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Approximation of Pseudo-Inverse Operators by g-Frames

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Abstract

Let T denote an operator on a Hilbert space $\mathcal{H}$ and let $\{\Lambda_j\}$ be a g-frame for the orthogonal complement of the kernel N_T. We construct a sequence of operators $\{\phi_n\}$ of the form $\phi_n(.) = \sum_{j=1}^{n} g_j^n(.)\Lambda_j$ which converges to the psuedoinverse $T^\dagger$ of T in the strong operator topology as $n \to \infty$. The operators $\{\phi_n\}$ can be found using finite-dimensional methods. We also prove an adaptive iterative version of the result.

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1 Introduction

Let $\mathcal{H}, \mathcal{K}$ be two separable Hilbert spaces and $\{W_j\}_{j\in J}$ be a sequence of closed subspaces of $\mathcal{K}$, where J is a subset of $\mathbb{Z}$. Let $\mathcal{B}(\mathcal{H}, W_j)$ be the collection of all bounded linear operators from $\mathcal{H}$ into W_j. For each sequence $\{W_j\}_{j\in J}$, we define the space $\Big(\sum_{j\in J} \oplus W_j\Big)_{\ell^2}$ by

$$\Big(\sum_{j\in J} \oplus W_j\Big)_{\ell^2} = \Big\{\{g_j\}_{j\in J} \big|\ g_j \in W_j \text{ and } \sum_{j\in J} \|g_j\|^2 < \infty\Big\}. \tag{1}$$

With the inner product defined by

$$\langle \{f_j\}, \{g_j\}\rangle = \sum_{j\in J} \langle f_j, g_j\rangle.$$

It is clear that $\Big(\sum_{j\in J} \oplus W_j\Big)_{\ell^2}$ is a Hilbert space.

Definition 1.1. Let $\Lambda_j \in \mathcal{B}(\mathcal{H}, W_j)$ for all $j \in J$. A family $\Lambda = \{\Lambda_j\}_{j\in J}$ is called a generalized frame or simply a g-frame for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{j\in J} \|\Lambda_j f\|^2 \leq B\|f\|^2 \qquad \forall f \in \mathcal{H}. \tag{2}$$

The constants A and B are called g-frame bounds. The synthesis operator of Λ given by

$$T_\Lambda : \Big(\sum_{j\in J} \oplus W_j\Big)_{\ell^2} \to \mathcal{H} \qquad T_\Lambda(\{g_j\}_{j\in J}) = \sum_{j\in J} \Lambda_j^* g_j.$$

The adjoint operator of T_Λ, which is called the analysis operator also obtain as follows

$$T_\Lambda^* : \mathcal{H} \to \Big(\sum_{j\in J} \oplus W_j\Big)_{\ell^2} \qquad T_\Lambda^* f = \{\Lambda_j f\}_{j\in J}.$$

By composing T_Λ with its adjoint T_Λ^*, we obtain the generalized frame operator

$$S_\Lambda : \mathcal{H} \to \mathcal{H}, \quad S_\Lambda f = T_\Lambda, \quad T_\Lambda^* f = \sum_{j\in J} \Lambda_j^* \Lambda_j f$$

which is a bounded, self-adjoint, positive and invertible operator and $CI_{\mathcal{H}} \leq S_\Lambda \leq DI_{\mathcal{H}}$. We call the operators T and T^*, synthesis and analysis operators, respectively. A g-frame for a subspace yields a representation of the orthogonal projection onto the subspace. Given a sequence $\{\Lambda_j\}_{j\in J}$, let P_J denote the orthogonal projection onto $\overline{span}\{\Lambda_j^*(W_j)\}_{j\in J}$. Let $\{\Lambda_j \in \mathcal{B}(\mathcal{H}, W_j) : j \in J\}$ be a g-Bessel sequence for $\mathcal{H}$. The operator

$$S : \mathcal{H} \longrightarrow \mathcal{H}\ , \quad Sf = \sum_{j\in J} \Lambda_j^* \Lambda_j f \tag{3}$$

is a positive and bounded operator.
A simple computation shows that

$$\langle Sf, f\rangle = \langle \sum_{j\in J} \Lambda_j^* \Lambda_j f, f\rangle = \sum_{j\in J} \langle \Lambda_j f, \Lambda_j f\rangle = \sum_{j\in J} \parallel \Lambda_j f \parallel^2 \qquad \forall f \in \mathcal{H}. \tag{4}$$

Therefore

$$A\langle f, f\rangle \leq \langle Sf, f\rangle \leq B\langle f, f\rangle \tag{5}$$

i.e.,

$$AI \leq S \leq BI. \tag{6}$$

This implies that S is an invertible operator if and only if $\{\Lambda_j \in \mathcal{B}(\mathcal{H}, W_j) : j \in J\}$ is a g-frame for $\mathcal{H}$. For more details about the theory and applications of generalized frames we refer the readers to [4,5].

