On Asymptotic Distribution of Sample Central Moments in Normal-Uniform Distribution

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Abstract

The central moments are very useful in topics identified distribution as they reveal the measure of skewness. In this article, we present the asymptotic distribution of estimator of third central moment as well as the distribution of estimator of skewed measure by employing of the theory of U-statistics in the family of skewed normal-uniform distribution. By changing the three parameters of this distribution, one related to normal distribution, the other related to uniform distribution, and the third to skew parameter, we characterize this kind of distribution.

Keywords: Asymptotic Distribution, Moments, Skew Normal-Uniform Distribution, U-statistics

1 Introduction

Valle and et al (2003) introduced a new family of asymmetric normal distributions that contains Azzalini's skew-normal distribution as a special case. The emergence of this new field resulted in the applied research conducted by Arnold (1983). There have also been research focused on robustness and Bayes estimation (O'Hagan(1976)). Gupta, et al (2002) found and discussed the models which include normal, t, Cauchy, Laplace, and logistic distribution. In all of these findings, a new parameter such as λ , played a key role for controlling the extent of kurtosis and skew.

Suppose the density uniform function has been defined on the interval of $[-h, h]$, then the skew normal-uniform density function will be as follows:

$$f_X(x) = \begin{cases} 0, & \lambda x \leq -h, \\ \frac{\lambda x + h}{\sqrt{2\pi} \sigma h} \exp\left(-\frac{x^2}{2\sigma^2}\right), & -h \leq \lambda x \leq h, \\ \frac{2}{\sqrt{2\pi} \sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right), & \lambda x \geq h, \end{cases} \quad \sigma > 0, h > 0, \lambda \in \mathbb{R}. \quad (1)$$

Nadarajah and Kotz (2003) had presented the moments for this distribution. By employing the gamma function as well as the incomplete gamma function, these moments can directly be obtained. For odd n

$$E(X^n) = \frac{2^{(n+1)/2} \sigma^{n+1} \lambda}{\sqrt{\pi} h} \gamma\left(\frac{n}{2} + 1, \frac{h^2}{2\sigma^2 \lambda^2}\right) + \frac{2^{n/2} \sigma^n}{\sqrt{\pi}} \Gamma\left(\frac{n+1}{2}, \frac{h^2}{2\sigma^2 \lambda^2}\right),$$

and in especial cases,

$$E(X) = \frac{\sigma^2 \lambda}{h} \left\{ 2 \Phi\left(\frac{h}{\sigma \lambda}\right) - 1 \right\},$$

$$E(X^3) = -\sqrt{2} \frac{\sigma^3}{\sqrt{\pi}} \exp\left(-\frac{h^2}{2\sigma^2 \lambda^2}\right) + \frac{3\sigma^4 \lambda}{h} \left\{ 2 \Phi\left(\frac{h}{\sigma \lambda}\right) - 1 \right\},$$

where $\gamma(a, x) = \int_0^x t^{a-1} \exp(-t) dt$, $\Gamma(a, x) = \int_x^\infty t^{a-1} \exp(-t) dt$, and $\Phi(\cdot)$ is density of normal distribution.

What results in this paper can be used to obtain a combination of theory of U-statistics and characteristics of skew normal-uniform distribution.

Many statisticians have studied the theory of U-statistics. It is referred to the generalization of U-statistics to Hoeffding (1946). Characterization and theorems of this groups of statistics were presented by Lee (1985) and Denker (1990). Lehmann (1999) and Serfling (1980) also reached to these discussions. We show that a rather complicated formulas of second, third, and fourth central estimator moments are summarized according to their forms, respectively [1].

$$U_2 = S^2 = \frac{n}{n-1} (\bar{x^2} - \bar{x}^2) = \frac{n}{n-1} M_2,$$

$$U_3 = \frac{n^2}{(n-1)(n-2)} \bar{x^3} - 3 \frac{n^2}{(n-1)(n-2)} \bar{x^2} \bar{x} + 2 \frac{n^2}{(n-1)(n-2)} \bar{x}^3$$

$$= \frac{n^2}{(n-1)(n-2)} M_3$$

$$U_4 = \frac{n^3}{(n-1)(n-2)(n-3)} M_4 + \frac{-n^2 + 3n}{(n-1)(n-2)(n-3)} \bar{x^4}$$

$$- \frac{8n^2 - 12n}{(n-1)(n-2)(n-3)} \bar{x^3} \bar{x} + \frac{-6n^2 + 9n}{(n-1)(n-2)(n-3)} \bar{x^2}^2,$$

where for $M_i = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^i$, for $i = 2, 3, 4$ is the i th central moments in the sample. Asymptotic distributions for estimators of central moments are:

$$(1) \sqrt{n}(S^2 - \mu_2) \sim \text{AN}(0, \mu_4 - \mu_2^2),$$

$$(2) \sqrt{n}(U_3 - \mu_3) \sim \text{AN}(0, 9\xi_3),$$

where μ_i is i th central moment, ξ_3 is related to the characteristics of a distribution, and calculated in Abbasi (2008).

