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Almost Homeomorphisms on Bigeneralized Topological Spaces

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Abstract

The aim of this paper to introduce the concept of almost $(\mu, \mu')^{(m,n)}$-homeomorphism on bigeneralized topological space. Also, we introduce the concept of $(\mu, \mu')^{(m,n)}$-homeomorphism on bigeneralized topological space. Basic properties, characterizations and relationships between $(\mu, \mu')^{(m,n)}$-homeomorphism and almost $(\mu, \mu')^{(m,n)}$-homeomorphism are obtained.

Keywords: bigeneralized topological space, $(\mu, \mu')^{(m,n)}$-homeomorphism, almost $(\mu, \mu')^{(m,n)}$-homeomorphism

1 introduction

Á. Csaŕszaŕ [3] introduced the concepts of generalized neighborhood systems and generalized topological spaces. He also introduced the concepts of continuous functions and associated interior and closure operators on generalized neighborhood systems and generalized topological spaces. In particular, he investigated characterizations for the generalized continuous function by using a closure operator defined on generalized neighborhood systems. In [4] , he introduced and studied the notions of g-α-open sets, g-semi-open sets, g-preopen sets and g-β-open sets in generalized topological spaces. W. K. Min [6] introduced the notion of almost (g, g')-continuity and investigated properties of such functions and relationships among (g, g')-continuity, almost (g, g')-continuity and weak (g, g')-continuity. C. Boonpok [1] introduced the concept of bigeneralized topological spaces and studied $(m.n)$-closed sets and $(m.n)$-open sets in bigeneralized topological spaces. He also introduced the notion of weakly open functions on bigeneralized topological spaces and investigated properties of such functions. Recently, in 2011, T. Duangphui and at al [5] introduced the notions of $(\mu,\mu^{'})^{(m,n)}$-continuous, almost $(\mu,\mu^{'})^{(m,n)}$-continuous and weakly $(\mu,\mu^{'})^{(m,n)}$-continuous functions. They obtained many characterizations and properties of such functions. In this paper, we introduce the concepts of $(\mu,\mu^{'})^{(m,n)}$-homeomorphism and almost $(\mu,\mu^{'})^{(m,n)}$-homeomorphism on bigeneralized topological spaces. We obtain several characterizations and properties of almost $(\mu,\mu^{'})^{(m,n)}$-homeomorphism. Moreover, many terms are reduced when we use the term of pairwise almost $(\mu,\mu^{'})^{(m,n)}$-homeomorphism instead of the term of almost $(\mu,\mu^{'})^{(m,n)}$-homeomorphism as we see through this paper.

2 Preliminaries

Definition 2.1. [3] Let X be a nonempty set and μ a collection of subsets of X. Then μ is called a *generalized topology* (briefly GT) on X if and only if $\emptyset \in \mu$ and $V_i \in \mu$ for $i \in I \neq \emptyset$ implies $\bigcup_{i\in I} V_i \in \mu$. We call the pair (X,μ) a *generalized topological space* (briefly GTS) on X. The elements of μ are called *μ-open* sets and the complements are called *μ-closed sets*.

The closure of a subset A in a generalized topological space (X,μ), denoted by $c_\mu(A)$, is the intersection of generalized closed sets including A, i.e., the smallest μ-closed set containing A. The interior of A, denoted by $i_\mu(A)$, is the union of generalized open sets contained in A, i.e., the largest μ-open set contained in A.

Proposition 2.1. [6] *Let (X,μ) be a generalized topological space. For subsets A and B of X, the following properties hold.*
(1) $c_\mu(X-A) = X - i_\mu(A)$ and $i_\mu(X-A) = X - c_\mu(A)$;
(2) if $X - A \in \mu$, then $c_\mu(A) = A$ and if $A \in \mu$, then $i_\mu(A) = A$;
(3) If $A \subseteq B$, then $c_\mu(A) \subseteq c_\mu(B)$ and $i_\mu(A) \subseteq i_\mu(B)$;
(4) $A \subseteq c_\mu(A)$ and $i_\mu(A) \subseteq A$;
(5) $c_\mu(c_\mu(A)) = c_\mu(A)$ and $i_\mu(i_\mu(A)) = i_\mu(A)$.

Proposition 2.2. [6] *Let (X,μ) be a generalized topological space and $A \subseteq X$. Then*
(1) $x \in i_\mu(A)$ if and only if there exists $V \in \mu$ such that $x \in V \subseteq A$;
(2) $x \in c_\mu(A)$ if and only if $V \cap A \neq \emptyset$ for every μ-open set V containing x.