2 Linear approximation of pseudo-inverse operators

Let $\mathcal{H},\mathcal{K}$ be two Hilbert spaces and $\mathcal{B}(\mathcal{H},\mathcal{K})$ the set of bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$. The range and the null space of $T\in\mathcal{B}(\mathcal{H},\mathcal{K})$ are denoted by R_T and N_T ,respectively. Suppose that the operator $T\in\mathcal{B}(\mathcal{H},\mathcal{K})$ has a closed range. Then there exists a unique bounded operator $T^\dagger$: $\mathcal{K}\longrightarrow\mathcal{H}$ satisfying:

$$TT^\dagger T=T \quad ,T^\dagger TT^\dagger=T^\dagger \quad ,(T^\dagger T)^*=T^\dagger T \quad ,(TT^\dagger)^*=TT^\dagger \tag{7}$$

The operator $T^\dagger$ is called the pseudo-inverse operator of T. If T is a bounded invertible operator, then $T^\dagger=T^{-1}$. Alternatively, $T^\dagger$ can be characterized as the unique linear operator from$\mathcal{H}$ to $\mathcal{H}$ for which

$$N_T^\dagger=R_T^\perp \quad ,R_T^\dagger=N_T^\perp \quad ,TT^\dagger f=f \qquad \forall f\in R_T. \tag{8}$$

It is well known that $TT^\dagger$ is the orthogonal projection of $\mathcal{H}$ onto R_T and that $T^\dagger T$ is the orthogonal projection of $\mathcal{H}$ onto $N_T^\perp$. Let P_{R_T} denote the orthogonal projection of $\mathcal{H}$ onto R_T and observe that for arbitrary $f\in\mathcal{H}$ we have

$$(I-P_{R_T})f\in R_T^\perp=N_T\,\dagger \tag{9}$$

Therefore

$$T^\dagger f=T^\dagger P_{R_T}f+T^\dagger(I-P_{R_T})f=T^\dagger P_{R_T}f. \tag{10}$$

The purpose of this note is to present a method for approximation of $T^\dagger$. Let $T:\mathcal{H}\longrightarrow\mathcal{H}$ be a bounded linear operator with closed range R_T . let $\{\Lambda_i\}_{i\in I}$ be a g-frame for the subspace $N_T^\perp$ with respect to $\{W_i\}_{i\in I}$. For each finite subset $J\subseteq I$, let $\mathcal{H}_J=\overline{span}\{\Lambda_j^*(W_j)\}_{j\in J}$. Then $\{\Lambda_j\}_{j\in J}$ is a g-frame for $\mathcal{H}_J$ with respect to $\{W_j\}_{j\in J}$ with g-frame operator given by

$$S_J:\mathcal{H}_J\longrightarrow\mathcal{H}_J \qquad S_Jf=\sum_{j\in J}\Lambda_j^*\Lambda_jf$$

Also, $\{\Lambda_jT^*\}_{j\in J}$ is a g-frame for $T(\mathcal{H}_J)$ with respect to $\{W_j\}_{j\in J}$ with g-frame operator given by

$$V_J:\overline{T(\mathcal{H}_J)}\longrightarrow\overline{T(\mathcal{H}_J)} \qquad V_Jf=\sum_{j\in J}T\Lambda_j^*\Lambda_jT^*f \qquad \forall f\in\overline{T(\mathcal{H}_J)}$$

where V_J is a invertible, bounded linear operator onto $T(\mathcal{H}_J)$. When in the following we write V_J^{-1}, it is understood that we invert V_J as an operator from $\overline{T(\mathcal{H}_J)}$ onto $\overline{T(\mathcal{H}_J)}$. Also for all $f\in\overline{T(\mathcal{H}_J)}$ we have

$$V_Jf=\sum_{j\in J}T\Lambda_j^*\Lambda_jT^*f=T(\sum_{j\in J}\Lambda_j^*\Lambda_j)(T^*f)=TS_JP_JT^*f$$

or

$$V_Jf=\sum_{j\in J}T\Lambda_j^*\Lambda_jT^*f=T(\sum_{j\in J}\Lambda_j^*\Lambda_j)(T^*f)=TP_JS_JT^*f$$

