Experimenting with Identities $x(yz) = y(xz)$ and $(xy)z = y(zx)$ in Left Alternative Ring

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Abstract

A semiprime ring $R$ of characteristic $\neq 2$ satisfying the identities $x(yz) = y(xz)$ and $(xy)z = y(zx)$ must be associative and commutative. If $(y, x, z)$ is replaced by $-(y, x, z)$ even then $R$ is associative.

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INTRODUCTION

Results on Novikov rings have been obtained by several authors [1], [2], [3], and various well behaved nonassociative examples exist. Novikov algebra is a special case of an LSA a left symmetric algebra. It was introduced in the study of Hamiltonian operators concerning integrability of certain nonlinear partial differential equation [1]. In particular, Novikov algebras bijectively correspond to a special class of Lie conformal algebra.

Kleinfeld and Smith [3] have studied on the generalization of Novikov rings. They obtained a subverity in the join of associative and Novikov rings by generalizing the theorem on simple Novikov algebras. In [6] the authors states that semiprime rings satisfying \(xy(z) = y(xz)\) and \((x, y, x) = 0\) must be associative.

In any nonassociative ring \(R\), the associator is defined as \((x, y, z) = (xy)z - x(yz)\) and the commutator as \([x, y] = xy - yx\) for all \(x, y, z \in R\). A ring \(R\) is said to be semiprime if for any ideal \(A\) of \(R\) if \(A^2 = 0\) implies \(A = 0\). The Left nucleus \(N_l\) is stated as \(N_l = \{n \in R / (n, R, R) = 0\}\). In this paper using the results of [1, 2, 3] we show that if \(R\) is a ring of char. \(\neq 2\) which is semiprime and satisfies the identities \(xy(z) = y(xz)\) and \((x, y, z) = -(y, x, z)\), then \(R\) must be associative.

MAIN RESULTS

The Novikov identities in a ring \(R\) consist of \(xy(z) = y(xz)\) \(\ldots \) (1)
and \((x, y, z) = (y, x, z)\). \(\ldots \) (2)
Variation of these identities might involve substituting \((xy)z = y(xz)\) \(\ldots \) (3)
for (1) and replacing (2) by the left alternative identity \((x, y, z) = -(y, x, z)\). \(\ldots \) (4)

**THEOREM 2.1:** If \(R\) is a ring satisfying identities \((x, y, z) = (y, x, z)\) and \((xy)z = y(xz)\) which is semiprime, then \(R\) must be associative and commutative.

**PROOF:** From (4) and (1) we have \(xy(z) - (xy)z = y(xz) - (yx)z\) which implies that \((xy)z = (yx)z\).
That is \([x, y]z = 0\) by using (3) which is nothing but \([x, y]R = 0\). \(\ldots \) (5)
Defining \(J = \Sigma [x, y]\), all sums of a finite number of commutators. We know already that \(JR = 0\), using (5). Then \(w[x, y] = [w, [x, y]] \in J\) and so \(RJ \subset J\). Thus \(J\) is an ideal of \(R\) and \(JR = 0\) implies \(J^2 = 0\). Since we are assuming \(R\) to be the semiprime, it
follows that \( J = 0 \). Thus \( R \) must be commutative. Then it follows from (3) and commutativity that \( x(yz) = y(xz) = (xy)z \).

Hence we have proved that \((x, y, z) = 0 \) which shows that the ring is associative aswell. ♦

The Teichmuller identity is given by

\[
f(x, y, z) = (xy, y, z) + (w, x, yz) - w(y, x, z).
\]

The following identity is valid in any left alternative ring:

\[
g(x, y, z) = (xy, w, z) + (w, x, yz) - y(w, x, z).
\]

Now we perform \( f(w, x, y, z) = (wx, y, z) - (w, x, yz) + (w, x, yz) - w(y, x, z) - (w, x, y)z \).

But from (7) we have

\[
\mathcal{h}(w, x, y, z) = 2y(w, y, z)\]

which implies \((xy, y, z) = y(x, y, z)\).