Definition 2.2. [1] Let X be a nonempty set and μ_1,μ_2 be generalized topologies on X. A triple (X,μ_1,μ_2) is said to be a *bigeneralized topological space* (briefly $BGTS$).

Let (X, μ_1, μ_2) be a bigeneralized topological space and A a subset of X. The closure of A and the interior of A with respect to μ_m are denoted by $c_{\mu_m}(A)$ and $i_{\mu_m}(A)$, respectively, for $m = 1, 2$.

Definition 2.3. [1] A subset A of a bigeneralized topological space (X, μ_1, μ_2) is called (m, n)-*closed* if $c_{\mu_m}(c_{\mu_n}(A)) = A$, where $m, n = 1, 2$ and $m \neq n$. The complement of (m, n)-closed set is called (m, n)-*open*.

Definition 2.4. [1] A subset H of a bigeneralized topological space (X, μ_1, μ_2) is said to be: (i) $\mu_{(m,n)}$-*regular open* if $H = i_{\mu_m}(c_{\mu_n}(H))$, where $m, n = 1, 2$ and $m \neq n$.
(ii) $\mu_{(m,n)}$-*semi-open* if $H \subseteq c_{\mu_n}(i_{\mu_m}(H))$, where $m, n = 1, 2$ and $m \neq n$.
(iii) $\mu_{(m,n)}$-*peropen* if $H \subseteq i_{\mu_m}(c_{\mu_n}(H))$, where $m, n = 1, 2$ and $m \neq n$.
(iv) $\mu_{(m,n)}$-α-*open* if $H \subseteq i_{\mu_m}(c_{\mu_n}(i_{\mu_m}(H)))$, where $m, n = 1, 2$ and $m \neq n$.
(v) $\mu_{(m,n)}$-β-*open* if $H \subseteq c_{\mu_n}(i_{\mu_m}(c_{\mu_n}(H)))$, where $m, n = 1, 2$ and $m \neq n$.

Definition 2.5. [5] Let (X, μ_X^1, μ_X^2) and (Y, μ_Y^1, μ_Y^2) be bigeneralized topological spaces. A function $f : (X, \mu_X^1, \mu_X^2) \longrightarrow (Y, \mu_Y^1, \mu_Y^2)$ is said to be $(\mu, \mu^{'})^{(m,n)}$-*continuous* at a point $x \in X$ if for each μ_Y^m-open set V containing $f(x)$, there exists a μ_X^n-open set U containing x such that $f(U) \subseteq V$, where $m, n = 1, 2$ and $m \neq n$.

A function $f : (X, \mu_X^1, \mu_X^2) \longrightarrow (Y, \mu_Y^1, \mu_Y^2)$ is said to be $(\mu, \mu^{'})^{(m,n)}$-*continuous* if it has this property at each point $x \in X$.

A function $f : (X, \mu_X^1, \mu_X^2) \longrightarrow (Y, \mu_Y^1, \mu_Y^2)$ is said to be *pairwise* $(\mu, \mu^{'})^{(m,n)}$-*continuous* if f is $(\mu, \mu^{'})^{(1,2)}$-continuous and $(\mu, \mu^{'})^{(2,1)}$-continuous.

Theorem 2.1. [5] *For a function $f : (X, \mu_X^1, \mu_X^2) \longrightarrow (Y, \mu_Y^1, \mu_Y^2)$ the following properties are equivalent:*
(1) f is $(\mu, \mu')^{(m,n)}$-continuous;
(2) $f^{-1}(V) = i_{\mu_X^n}(f^{-1}(V))$ for every $V \in \mu_Y^m$;
(3) $f^{-1}(i_{\mu_Y^m}(B)) \subseteq i_{\mu_X^n}(f^{-1}(B))$ for every subset B of Y;
(4) $c_{\mu_X^n}(f^{-1}(F)) = f^{-1}(F)$ for every μ_Y^m-closed subset F of Y.

Definition 2.6. [5] Let (X, μ_X^1, μ_X^2) and (Y, μ_Y^1, μ_Y^2) be bigeneralized topological spaces. A function $f : (X, \mu_X^1, \mu_X^2) \longrightarrow (Y, \mu_Y^1, \mu_Y^2)$ is said to be *almost* $(\mu, \mu^{'})^{(m,n)}$-*continuous* at a point $x \in X$ if for each μ_Y^m-open set V containing $f(x)$, there exists a μ_X^n-open set U containing x such that $f(U) \subseteq i_{\mu_Y^m}(c_{\mu_Y^n}(V))$, where $m, n = 1, 2$ and $m \neq n$.

