International Mathematical Forum, Vol. 8, 2013, no. 40, 1955 - 1962
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/imf.2013.311209

On a Fuzzy Urysohn Integral Equation

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Abstract

The method of upper and lower solutions is used to prove the unique solvability of a fuzzy Urysohn integral equation. This equation involving fuzzy set valued mappings of a real variable whose values are normal, convex, upper semi-continuous and compactly supported fuzzy sets in $\mathbb{R}^n$.

Keywords: Maximal and minimal solutions, fuzzy, Urysohn integral equation.

1 Introduction

Fuzzy sets is one of the most fundamental and in influential tools in computational intelligence. Fuzzy sets can provide solutions to a broad range of problems of control, pattern classification, reasoning, planning, and computer vision, see [20]. In [8] and [9], the authors introduced the concept of integration of fuzzy functions. In [14], the authors suggested and used a Riemann integral type approach while in [15] the author defines the integral of fuzzy function by using the Lebesgue-type concept of integration. We can find more information about fuzzy set, integration of fuzzy functions and fuzzy integral equations in [1-3, 5-16, 18-23] and references therein. Throughout this paper, Ω denotes a nonempty bounded open subset of $\mathbb{R}^n$. The closure of Ω is denoted by $\overline{\Omega}$, and its boundary is denoted by $\partial\Omega$.

On the other hand, nonlinear integral equations appear in many applications. For example, they occur in solving several problems arising in economics, engineering and physics. The most frequently investigated nonlinear integral equations are the Hammerstein integral equation and its generalization, the Urysohn integral equation, see for example [6, 7, 10, 11, 12] and references therein.

In this note, we study the fuzzy integral equation of Urysohn type by means of the fuzzy integral due to Kaleva [15]. The equation under our consideration has the form

$$y(t) = f(t) + \int_{\Omega} u(t, s, y(s))\, ds, \quad t \in \overline{\Omega}. \tag{1}$$

We prove that Eq.(1) has a unique solution by using the monotone iterative method and the notion of upper and lower solutions.

2 Auxiliary facts and results

This section is devoted to collect some definitions and results which will be needed further on.

Definition 2.1 *A fuzzy set Λ in a nonempty set X is characterized by its membership function $\Lambda : X \to [0, 1]$. $\Lambda(x)$ is called the membership function of fuzzy set Λ, it is interpreted as the degree of membership of element x in fuzzy set Λ for each $x \in X$.*

Remark 2.2 *The value zero represents a complete non-membership while the value one is represents a complete membership. The values between zero and one represent intermediate degrees of membership.*

Example 2.3 *The membership function of fuzzy set of real numbers, close to zero, can be given in the form*

$$A(x) = \frac{1}{1 + x^3}$$

Example 2.4 *The membership function of fuzzy set of real numbers, close to one can be given in the form*

$$B(x) = \exp(-\gamma(x-1)^2),$$

where γ *is a positive real number.*

Take $P_j(\mathbb{R}^n)$ denote the collection of all nonempty compact convex subsets of $\mathbb{R}^n$ and define the addition and scalar multiplication in $P_j(\mathbb{R}^n)$ as usual. Let A and B be two nonempty bounded subsets of $\mathbb{R}^n$. The distance between A and B is defined by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\right\}$$

where $d(b,A) = \inf\{d(b,a) : a \in A\}$. This distance is called Hausdorff metric. It is easy to show that the pair $(P_j(\mathbb{R}^n), d)$ form a metric space. This metric space this complete [21].

A fuzzy set $u \in \mathbb{R}^n$ is a function $u : \mathbb{R}^n \to [0,1]$ satisfies the following:

(i) There exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$, i.e., u is normal,

(ii) for $x,\ y \in \mathbb{R}^n$ and $\beta \in [0,1]$, $u(\beta x + (1-\beta)y) \geq \min(u(x), u(y))$, i.e., u is fuzzy convex,

(iii) u is upper semi-continuous, and

(iv) the closure $[u]^0$ of $\{x \in \mathbb{R}^n\ :\ u(x) > 0\}$ is compact.

For $0 < \alpha \leq 1$, the $\alpha-$level set $[u]^\alpha$ is define by $[u]^\alpha = \{x \in \mathbb{R}^n : u(x) \geq \alpha\}$. Then from $(i) - (iv)$, it follows that $[u]^\gamma \in P_j(\mathbb{R}^n)$ for all $0 \leq \alpha \leq 1$.