Thus \((x, y, z) = 0\) which shows that the ring is associative aswell. ♦

And therefore for all \( x, y, z \) in \( R \), \( bc + cb \in N_l \). We observe that \( w(x \cdot yz) = w(y \cdot xz) = y(w \cdot xz) \), follows from (1), as does \( (wx)(yz) = y(w \cdot xz) \).

Substituting \( w, x, yz = y(w, x, z) \) in \( h(w, x, y, z) \), we obtain \( ([w, x], y, z) + (w, x, yz) = (w, x, yz) + y(w, x, z) \).

Thus \( (a, b^2, c) = 0 \). This shows that for all \( b \in R \) we have \( b^2 \in N_l \).

Now let \( T = \{ t \in R / tR = 0 \} \).

**Lemma 2.1:** \( N_l = \{ n \in R / (n, R, R) = 0 \} \) is an ideal of \( R \).

**Proof:** By the definition of \( T \), we see that \( T \) is a right ideal since \( TR = 0 \), \( t \in T \), \( x, y \in R \). Replacing \( y \) by \( t \) in (4) we obtain \( (x, t, z) = -(t, x, z) \). Implies \( (xt)z - x(tz) = (tx)z - t(xz) \).

That is \( (tx)z = (tx)z - x(tz) \). But \( (tx)z = 0 \) and \( x(tz) = 0 \). Hence \( TR \subseteq T \). This shows \( T \) as makes \( T \) an ideal of \( R \). But \( T^2 \subseteq TR = 0 \). Assuming \( R \) is semiprime ring we obtain \( T = 0 \). Then in the light of this equation, (12) becomes \( (R, R, N_l) = 0 \).

Now let \( T = \{ t \in R / tR = 0 \} \).

Now the left nucleus of \( R \) is actually the nucleus of \( R \). In (11) let \( y = n \in N_l \), then \( ([w, n], y, z) = 0 \), so that \([w, n] \in N_l \) from (9) it follows that \( zn + nz \in N_l \). Therefore \( 2zn \)
\[ N_l \in \mathbb{N} \text{ and } 2nz \in N_l. \] Assuming \( \text{char.} \neq 2 \), it follows that \( zn \in N_l \) and \( nz \in N_l \). Hence \( N_l \) is an ideal of \( R \).

**THEOREM 2.2:** If \( R \) is a ring of char. \( \neq 2 \) which is semiprime and satisfies the identities \((xy)z = y(zx)\) and \((x, y, z) = -(y, x, z)\) then \( R \) must be associative.

**PROOF:** In (10), let \( y = n \in N_l \). Then \((w, x, nz) = n(w, x, z)\). Because of the Lemma and \((w, x, nz) = (w, x, n^*) = 0\).

Thus \( n(w, x, z) = 0 \) or \( N(R, R, R) = 0 \). … (14)

Let \( a \in (a, a, a) \) stand for an arbitrary associator \( R \).

Then (14) and (9) implies \( ub^2 = 0 = u(xy + yx)\). … (15)

Using the Lemma and equation we get (9), we get \( z(xy + yx) \in N_l \), so that (14) implies \( u\{z(xy + yx)\} = 0 \). … (16)

Thus using (16) and (1) we obtain \( u(z \cdot xy) = -u(z \cdot yx) = -u(y \cdot zx) \), or \( u(z \cdot xy) = -u(y \cdot zx) \). … (17)

Applying the cyclic permutation on \( x, y, \) and \( z \) in (17) twice more yields \( 2u(z \cdot xy) = 0 \). … (18)

Then by char \( \neq 2 \) we obtain \( u(z \cdot xy) = 0 \). … (19)

Now using (19) and (15) we see that \( u(xy \cdot z) = 0 \). … (20)

In (19) & (20) combined result in \( (R, R, R) (R, R, R) = 0 \). … (21)

If \( A \) is the associator ideal of \( R \), then (10) implies that \( A \) consists of all finite sums of associators and (21) then shows that \( A^2 = 0 \). Since we are assuming \( R \) is semi prime, this implies \( A = 0 \), so that \( R \) must be associative. ♦

At this stage we may observe that an associative ring which satisfies (3) automatically satisfies (2) and (3) and so Theorem 2.1 tells us that \( R \) must also be commutative under the hypotheses of Theorem 2.2.

**REFERENCES**


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