A function $f : (X, \mu_X^1, \mu_X^2) \longrightarrow (Y, \mu_Y^1, \mu_Y^2)$ is said to be *almost* $(\mu, \mu^{'})^{(m,n)}$-*continuous* if it has this property at each point $x \in X$.

A function $f : (X, \mu_X^1, \mu_X^2) \longrightarrow (Y, \mu_Y^1, \mu_Y^2)$ is said to be *pairwise almost* $(\mu, \mu^{'})^{(m,n)}$-*continuous* if f is almost $(\mu, \mu^{'})^{(1,2)}$-continuous and almost $(\mu, \mu^{'})^{(2,1)}$-continuous.

Theorem 2.2. [5] *For a function* $f : (X, \mu_X^1, \mu_X^2) \longrightarrow (Y, \mu_Y^1, \mu_Y^2)$ *the following properties are equivalent:*
(1) f *is almost* $(\mu, \mu')^{(m,n)}$*-continuous;*
(2) $f^{-1}(V) \subseteq i_{\mu_X^n}(f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(V))))$ *for every* $V \in \mu_Y^m$*;*
(3) $c_{\mu_X^n}(f^{-1}(c_{\mu_Y^m}(i_{\mu_Y^n}(F)))) \subseteq f^{-1}(F)$ *for every* μ_Y^m*-closed subset* F *of* Y *;*
(4) $c_{\mu_X^n}(f^{-1}(c_{\mu_Y^m}(i_{\mu_Y^n}(c_{\mu_Y^m}(B))))) \subseteq f^{-1}(c_{\mu_Y^m}(B))$ *for every subset* B *of* Y *;*
(5) $f^{-1}(i_{\mu_Y^m}(B)) \subseteq i_{\mu_X^n}(f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(i_{\mu_Y^m}(B)))))$ *for every subset* B *of* Y *;*
(6) $f^{-1}(V)$ *is a* μ_X^n*-open subset of* X *for every* $\mu_{(m,n)}$*-regular open subset* V *of* Y*;*
(7) $f^{-1}(F)$ *is a* μ_X^n*-closed subset of* X *for every* $\mu_{(m,n)}$*-regular closed subset* F *of* Y*.*

3 Almost $(\mu, \mu')^{(m,n)}$-homeomorphisms

Definition 3.1. Let (X, μ_X^1, μ_X^2) and (Y, μ_Y^1, μ_Y^2) be bigeneralized topological spaces. A bijection $f : (X, \mu_X^1, \mu_X^2) \longrightarrow (Y, \mu_Y^1, \mu_Y^2)$ is said to be $(\mu, \mu')^{(m,n)}$-*homeomorphism* if f is $(\mu, \mu')^{(m,n)}$-continuous and f^{-1} is $(\mu, \mu')^{(n,m)}$-continuous.

A function $f : (X, \mu_X^1, \mu_X^2) \longrightarrow (Y, \mu_Y^1, \mu_Y^2)$ is said to be *pairwise* $(\mu, \mu')^{(m,n)}$-*homeomorphism* if f is $(\mu, \mu')^{(1,2)}$-homeomorphism and $(\mu, \mu')^{(2,1)}$-homeomorphism.

Proposition 3.1. *A bijection* $f : (X, \mu_X^1, \mu_X^2) \longrightarrow (Y, \mu_Y^1, \mu_Y^2)$ *is pairwise* $(\mu, \mu')^{(m,n)}$*-homeomorphism if and only if both* f *and* f^{-1} *are pairwise* $(\mu, \mu')^{(m,n)}$*-continuous.*

Proof. Obvious from Definition 2.5 and Definition 3.1. □

Lemma 3.1. *For a function* $f : (X, \mu_X^1, \mu_X^2) \longrightarrow (Y, \mu_Y^1, \mu_Y^2)$ *the following properties are equivalent:*
(1) f^{-1} *is* $(\mu, \mu')^{(n,m)}$*-continuous;*
(2) $f(U) = i_{\mu_Y^m}(f(U))$ *for every* $U \in \mu_X^n$*;*
(3) $f(i_{\mu_X^n}(A)) \subseteq i_{\mu_Y^m}(f(A))$ *for every subset* A *of* X*;*
(4) $c_{\mu_Y^m}(f(H)) = f(H)$ *for every* μ_X^n*-closed subset* H *of* X*.*