Hence for all $f \in \overline{T(\mathcal{H}_J)}$

$$V_J f = TP_J S_J T^* f = TS_J P_J T^* f.$$

Lemma 2.1. Let $\{\Lambda_j\}_{j\in J}$ be a g-frame sequence for $\mathcal{H}$ with respect to $\{w_j\}_{j\in J}$. Then the orthogonal projection onto $H_0 = \overline{span}\{\Lambda_j^*(w_j)\}_{j\in J}$ is given by

$$P_{H_0} = \sum_{j\in J} S_\Lambda^{-1}\Lambda_j^*\Lambda_j f \qquad \forall f \in \mathcal{H}. \tag{11}$$

Proof. By assumption we have $S^{-1}(H_0) = H_0$. Hence for all $g \in \mathcal{H}$ we have

$$\begin{aligned}\langle\sum_{j\in J} S_\Lambda^{-1}\Lambda_j^*\Lambda_j f, g\rangle &= \sum_{j\in J}\langle S_\Lambda^{-1}\Lambda_j^*\Lambda_j f, g\rangle = \sum_{j\in J}\langle P_{H_0}S_\Lambda^{-1}\Lambda_j^*\Lambda_j f, g\rangle = \langle f, \sum_{j\in J}\Lambda^* j\Lambda_j S_\Lambda^{-1}P_{H_0}g\rangle \\ &= \langle f, P_{H_0}g\rangle = \langle P_{H_0}f, g\rangle\end{aligned}$$

Hence $P_{H_0}f = \sum_{j\in J} S_\Lambda^{-1}\Lambda_j^*\Lambda_j f$. □

Lemma 2.2. Let $f \in \mathcal{H}$ be and $I \subseteq J$ be finite.Then

$$inf_{\phi\in\mathcal{H}_I}\|f - T\phi\| = \|f - T\psi\|$$

where ψ is defined by

$$\psi = \sum_{j\in I}\Lambda_j^*\Lambda_j T^* V_I^{-1} f \qquad for\ all \quad f \in \mathcal{H}.$$

Proof. Let $f \in \mathcal{H}$ be arbitary and $W \subseteq \mathcal{H}$. We have $\|f - P_W f\| = \inf_{h\,\in W}\|f - h\|$ where $P_W : \mathcal{H} \longrightarrow W$ is orthogonal projection. Then

$$\inf_{\phi\in\mathcal{H}_I}\|f - T\phi\| = \inf_{h\in T\mathcal{H}_I}\|f - h\| = \|f - P_{T\mathcal{H}_I}f\|.$$

In addition

$$P_{T\mathcal{H}_I}f = \sum_{j\in I} T\Lambda_j^*\Lambda_j T^*\Lambda_j^{-1} f.$$

Hence

$$P_{T\mathcal{H}_I}f = \sum_{j\in I} T\Lambda_j^*\Lambda_j T^* V_j^{-1} f = T(\sum_{j\in I}\Lambda^* j\Lambda_j T^* V_j^{-1} f).$$

Put

$$\psi = \sum_{j\in I}\Lambda_j^*\Lambda_j T^* V_j^{-1} f = T(\sum_{j\in I}\Lambda_j^*\Lambda_j T^* V_j^{-1} f)$$

Therefore we obtain

$$\inf_{\phi\in\mathcal{H}_I}\|f - T\phi\| = \inf_{h\in T\mathcal{H}_I}\|f - h\| = \|f - P_{T\mathcal{H}_I}f\| = \|f - T\psi\|.$$

□

Lemma 2.2 leads to a method for approximation of $T^\dagger f$. Let $\{I_n\}_{n\in\mathbb{N}}$be a family of finite subsets of I such that$I_1 \subseteq I_2 \ldots \subseteq I_n \nearrow I$. Abusing the notation, we will write $\mathcal{H}_n, S_n, V_n, P_n$ instead of $\mathcal{H}_{I_n}, S_{I_n}, V_{I_n}, P_{I_n}$. The following lemma states that for $f \in R_T$ we can make $\inf_{\phi\, \in\mathcal{H}_n} \|f - T\phi\|$ arbitrarily small by choosing n large enough.