We can define addition and scalar multiplication in E^n by the aid of Zadeh's extension principle in the forms:

$$\begin{aligned} [u+v]^\alpha &= [u]^\alpha + [v]^\alpha, \\ [\lambda\, u]^\alpha &= \lambda\, [u]^\alpha, \end{aligned}$$

where $u,\ v \in E^n$, $\lambda \in \mathbb{R}$ and $0 \leq \alpha \leq 1$. Let's define $\hat{0} : \mathbb{R}^n \to [0,1]$ by

$$\hat{0}(t) = \begin{cases} 1 & \text{if } t = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Here, $\hat{0}$ is called the null element of E^n.

Now, let $D : E^n \times E^n \to [0,\infty)$ be define by

$$D(u,v) = \sup_{0\leq\alpha\leq 1} H_d\left([u]^\alpha,\ [v]^\alpha\right)$$

where d is the Hausdorff metric defined in $P_j(\mathbb{R}^n)$. Then (E^n, D) is a complete metric space [21]. Also,

(1) $D(u+w, v+w) = D(u,v)$ for $u,\ v,\ w \in E^n$

(2) $D(\lambda u, \lambda v) = |\lambda|\ D(u,v)$ for all $u,\ v \in E^n$ and $\lambda \in \mathbb{R}$

In the next, we sate some definitions and theorems concerning integrability properties for the set-valued mapping of a real variable whose values are in (E^n, D), (see [15, 21]).

Definition 2.5 *A mapping $T : \overline{\Omega} \to E^n$ is said to be a strongly measurable map if for $\gamma \in [0,1]$ the set-valued mapping $T_\gamma : \overline{\Omega} \to P_j(\mathbb{R}^n)$ defined by $T_\gamma(t) = [f(t)]^\gamma$ is Lebesgue measurable, when $P_j(\mathbb{R}^n)$ is endowed with the topology generated by the Hausdorff metric H_d.*

Definition 2.6 *A mapping $T : \overline{\Omega} \to E^n$ is said to be a strongly bounded map if there exists an integrable function h with $\|y\| \le h(t)$ for all $y \in T_0(t)$.*

Definition 2.7 *The integral of $T : \overline{\Omega} \to E^n$ over $\overline{\Omega}$ is given by*

$$\left(\int_\Omega T(t)\, dt\right)^\gamma = \int_\Omega T_\gamma(t)\, dt = \left\{\int f(t)\, dt \mid f : \Omega \to \mathbb{R}^n \text{ is a measurable selection for } T_\gamma\right\}$$

Definition 2.8 *Let $T : \overline{\Omega} \to E^n$ be both strongly measurable and integrably bounded map. The mapping T is said to be integrable over $\overline{\Omega}$ if $\int\limits_\Omega T(t)\, dt \in E^n$.*

Theorem 2.9 *If $T : \overline{\Omega} \to E^n$ be strongly measurable and integrably bounded, then T is integrable.*

Theorem 2.10 *If $T : \overline{\Omega} \to E^n$ is continuous then it is integrable.*

Theorem 2.11 *If $T : \overline{\Omega} \to E^n$ is integrable and $b \in \overline{\Omega}$. Then*

$$\int_{t_0}^{t_0+a} T(t)\, dt = \int_{t_0}^{b} T(t)\, dt + \int_{b}^{t_0+a} T(t)\, dt$$

Theorem 2.12 *If $T,\ S : \overline{\Omega} \to E^n$ be integrable and $\lambda \in \mathbb{R}$. Then*

(1) $\int\limits_\Omega (T(t) + S(t))\, dt = \int\limits_\Omega T(t)\, dt + \int\limits_\Omega S(t)\, dt,$

(2) $\int\limits_\Omega \lambda\, T(t)\, dt = \lambda \int\limits_\Omega T(t)\, dt,$

(3) *$D(T, S)$ is integrable,*

(4) $D\left(\int\limits_{\Omega} T(t)\,dt, \int\limits_{\Omega} S(t)\,dt\right) \leq \int\limits_{\Omega} D(T(t), S(t))\,dt.$

Now, we state the definitions of upper and lower and maximal and minimal solutions, respectively.