Proposition 3.2. *For any bijection* $f : (X, \mu_X^1, \mu_X^2) \longrightarrow (Y, \mu_Y^1, \mu_Y^2)$ *the following properties are equivalent:*
(1) f *is* $(\mu, \mu')^{(m,n)}$*-homeomorphism;*
(2) $f^{-1}(i_{\mu_Y^m}(B)) = i_{\mu_X^n}(f^{-1}(B))$ *for every subset* B *of* Y*;*
(3) $f(i_{\mu_X^n}(A)) = i_{\mu_Y^m}(f(A))$ *for every subset* A *of* X*.*

Proof. (1) $\Longrightarrow$ (2) Let B be any subset of Y. By Theorem 2.1(3), $f^{-1}(i_{\mu_Y^m}(B)) \subseteq i_{\mu_X^n}(f^{-1}(B))$. By Lemma 3.1(3), $f(i_{\mu_X^n} f^{-1}((B)) \subseteq i_{\mu_Y^m}(f(f^{-1}(B))$ since f^{-1} is $(\mu, \mu')^{(n,m)}$-continuous and $f^{-1}(B)$ is a subset of X. Thus, $i_{\mu_X^n}(f^{-1}(B)) \subseteq f^{-1}(i_{\mu_Y^m}(B))$. Hence, $f^{-1}(i_{\mu_Y^m}(B)) = i_{\mu_X^n}(f^{-1}(B))$.
(2) $\Longleftrightarrow$ (3) Obvious.
(2) $\Longrightarrow$ (1) It follows immediately from Theorem 2.1(3) and Lemma 3.1(3). □

Definition 3.2. Let (X, μ_X^1, μ_X^2) and (Y, μ_Y^1, μ_Y^2) be bigeneralized topological spaces. A bijection $f : (X, \mu_X^1, \mu_X^2) \longrightarrow (Y, \mu_Y^1, \mu_Y^2)$ is said to be *almost* $(\mu, \mu')^{(m,n)}$*-homeomorphism* if f is almost $(\mu, \mu')^{(m,n)}$-continuous and f^{-1} is almost $(\mu, \mu')^{(n,m)}$-continuous.

A function $f : (X, \mu_X^1, \mu_X^2) \longrightarrow (Y, \mu_Y^1, \mu_Y^2)$ is said to be *pairwise almost* $(\mu, \mu')^{(m,n)}$*-homeomorphism* if f is almost $(\mu, \mu')^{(1,2)}$-homeomorphism and almost $(\mu, \mu')^{(2,1)}$-homeomorphism.

From the above definitions of $(\mu, \mu')^{(m,n)}$-homeomorphism and almost $(\mu, \mu')^{(m,n)}$-homeomorphism, we have the following implication but the reverse relation may not be true in general:

$$(\mu, \mu')^{(m,n)}\text{-homeomorphism} \Longrightarrow \text{almost } (\mu, \mu')^{(m,n)}\text{-homeomorphism.}$$

Example 3.1. Let $X = Y = \{a, b, c\}$, $\mu_X^1 = \{\emptyset, \{a\}, \{b, c\}, X\}$, $\mu_X^2 = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$, $\mu_Y^1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$, $\mu_Y^2 = \{\emptyset, \{c\}, \{a, c\}\}$. Let $f : (X, \mu_X^1, \mu_X^2) \longrightarrow (Y, \mu_Y^1, \mu_Y^2)$ be the identity function. Then f is almost $(\mu, \mu')^{(1,2)}$-homeomorphism but it is not $(\mu, \mu')^{(1,2)}$-homeomorphism.

Theorem 3.1. *A bijection* $f : (X, \mu_X^1, \mu_X^2) \longrightarrow (Y, \mu_Y^1, \mu_Y^2)$ *is pairwise almost* $(\mu, \mu')^{(m,n)}$*-homeomorphism if and only if both* f *and* f^{-1} *are pairwise almost* $(\mu, \mu')^{(m,n)}$*-continuous.*