Now, in next section, we obtain parameters of asymptotic distribution in (2) for skewed normal-uniform distribution. Since the results yield rather complicated formulas, special cases which ends in simpler forms are shown. Also, we present the numerical results.

2 Bayesian inference technique

Variance of asymptotic distribution of U_3 in skew normal uniform distribution is

$$\begin{aligned} 9\xi_3 = \frac{\sigma^5}{(\pi^{\frac{3}{2}}h^6\lambda)} & \left(15\sigma\pi^{\frac{3}{2}}h^6\lambda - 2\sqrt{\pi}\sigma h^6\lambda e^{-\frac{h^2}{\sigma^2\lambda^2}} \right. \\ & + 3\sqrt{2}\pi h^5\sigma^2\lambda^2 e^{-\frac{1}{2}\frac{h^2}{\sigma^2\lambda^2}} \operatorname{erf}\left(\frac{\sqrt{2}}{2}\frac{h}{\sigma\lambda}\right) - 9\pi^{\frac{3}{2}}h^4\sigma^3\lambda^3 \operatorname{erf}\left(\frac{\sqrt{2}}{2}\frac{h}{\sigma\lambda}\right)^2 \\ & - 18\pi^{\frac{3}{2}}h^2\sigma^5\lambda^5 \operatorname{erf}\left(\frac{\sqrt{2}}{2}\frac{h}{\sigma\lambda}\right)^4 - 36\pi^{\frac{3}{2}}\sigma^7\lambda^7 \operatorname{erf}\left(\frac{\sqrt{2}}{2}\frac{h}{\sigma\lambda}\right)^6 \\ & + 3\sqrt{2}\pi h^7 e^{-\frac{1}{2}\frac{h^2}{\sigma^2\lambda^2}} \operatorname{erf}\left(\frac{\sqrt{2}}{2}\frac{h}{\sigma\lambda}\right) \\ & \left. + 42\sqrt{2}\pi h^3\sigma^4\lambda^4 e^{-\frac{1}{2}\frac{h^2}{\sigma^2\lambda^2}} \operatorname{erf}\left(\frac{\sqrt{2}}{2}\frac{h}{\sigma\lambda}\right)^3 \right). \end{aligned}$$

The calculation is done using Maple software. Regarding the employing of a special kind of the three parameters of SNU distribution in the above statement, the following cases can be observed.

1. For $h = \sigma\lambda$,

$$\xi_3 = \frac{1}{36} \times 45.8648 \sigma^4$$

2. For $h = \lambda$,

$$\begin{aligned}\xi_3 = & \frac{\sigma^5}{9\sqrt{\pi^3}} (15\sqrt{\pi^3} \sigma - 2\sigma e^{-\frac{1}{\sigma^2}} \sqrt{\pi} + 3\sqrt{2}\pi\sigma^2 e^{-\frac{1}{2\sigma^2}} \operatorname{erf}\left(\frac{\sqrt{2}}{2\sigma}\right) \\ & - 9\pi^{\frac{3}{2}} \sigma^3 \operatorname{erf}\left(\frac{\sqrt{2}}{2\sigma}\right)^2 - 18\sqrt{\pi^3} \sigma^5 \operatorname{erf}\left(\frac{\sqrt{2}}{2\sigma}\right)^4 \\ & - 36\sqrt{\pi^3} \sigma^7 \operatorname{erf}\left(\frac{\sqrt{2}}{2\sigma}\right)^6 + 3\sqrt{2}\pi e^{-\frac{1}{2\sigma^2}} \operatorname{erf}\left(\frac{\sqrt{2}}{2\sigma}\right) \\ & + 42\sqrt{2}\pi\sigma^4 e^{-\frac{1}{2\sigma^2}} \operatorname{erf}\left(\frac{\sqrt{2}}{2\sigma}\right)^3)\end{aligned}$$

Table 1 shows the differences of this quantity for case 1 according to σ amounts.

3. For $\lambda = i/\sigma$,

$$\xi_3 = \frac{1}{36} \sigma^6 f(i, h)$$

where $f(i, h)$ is a complicated function of i and h . In limit case, whenever the value of h tends towards infinity, $f(i, h)$ tends to the number three. Table 2 shows the amounts of $f(i, h)$ for different amount of two arguments.

Table 1. Variance differences for U_3 in different normal distributions parameters when $h = \lambda$

σ	1	2	3	4	5
$36 \times \xi_3$	45.86	2835.40	32134.59	180267.77	687188.13
σ	6	7	8	9	10
$36 \times \xi_3$	205117.68	5171170.39	11520716.38	23353557.48	43940843.87

Table 2. Values of the $f(i, h)$ with the two different values of arguments

$\begin{matrix} h \rightarrow \\ i \downarrow \end{matrix}$	1	3	5	7	9
1	45.8648	55.3558	58.4357	59.2341	59.5443
3	44.0803	45.8648	49.5557	52.9553	55.3558
5	43.9800	44.5008	45.8648	47.9961	50.3117
7	43.9542	44.1926	44.7948	45.8648	47.3397
9	43.9438	44.0804	44.40644	44.9909	45.8648

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