\square$

By Example 1, a right alternative AG-groupoid needs not to be nuclear square. However the following theorem shows that it is true for AG^{**} -groupoid which is a special class of AG-groupoid. This theorem also shows that how useful Theorem 3.2 is.

Theorem 3.3. *A right alternative AG^{**} -groupoid S is nuclear square.*

Proof. Let S be a right alternative AG^{**} -groupoid. Then

$$\begin{aligned}
 a \cdot bc^2 &= b \cdot ac^2 && \text{by definition of } AG^{**}\text{-groupoid} \\
 &= b(ac \cdot c) && \text{by right alternative property} \\
 &= ac \cdot bc && \text{by definition of } AG^{**}\text{-groupoid} \\
 &= ab \cdot c^2 && \text{by medial law} \\
 \Rightarrow a \cdot bc^2 &= ab \cdot c^2.
 \end{aligned}$$

Now Theorem 3.2 and Lemma 3.1 complete the proof. $\square$

The following example proves that such non-associative class exists.

Example 2. *A non-associative right alternative AG^{**} -groupoid of order 3.*

$\cdot$	1	2	3
1	3	2	3
2	1	3	3
3	3	3	3

Next we find a subclass of AG^{**} -groupoid.

Theorem 3.4. *A T^1 -AG-groupoid S is AG^{**} -groupoid.*

Proof. Let S be a T^1 -AG-groupoid, and let $a, b, c \in S$. Then

$$\begin{aligned}
ab \cdot c &= cb \cdot a && \text{by left invertive law} \\
\Rightarrow c \cdot ab &= a \cdot cb && \text{by definition of } T^1\text{-AG-groupoid}
\end{aligned}$$

Hence S is AG^{**}-groupoid. $\square$

The converse of the above theorem is not true even if we add right alternativity as the following example shows.

Example 3. *A right alternative AG^{**}-groupoid of order 4 which is not T^1 -AG-groupoid.*

$\cdot$	1	2	3
1	3	2	3
2	3	3	3
3	3	3	3

Corollary 3.5. *A right alternative T^1 -AG-groupoid S is nuclear square.*

The following theorem provides a relation among the three classes of AG-groupoids namely cancellative AG-groupoid, left nuclear square AG-groupoid and T^1 -AG-groupoid.

Theorem 3.6. *A cancellative left nuclear square AG-groupoid S is T^1 -AG-groupoid.*

Proof. Let S be a cancellative left nuclear square AG-groupoid, and let $a, b, c, d \in S$ such that $ab = cd$. Then

$$\begin{aligned}
c^2 \cdot ba &= c^2b \cdot a && \text{by left nuclear square property} \\
&= ab \cdot c^2 && \text{by left invertive law} \\
&= cd \cdot c^2 && \text{by assumption} \\
&= c^2d \cdot c && \text{by left invertive law} \\
&= c^2 \cdot dc && \text{by left nuclear square property} \\
\Rightarrow c^2 \cdot ba &= c^2 \cdot dc \\
ba &= dc. && \text{by cancellativity}
\end{aligned}$$

Hence S is T^1 -AG-groupoid. $\square$

The following example shows the existence of such a non-associative class.

Example 4. *A non-associative cancellative left nuclear square of order 3.*

$\cdot$	1	2	3
1	2	3	1
2	1	2	3
3	3	1	2

An AG-group is always cancellative [6] or [7] and left nuclear square [3]. So we have the following result.

Corollary 3.7. *An AG-group S is T^1 -AG-groupoid.*

A cancellative AG-groupoid is not necessarily an AG-group as the following example shows. We give the conjecture:

Conjecture 3.8. *A cancellative left nuclear square T^1 -AG-groupoid is an AG-group.*

Example 5. *A cancellative non-associative AG-groupoid of order 4 which is not an AG-group.*

$\cdot$	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

References

- [1] M. Shah, I. Ahmad, A. Ali. Discovery of new classes of AG-groupoids, *Res. J. Recent Sciences*, Vol. 1(11), 1-8, November (2012) 47-49.
- [2] M. Shah, A. Ali, and I. Ahmad. On introduction of new classes of AG-groupoids, *Res. J. Recent Sciences*, to appear.
- [3] M. Shah. A theoretical and computational investigations in AG-groups, Ph.D. Thesis, Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan, 2012.
- [4] J. D. Philips and P. Vojtechovsky. C-loops: an introduction, *Publicationes Mathematicae Debrecen* 68(1-2) (2006), 115-137.
- [5] M. Shah, A. Ali, and V. Sorge. Nuclei and commutants of C-loops, *Quasi-groups and Related systems*, to appear.
- [6] M. Shah, T. Shah, A. Ali. On the cancellativity of AG-groupoids, *International Mathematical Forum* 6(2011), No. 44, 2187-2194.
- [7] Q. Mushtaq and M.S. Kamran. On left almost groups, *Proc. Pak. Acad. of Sciences*, Vol. 33(1996): 1-2.
- [8] M. Rashad, I. Ahmad and M. Shah. Some new results on T^1, T^2 and T^4 -AG-groupoids, *Res. J. Recent Sciences*, to appear.

- [9] M. Shah, I. Ahmad and A. Ali. On quasi-cancellativity of AG-groupoids, *Int. J. Contemp. Math. Sciences*, Vol. 7, 2012, no. 42, 2065 - 2070.
- [10] I. Ahmad, M. Rashad and M. Shah. Some properties of AG^* -groupoid, *Int. J. Recent Sci.*, to appear.

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