Proof. Obvious from Definition 2.6 and Definition 3.2. □

Lemma 3.2. *For any bijection* $f : (X, \mu_X^1, \mu_X^2) \longrightarrow (Y, \mu_Y^1, \mu_Y^2)$ *the following properties are equivalent:*
(1) f^{-1} *is almost* $(\mu, \mu')^{(n,m)}$*-continuous;*
(2) $f(U) \subseteq i_{\mu_Y^m}(f(i_{\mu_X^n}(c_{\mu_X^m}(U))))$ *for every* $U \in \mu_X^n$*;*
(3) $c_{\mu_Y^m}(f(c_{\mu_X^n}(i_{\mu_X^m}(H)))) \subseteq f(H)$ *for every* μ_X^n*-closed subset* H *of* X *;*
(4) $c_{\mu_Y^m}(f(c_{\mu_X^n}(i_{\mu_X^m}(c_{\mu_X^n}(A))))) \subseteq f(c_{\mu_X^n}(A))$ *for every subset* A *of* X *;*
(5) $f(i_{\mu_X^n}(A)) \subseteq i_{\mu_Y^m}(f(i_{\mu_X^n}(c_{\mu_X^m}(i_{\mu_X^n}(A)))))$ *for every subset* A *of* X *;*
(6) $f(U)$ *is a* μ_Y^m*-open subset of* Y *for every* $\mu_{(n,m)}$*-regular open subset* U *of* X*;*
(7) $f(H)$ *is a* μ_Y^m*-closed subset of* Y *for every* $\mu_{(n,m)}$*-regular closed subset* H *of* X *.*

Theorem 3.2. *For any bijection* $f : (X, \mu_X^1, \mu_X^2) \longrightarrow (Y, \mu_Y^1, \mu_Y^2)$ *the following properties are equivalent:*
(1) f *is pairwise almost* $(\mu, \mu')^{(m,n)}$*-homeomorphism;*
(2) $f(c_{\mu_X^n}(U)) = c_{\mu_Y^m}(f(U))$ *and* $f^{-1}(c_{\mu_Y^n}(V)) = c_{\mu_X^m}(f^{-1}(V))$ *for every* $\mu_{(m,n)}$*-regular open subset* U *of* X *and for every* $\mu_{(m,n)}$*-regular open subset* V *of* Y*, where* $m, n = 1, 2$ *and* $m \neq n$*;*
(3) $f(i_{\mu_X^n}(H)) = i_{\mu_Y^m}(f(H))$ *and* $f^{-1}(i_{\mu_Y^n}(F)) = i_{\mu_X^m}(f^{-1}(F))$ *for every* $\mu_{(m,n)}$*-regular closed subset* H *of* X *and for every* $\mu_{(m,n)}$*-regular closed subset* F *of* Y*, where* $m, n = 1, 2$ *and* $m \neq n$*.*

Proof. (1) $\Longrightarrow$ (2) Let U be any $\mu_{(m,n)}$-regular open subset of X, where $m,n=1,2$ and $m\neq n$. By Lemma 3.2(4), we have $c_{\mu_Y^m}(f(U))\subseteq f(c_{\mu_X^n}(U))$, since f^{-1} is pairwise almost $(\mu,\mu')^{(m,n)}$-continuous. For $f(U)$ is μ_Y^n-open subset of Y, then $c_{\mu_Y^m}(f(U))$ is $\mu_{(m,n)}$-regular closed subset of Y and hence $f^{-1}(c_{\mu_Y^m}(f(U)))$ is μ_X^n-closed subset of X. This implies $c_{\mu_X^n}(U)\subseteq f^{-1}(c_{\mu_Y^m}(f(U)))$ and hence $f(c_{\mu_X^n}(U))\subseteq c_{\mu_Y^m}(f(U))$. Consequently, $f(c_{\mu_X^n}(U))=c_{\mu_Y^m}(f(U))$.

Let V be any $\mu_{(m,n)}$-regular open subset of Y, where $m,n=1,2$ and $m\neq n$. By Theorem 2.2(4), we have $c_{\mu_X^m}(f^{-1}(V))\subseteq f^{-1}(c_{\mu_Y^n}(V))$, since f is pairwise almost $(\mu,\mu')^{(m,n)}$-continuous. From Theorem 2.2(6), $f^{-1}(V)$ is μ_X^n-open subset of X and then $c_{\mu_X^m}(f^{-1}(V))$ is $\mu_{(m,n)}$-regular closed subset of X. By the proof above, we have $f^{-1}(c_{\mu_Y^n}(V))=c_{\mu_X^m}(f^{-1}(V))$.

(2) $\Longrightarrow$ (3) Obvious.