Definition 2.13 *A function* $u \in C(\overline{\Omega}, E^n)$ *is called an upper solution of equation (1) if*

$$u(t) \geq f(t) + \int_{\Omega} u(t, s, y(s))\,ds, \quad t \in \overline{\Omega}.$$

Similarly, a function $l \in C(\overline{\Omega}, E^n)$ *is called a lower solution of equation (1) if*

$$l(t) \leq f(t) + \int_{\Omega} u(t, s, y(s))\,ds, \quad t \in \overline{\Omega}.$$

Definition 2.14 *The function* $\overline{y}(t)$ $(\underline{y}(t))$ *is said to be a maximal (minimal) solution of equation (1), if* $y(t) \leq \overline{y}(t)$ $(\underline{y}(t) \leq y(t))$ *for all* $t \in \overline{\Omega}$.

3 Main Theorems

In this section, we will study Eq.(1) assuming that the following assumptions are satisfied.

(a_1) The functions $u,\ l \in C(\overline{\Omega}, E^n)$ with $u(t) \geq l(t)$ for $t \in \overline{\Omega}$ are upper and lower solutions, respectively, of equation (1).

(a_2) For each i, $u_i(t, s, \phi)$ is monotone increasing in ϕ for fixed t and s in $\overline{\Omega}$.

(a_3) There exists a continuous function $m : \overline{\Omega} \to \mathbb{R}_+$ such that

$$H_d\left([u(t, s, y)]^\alpha, [u(t, s, x)]^\alpha\right) \leq m(t)\ H_d\left([y]^\alpha, [x]^\alpha\right)$$

for all $x,\ y \in E^n$ and $t,\ s \in \overline{\Omega}$.

Let's define the sequences $\{u_n\}$ and $\{l_n\}$ by

$$u_n(t) = f(t) + \int_{\Omega} u(t, s, u_{n-1}(s))\,ds,\ t \in \bar{\Omega},$$

$$l_n(t) = f(t) + \int_{\Omega} u(t, s, l_{n-1}(s))\,ds,\ t \in \bar{\Omega},$$

with $u_0(t) = u(t)$ and $l_0(t) = l(t)$, $t \in \bar{\Omega}$.

Theorem 3.1 *Let the assumptions* (a_1) *and* (a_2) *be satisfied. Then the sequence* $\{u_n\}$ *converges uniformly from above to the maximal solution* $\bar{y}(t)$ *of equation (1) while the sequence* $\{l_n\}$ *converges uniformly from below to the minimal solution* $\underline{y}(t)$ *of equation (1). Moreover, if* $y(t)$ *is any solution of equation (1) such that* $l(t) \leq y(t) \leq u(t)$ *on* $\bar{\Omega}$*, then*

$$l \leq l_1 \leq l_2 \leq ... \leq l_n \leq ... \leq \underline{y} \leq y \leq \bar{y} \leq ... \leq u_n \leq u_{n-1} \leq ... \leq u_1 \leq u$$

on $\overline{\Omega}$*.*

Proof: The proof is straight forward as the proof of Theorem 1 in [17]. □

Now, if the function u satisfy the assumption (a_1), then we have the following result

Theorem 3.2 *Let the assumptions of Theorem* 3.1 *be satisfied. In addition assume* (a_3) *holds. Then both the maximal solution* $\overline{y}(t)$ *and the minimal solution* $\underline{y}(t)$ *are coincide on* $\overline{\Omega}$*.*

Proof: Define $z(t) = H_d\left([\overline{y}(t)]^\alpha, [\underline{y}(t)]^\alpha\right)$, $t \in \overline{\Omega}$. Then we have

$$\begin{aligned} z(t) &= H_d\left(\left[f(t) + \int_\Omega u(t,s,\overline{y}(s))\ ds\right]^\alpha, \left[f(t) + \int_\Omega u(t,s,\underline{y}(s))\ ds\right]^\alpha\right) \\ &\leq \int_\Omega H_d\left([u(t,s,\overline{y}(s))]^\alpha, \left[u(t,s,\underline{y}(s))\right]^\alpha\right)\ ds \\ &\leq \int_\Omega m(s)\ H_d\left([\overline{y}(s)]^\alpha, \left[\underline{y}(s)\right]^\alpha\right)\ ds \\ &\leq \int_\Omega m(s) z(s)\ ds \\ &\leq \epsilon \int_\Omega m(s) z(s)\ ds. \end{aligned}$$

Now, by a suitable application of Theorem 1 in [4], we obtain

$$z(t) \leq \epsilon \exp\left(\int_\Omega m(s)\ ds\right) \text{ for } t \in \overline{\Omega}.$$

Therefore, $\overline{y}(t) = \underline{y}(t)$ on $\overline{\Omega}$ as $\epsilon \to 0$ and this completes the proof. □

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Received: November 5, 2013