Lemma 2.3. Let $f \in R_T$ be , then

$$lim_{n\to\infty} \inf_{\phi} {}_{\in\mathcal{H}_n} \|f - T\phi\| = 0.$$

Proof. By lemma (2.2) we have

$$lim_{n\to\infty} \inf_{\phi} {}_{\in\mathcal{H}_n} \|f - T\phi\| = lim_{n\to\infty} \|f - P_{T\mathcal{H}_n} f\|$$

we show given $f \in \mathcal{H}$,

$$lim_{n\to\infty} P_{T\mathcal{H}_n} f = f$$

Since $I_1 \subseteq I_2 \subseteq ... \subseteq I_n \nearrow I$ and $\{\Lambda_j\}_{j\in J}$ is a g-frame for $N_T^\perp$ with respect to $\{W_j\}_{j\in J}$ therefore

$$N_T^\perp = \overline{span}\{\Lambda_j^*(W_j)\}$$

and

$$\mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq ... \subseteq \mathcal{H}_n \nearrow N_T^\perp.$$

Then

$$N_T^\perp = \overline{span}\{\mathcal{H}_n\}$$

and

$$TN_T^\perp = \overline{span}\{T\mathcal{H}_n\}_{n\in\mathbb{N}}.$$

On the other hand

$$T(\mathcal{H}) = T(N_T \oplus N_T^\perp) = T(N_T^\perp) = \overline{span}\{T(\mathcal{H}_n)\}_{n\in\mathbb{N}}$$

Since

$$T(\mathcal{H}_1) \subseteq T(\mathcal{H}_2) \subseteq \ldots \subseteq T(\mathcal{H}_n) \nearrow T(\mathcal{H})$$

Therefore given $f \in \mathcal{H}$

$$lim_{n\to\infty} P_{T(\mathcal{H}_n)} P_{T(\mathcal{H})} f = P_{T(\mathcal{H})} f \quad and \quad lim_{n\to\infty} P_{T(\mathcal{H}_n)} f = f \quad forall \quad f \in R_T$$

Then

$$lim_{n\to\infty} inf_{\phi\in\mathcal{H}_n} \|f - T\phi\| = lim_{n\to\infty} \|f - P_{T\mathcal{H}_n} f\| = 0.$$

□

The next theorem shows how we can obtain a family of operators $\{\phi_n\}_{n\in\mathbb{N}}$ that converges to $T^\dagger$ in the strong operator topology.

Lemma 2.4. For $n \in \mathbb{N}$, define $\phi_n : \mathcal{H} \longrightarrow \mathcal{H}$ by

$$\phi_n f = \sum_{j\in J_n} \Lambda_j^* \Lambda_j T^* V_n^{-1} f$$

then

$$lim_{n\to\infty} \phi_n f = T^\dagger f.$$

Proof. Let $f \in \mathcal{H}$. We have $T\phi_n f = \sum_{j\in J_n} T\Lambda_j^*\Lambda_j T^* V_n^{-1} f {=} P_{T(\mathcal{H}n)} f$ as $n \to \infty$. Then

$$lim_{n\to\infty} T\phi_n f = lim_{n\to\infty} P_{T(\mathcal{H}n)} f = P_{T(\mathcal{H})} f = TT^\dagger f.$$

Since

$$\overline{span}\{\Lambda_j^*(W_j)\}_{j\in J} = N_T^\perp$$

and given $n \in \mathbb{N}$, we have $\phi_n f \in N_T^\perp$. Then

$$T^\dagger T\phi_n f = \phi_n f.$$

It follows that

$$lim_{n\to\infty} \phi_n f = lim T^\dagger TT^\dagger f = T^\dagger f.$$

□

3 Nonlinear iterative approximation of $T^\dagger f$

In the previous section, the index sets $\{I_n\}_{n\in\mathbb{N}}$ were fixed independentlyof f . As a consequence, we obtained a family of operators $\{\phi_n\}$ convergingto $T^\dagger$ in the strong operator topology. We now describe an elementdependent method for approximation of $T^\dagger f$. This means that we fix $f \in \mathcal{H}$ and that the choice of $\{I_n\}_{n\in\mathbb{N}}$ depends on f . The advantage is that the choice of In at the nth step of the approximation might fit f better, but the disadvantage is that the method becomes nonlinear. This method is inspired by various versions of matching pursuit algorithms; cf. [1,2, 3]. Corresponding to an index set I_n, we use the notation $\mathcal{H}_n, S_n, V_n$ as in Section 2. First, fix $f \in \mathcal{H}$ and let $\varepsilon > 0$ be given. Choose the set I_1 such that