(3) $\Longrightarrow$ (1) Let V be any $\mu_{(m,n)}$-regular open subset of Y, where $m,n=1,2$ and $m\neq n$. Since $c_{\mu_Y^n}(V)$ is $\mu_{(n,m)}$-regular closed subset of Y. By (3), it follows $f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(V)))=i_{\mu_X^n}(f^{-1}(c_{\mu_Y^n}(V)))$. Thus, $f^{-1}(V)=i_{\mu_X^n}(f^{-1}(c_{\mu_Y^n}(V)))$ and hence $f^{-1}(V)$ is μ_X^n-open subset of X. By Theorem 2.2(6), f is pairwise almost $(\mu,\mu')^{(m,n)}$-continuous.

Let U be any $\mu_{(m,n)}$-regular open subset of X, where $m,n=1,2$ and $m\neq n$. Since $c_{\mu_X^n}(U)$ is $\mu_{(n,m)}$-regular closed subset of X. By (3), it follows $f(i_{\mu_X^m}(c_{\mu_X^n}(U)))=i_{\mu_Y^n}(f(c_{\mu_X^n}(U)))$. Thus, $f(U)=i_{\mu_Y^n}(f(c_{\mu_X^n}(U)))$ and hence $f(U)$ is μ_Y^n-open subset of Y. By Lemma 3.2(6), f^{-1} is pairwise almost $(\mu,\mu')^{(m,n)}$-continuous.

□

Theorem 3.3. *For any bijection $f:(X,\mu_X^1,\mu_X^2)\longrightarrow(Y,\mu_Y^1,\mu_Y^2)$ the following properties are equivalent:*

(1) f is pairwise almost $(\mu,\mu')^{(m,n)}$-homeomorphism;

(2) $f(A)$ is $\mu_{(m,n)}$-regular open subset of Y if and only if A is $\mu_{(n,m)}$-regular open subset of X, where $m,n=1,2$ and $m\neq n$;

(3) $f^{-1}(B)$ is $\mu_{(m,n)}$-regular open subset of X for every $\mu_{(n,m)}$-regular open subset B of Y, where $m,n=1,2$ and $m\neq n$;

(4) For every subset A of X, where $m,n=1,2$ and $m\neq n$, we have

$$f(i_{\mu_X^n}(c_{\mu_X^m}(A)))=i_{\mu_Y^m}(c_{\mu_Y^n}(f(A)));$$

(5) For every subset A of X, where $m,n=1,2$ and $m\neq n$, we have

$$f(c_{\mu_X^m}(i_{\mu_X^n}(A)))=c_{\mu_Y^n}(i_{\mu_Y^m}(f(A))).$$

Proof. We prove only the implications (1) $\Longrightarrow$ (2) and (3) $\Longrightarrow$ (1). The implications (2) $\Longrightarrow$ (3), (1) $\Longrightarrow$ (4), (4) $\Longleftrightarrow$ (5) and (4) $\Longrightarrow$ (1) are straightforward.

(1) $\Longrightarrow$ (2) Let A be any $\mu_{(m,n)}$-regular open subset of X, where $m, n = 1, 2$ and $m \neq n$. Thus, $f(A) = f(i_{\mu_X^m}(c_{\mu_X^n}(A)))$. For $c_{\mu_X^n}(A)$ is $\mu_{(n,m)}$-regular closed subset of X, then $f(A) = i_{\mu_Y^n}(f(c_{\mu_X^n}(A)))$, from Theorem 3.3(3). By using Theorem 3.3(2), we get $f(A) = i_{\mu_Y^n}(c_{\mu_X^m}(f(A)))$. Consequently, $f(A)$ is $\mu_{(n,m)}$-regular open subset of Y.

Conversely, Let A be any subset of X such that $f(A)$ is $\mu_{(n,m)}$-regular open subset of Y, where $m, n = 1, 2$ and $m \neq n$. By the proof above it follows A is $\mu_{(m,n)}$-regular open subset of X.

(3) $\Longrightarrow$ (1) Let $A = f^{-1}(B)$ be any $\mu_{(m,n)}$-regular open subset of X, where $m, n = 1, 2$ and $m \neq n$. By (3), $f(A) = B$ is $\mu_{(n,m)}$-regular open subset of Y and hence $f(A)$ is μ_Y^n-open subset of Y . By Lemma 3.2(6), f^{-1} is pairwise almost $(\mu, \mu^{'})^{(m,n)}$-continuous. Let B be $\mu_{(m,n)}$-regular open subset of Y, where $m, n = 1, 2$ and $m \neq n$. By (3), $f^{-1}(B)$ is $\mu_{(n,m)}$-regular open subset of X and hence $f^{-1}(B)$ is μ_Y^n-open subset of X . By Theorem 3.2(6), f is pairwise almost $(\mu, \mu^{'})^{(m,n)}$-continuous. Thus, f is pairwise almost $(\mu, \mu^{'})^{(m,n)}$-homeomorphism.