$$\|P_{R_T} f - P_{T\mathcal{H}_1} P_{R_T} f\| \le \varepsilon$$

Write $P_{R_T} f = P_{T\mathcal{H}_1} P_{R_T} f + R_1$ and observe that

$$\|P_{R_T} f\|^2 = \|P_{T\mathcal{H}_1} P_{R_T} f\|^2 + \|R_1\|^2$$

Now choose I_2 such that $\|R_1 - P_{T\mathcal{H}_2}R_1\| \leq \varepsilon$. Write $R_1 = P_{T\mathcal{H}_2}R_1 + R_2$ and observe that

$$\|R_2\| = \|R_1 - P_{T\mathcal{H}_2}R_1\| \leq \varepsilon$$

Thus

$$\|P_{R_T}f - (P_{T\mathcal{H}_1}P_{R_T}f + P_{T\mathcal{H}_2}R_1)\| = \|R_2\| \leq \varepsilon.$$

In general, after constructing R_n, chooseI_{n+1} such that

$$\|R_n - P_{TH_{n+1}}R_n\| \leq \varepsilon(2^{-n-1}).$$

Write $R_n = P_{T\mathcal{H}_{n+1}}R_n + R_{n+1}$ and observe that $\|R_{n+1}\| \leq \varepsilon(2^{-n-1})$. Thus, with $R_0 = P_{R_T}f$ we have

$$\|P_{R_T}f\|^2 = \sum_{k=0}^{n} \|P_{T\mathcal{H}_{k+1}}R_k\|^2 + \|R_{n+1}\|^2.$$

Since $\{\Lambda_i T^*\}_{i\in I_{k+1}}$ is a g-frame for $T\mathcal{H}_{k+1}$ and the corresponding g-frame operatoris V_{k+1}, we have by Lemma 2.1 that

$$P_{T\mathcal{H}_{k+1}}R_k = \sum_{j\in J_{k+1}} T\Lambda_j^*\Lambda_j T^* V_{k+1}^{-1}R_k$$

Put $g_k = \sum_{j\in J_{k+1}} \Lambda_j^*\Lambda_j T^* V_{k+1}^{-1}R_k$ then $P_{T\mathcal{H}_{k+1}}R_k = Tg_k$. We obtain

$$\|P_{R_T}f - T\sum_{k=0}^{n} g_k\| < \varepsilon(2^{-n-1})$$

The iterative approximation of $P_{R_T}f$ leads to the following result on approximation of $T^\dagger f$.

Lemma 3.1. Let $f \in \mathcal{H}$ be and construct $\{g_k\}_{k=0}^{\infty}$ as above. Then

$$\|T^\dagger f - \sum_{k=0}^{n} g_k\| \leq \varepsilon(2^{-n-1})\|T^\dagger\|$$

Proof. First observe that for all $k \in \mathbb{N}$ we have

$$g_k \in \overline{span}\{\Lambda_j^*(W_j)\}_{j\in J} = N_T^\perp$$

also $T^\dagger f \in R_{T^\dagger} = N_T^\perp$, and since $T^\dagger T : \mathcal{H} \longrightarrow N_T^\perp$ is the orthogonal projection onto $N_T^\perp$. We obtain

$$\|T^\dagger f - \sum_{k=0}^{n} g_k\| = \|T^\dagger T T^\dagger f - T^\dagger T(\sum_{k=0}^{n} g_k)\| \leq \|T^\dagger\|\|TT^\dagger f - T\sum_{k=0}^{n} g_k\|$$
$$= \|T^\dagger\|\|P_{R_T}f - T\sum_{k=0}^{n} g_k\| < \varepsilon(2^{-n-1})\|T^\dagger\|.$$

□

Let P denote the orthogonal projection onto span $\{\Lambda_j T^*\}_{j\in k+1}$. Then

$$\|f - Pf\| \leq \|f - T\sum_{k=0}^{n} g_k\|.$$

By writing $Pf = Tg$, g might approximate $T^\dagger f$ even better than $\sum_{k=0}^{n} g_k$. However, for large index sets, calculation of g becomes more involved because of the need to invert the frame operator corresponding to $\{\Lambda_j T^*\}_{j\in k+1}$.
The motivation behind the iterative method is to split the inversion into successive inversions of smaller matrices. However, in the worst case the index set I_n may have a lot of overlap with $I_1, I_2, \dots, I_{n-1}$ (or even include those sets) and then the iterative method is not appropriate.

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