□

Theorem 3.4. *For any bijection $f : (X, \mu_X^1, \mu_X^2) \longrightarrow (Y, \mu_Y^1, \mu_Y^2)$ the following properties are equivalent:*
(1) f is almost $(\mu, \mu^{'})^{(m,n)}$-homeomorphism;
(2) $f(i_{\mu_X^n}(c_{\mu_X^m}(U))) = i_{\mu_Y^m}(f(i_{\mu_X^n}(c_{\mu_X^m}(U))))$ and $f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(V))) = i_{\mu_X^n}(f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(V))))$ for every $\mu_{(n,m)}$-β-open subset U of X and for every $\mu_{(m,n)}$-β-open subset V of Y;
(3) $f(i_{\mu_X^n}(c_{\mu_X^m}(U))) = i_{\mu_Y^m}(f(i_{\mu_X^n}(c_{\mu_X^m}(U))))$ and $f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(V))) = i_{\mu_X^n}(f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(V))))$ for every $\mu_{(n,m)}$-semi-open subset U of X and for every $\mu_{(m,n)}$-semi-open subset V of Y;
(4) $f(i_{\mu_X^n}(c_{\mu_X^m}(U))) = i_{\mu_Y^m}(f(i_{\mu_X^n}(c_{\mu_X^m}(U))))$ and $f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(V))) = i_{\mu_X^n}(f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(V))))$ for every $\mu_{(n,m)}$-α-open subset U of X and for every $\mu_{(m,n)}$-α-open subset V of Y;
(5) For every $\mu_{(n,m)}$-preopen subset U of X and for every $\mu_{(m,n)}$-preopen subset V of Y we have

$$f(i_{\mu_X^n}(c_{\mu_X^m}(U))) = i_{\mu_Y^m}(f(i_{\mu_X^n}(c_{\mu_X^m}(U))))$$

and

$$f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(V))) = i_{\mu_X^n}(f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(V)))).$$

Proof. (1) $\Longrightarrow$ (2) Let U be any $\mu_{(n,m)}$-β-open subset of X and V be any $\mu_{(m,n)}$-β-open subset of Y. Since $i_{\mu_X^n}(c_{\mu_X^m}(U))$ is $\mu_{(n,m)}$-regular open subset of X and $i_{\mu_Y^m}(c_{\mu_Y^n}(V))$ is $\mu_{(m,n)}$-regular open subset of Y, by Lemma 3.2 (6) and Theorem 2.2(6), $f(i_{\mu_X^n}(c_{\mu_X^m}(U))) = i_{\mu_Y^m}(f(i_{\mu_X^n}(c_{\mu_X^m}(U))))$ and $f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(V))) = i_{\mu_X^n}(f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(V))))$.

(2) $\Longrightarrow$ (3) Since every $\mu_{(n,m)}$-semi-open subset U of X is $\mu_{(n,m)}$-β-open and every $\mu_{(m,n)}$-semi-open subset V of Y is $\mu_{(m,n)}$-β-open, it is obvious.
(3) $\Longrightarrow$ (4) Since every $\mu_{(n,m)}$-α-open subset U of X is $\mu_{(n,m)}$-semi-open and every $\mu_{(m,n)}$-α-open subset V of Y is $\mu_{(m,n)}$-semi-open, it is obvious.
(4) $\Longrightarrow$ (1) Let A be any subset of X and B be any subset of of Y. Since $i_{\mu_X^n}(A)$ is $\mu_{(n,m)}$-α-open subset of X and $i_{\mu_Y^m}(B)$ is $\mu_{(m,n)}$-α-open subset of Y, by (4) we have

$$f(i_{\mu_X^n}(c_{\mu_X^m}(i_{\mu_X^n}(A)))) = i_{\mu_Y^m}(f(i_{\mu_X^n}(c_{\mu_X^m}(i_{\mu_X^n}(A)))))$$

and

$$f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(i_{\mu_Y^m}(B)))) = i_{\mu_X^n}(f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(i_{\mu_Y^m}B)))).$$

Thus, $f(i_{\mu_X^n}(A)) \subseteq i_{\mu_Y^m}(f(i_{\mu_X^n}(c_{\mu_X^m}(i_{\mu_X^n}(A)))))$ and $f^{-1}(i_{\mu_Y^m}(B)) \subseteq i_{\mu_X^n}(f^{-1}(i_{\mu_Y^m}(c_{\mu_Y^n}(i_{\mu_Y^m}B)))))$. From Lemma 3.2(5) and Theorem 2.2(5), f is almost $(\mu, \mu')^{(m,n)}$-homeomorphism.
(1) $\Longrightarrow$ (5) Obvious.
(5) $\Longrightarrow$ (1) Let U be any $\mu_{(n,m)}$-regular open subset of X and V be any $\mu_{(m,n)}$-regular open subset of Y. Since U is $\mu_{(n,m)}$-preopen subset of X and V is $\mu_{(m,n)}$-preopen subset of Y, by (5), we have $f(U) = i_{\mu_Y^m}(f(U))$ and $f^{-1}(V) = i_{\mu_X^n}(f^{-1}(V))$. Thus $f(U)$ is μ_Y^m-open subset of Y and $f^{-1}(V)$ is μ_X^n-open subset of X. Consequently, f is almost $(\mu, \mu')^{(m,n)}$-homeomorphism. □

Theorem 3.5. *For any bijection $f : (X, \mu_X^1, \mu_X^2) \longrightarrow (Y, \mu_Y^1, \mu_Y^2)$ the following properties are equivalent:*
(1) f is pairwise almost $(\mu, \mu')^{(m,n)}$-homeomorphism;
(2) $f(i_{\mu_X^m}(c_{\mu_X^n}(U))) = i_{\mu_Y^n}(c_{\mu_Y^m}(f(U)))$ for every $\mu_{(m,n)}$-β-open subset U of X, where $m, n = 1, 2$ and $m \neq n$;
(3) $f(i_{\mu_X^m}(c_{\mu_X^n}(U))) = i_{\mu_Y^n}(c_{\mu_Y^m}(f(U)))$ for every $\mu_{(m,n)}$-semi-open subset U of X, where $m, n = 1, 2$ and $m \neq n$;
(4) $f(i_{\mu_X^m}(c_{\mu_X^n}(U))) = i_{\mu_Y^n}(c_{\mu_Y^m}(f(U)))$ for every $\mu_{(m,n)}$-preopen subset U of X, where $m, n = 1, 2$ and $m \neq n$.

Proof. We only prove the implication (4) $\Longrightarrow$ (1). The implications (1) $\Longrightarrow$ (2), (2) $\Longrightarrow$ (3), (3) $\Longrightarrow$ (1) and (1) $\Longrightarrow$ (4) are straightforward.
(4) $\Longrightarrow$ (1) Let A be any $\mu_{(m,n)}$-regular open subset of X, where $m, n = 1, 2$ and $m \neq n$. Since A is $\mu_{(m,n)}$-preopen subset of X, by (4), it follows $f(A) = i_{\mu_Y^n}(c_{\mu_Y^m}(f(A)))$ and hence $f(A)$ is $\mu_{(n,m)}$-regular open subset of Y. Let $f(A)$ be $\mu_{(n,m)}$-regular open subset of Y. Then $f(i_{\mu_X^m}(c_{\mu_X^n}(A))) = f(A)$ and hence $A = i_{\mu_X^m}(c_{\mu_X^n}(A)))$. Thus, A is $\mu_{(m,n)}$-regular open subset of X. By Theorem 3.3(2), f is pairwise almost $(\mu, \mu')^{(m,n)}$-homeomorphism.

□

References

[1] C. Boonpok, *Weakly open functions on bigeneralized topological spaces*, Int. Journal of Math. Analysis, 4 (18)(2010), 891-897.

[2] C. Boonpok, *M-continuous functions on biminimal structure spaces*, Far East J. of Math. Sci., 43 (1)(2010), 41-58.

[3] Á. Császár, *Generalized topology, generalized continuity*, Acta Math. Hungar., 96 (2002), 351-357.

[4] Á. Császár, *Extermally disconnected generalized topologies*, Annales Univ. Sci. Budapest., 47 (2004), 91-96.

[5] T. Duangphui, C. Boonpok and C. Viriyapong, *Continuous functions on bigeneralized topological spaces*, Int. Journal of Math. Analysis, 5 (24)(2011), 1165-1174.

[6] W. K. Min, *Almost continuity on generalized topological spaces*, Acta Math. Hungar., 125 (2009), 121-125.

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