Connectedness and its Applications

Hiranmay Dasgupta

Department of Mathematics, JIS College of Engineering
Phase-III, Block-A, Kalyani, Nadia
Pin-741235, West Bengal, India
hiranmaydg@yahoo.com

Sucharita Chakrabarti

Guru Nanak Institute of Technology
157/F Nilgunj Road, Panihati
Kolkata-700114, West Bengal, India
sucharitamath@gmail.com

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Abstract

In this paper we introduced and studied the notions of m -connecteness and its applications in spaces with minimal structures. We also explored and studied the concepts of m_X -components, m -quasi-nodal sets, m -quasi-directed spaces m_X -ending in this weaker space.

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1. Introduction.

Topological spaces have been generalized in different ways, for example, the abstract space [1], the supra-topological space [7], a space with minimal structure ([6], [9]). In this paper we have explored the notion of connectedness, a familiar

and important concept of topological spaces within the frame work of minimal structure ([6], [9]). We also introduced the concepts of m_X -components, m -quasi-nodal sets, m_X -ending, m -quasi-directed spaces in these weaker spaces and studied some basic properties of these notions. The advantage of considering our study through m -structure is that one can get easily the reflection of our exposition in other particular spaces of which m -space is a generalization through a single attempt.

For easy understanding of our work provided here we recall some basic definitions and results, for details of which we refer to [6], [9], [3], [4].

Throughout the paper (X, m_X) , (Y, m_Y) stand for m -spaces with minimal structures m_X , m_Y defined on X , Y respectively. Also $\mathbb{N}$, $\mathbb{R}$ are respectively the set of all natural numbers and the set of all reals, $\mathcal{P}(X)$ is the class of all subsets of X , ϕ being the empty set.

2. Preliminaries

Definition 2.1 ([6], cf. [9]) Let X be a non-empty set. A subfamily m_X of $\mathcal{P}(X)$ is called a minimal structure (briefly m -structure) on X if $\phi, X \in m_X$. (X, m_X) is called a space with minimal structure m_X or simply an m -space. Each member of m_X is called an m_X -open set and complement of an m_X -open set is called an m_X -closed set.

Remark 2.1. Topological spaces, supra-topological spaces [7], abstract spaces [1] are all spaces with minimal structures, but not conversely as (X, m_X) where $X = [0, 1]$ and $m_X = \{X, \phi, [\frac{1}{8}, \frac{1}{2}], [\frac{1}{4}, \frac{2}{3}]\}$ is a space with minimal structure, which is not any one of the above mentioned spaces.

Definition 2.2. [6] For a subset A of X in (X, m_X) , the m_X -closure of A and the m_X -interior of A are defined as follows:

- (1) $m_X\text{-Cl}(A) = \cap \{F : A \subset F, X - F \in m_X\}$
- (2) $m_X\text{-Int}(A) = \cup \{U : U \subset A, U \in m_X\}$

Lemma 2.1. [6] For subsets A and B of X in (X, m_X) the following results hold:

- (1) $m_X\text{-Cl}(X - A) = X - m_X\text{-Int} A$ and $m_X\text{-Int}(X - A) = X - m_X\text{-Cl} A$.
- (2) $m_X\text{-Cl} A = A$ if $(X - A) \in m_X$ and $m_X\text{-Int} A = A$ if $A \in m_X$.
- (3) $m_X\text{-Cl}(\phi) = \phi$, $m_X\text{-Cl}(X) = X$, $m_X\text{-Int}(\phi) = \phi$ and $m_X\text{-Int}(X) = X$.
- (4) If $A \subset B$, then $m_X\text{-Cl} A \subset m_X\text{-Cl} B$ and $m_X\text{-Int} A \subset m_X\text{-Int} B$.
- (5) $A \subset m_X\text{-Cl} A$ and $m_X\text{-Int} A \subset A$.
- (6) $m_X\text{-Cl}(m_X\text{-Cl}(A)) = m_X\text{-Cl} A$, $m_X\text{-Int}(m_X\text{-Int}(A)) = m_X\text{-Int}(A)$.

Lemma 2.2. [9] Let A be a subset of X in (X, m_X) . Then $x \in m_X\text{-Cl}(A)$ if and only if $U \cap A \neq \phi$ for every $U \in m_X$ containing x .

Remark 2.2. In topological space, the converse of Lemma 2.1.(2) is also true. But in (X, m_X) the converse of Lemma 2.1.(2) is not necessarily true as

shown by the following example:

Example 2.1. We consider the space (X, m_X) , where $X = \{a, b, c\}$, $m_X = \{X, \phi, \{a\}, \{b\}\}$. Now $m_X\text{-Int}\{a, b\} = \{a, b\}$ and $m_X\text{-Cl}\{c\} = \{c\}$ though $\{a, b\}$ is not m_X -open and $\{c\}$ is not m_X -closed.

To overcome this limitation and to make possible further study the following notions are introduced:

Definition 2.3.[3] Let $A \subset X$ in (X, m_X) . Then A is called m_X -pseudo-open if $m_X\text{-Int}(A) = A$ and A is called m_X -pseudo-closed if $m_X\text{-Cl}(A) = A$.

Remark 2.3. From Lemma 2.1(1) it follows that complement of an m_X -pseudo-open set is m_X -pseudo-closed and conversely. Also from Lemma 2.1(6) it follows that in (X, m_X) , for $A \subset X$, $m_X\text{-Cl}(A)$ and $m_X\text{-Int}(A)$ are respectively m_X -pseudo-closed and m_X -pseudo-open sets.

Remark 2.4. Every m_X -open (m_X -closed) set is m_X -pseudo-open (m_X -pseudo-closed) but the converse is not true as shown by the following example:

Example 2.2. We consider the space (X, m_X) , where $X = \{a, b, c\}$, $m_X = \{X, \phi, \{a\}, \{b\}\}$. Now $m_X\text{-Int}\{a, b\} = \{a\} \cup \{b\} = \{a, b\}$. Thus $\{a, b\}$ is an m_X -pseudo-open set, though it is not an m_X -open set. By Remark 2.3, $\{c\}$ is an m_X -pseudo-closed set, though $\{c\}$ is not an m_X -closed set.

Remark 2.5. If m_X is a topology on X then the notion of m_X -pseudo-open set (m_X -pseudo-closed set) coincides with the notion of open set (closed set) in the topological space (X, m_X) .

Definition 2.4.[4] A mapping $f : (X, m_X) \rightarrow (Y, m_Y)$ is called M^* -continuous if and only if $f^{-1}(V) \in m_X$ whenever $V \in m_Y$.

Proposition 2.1.[4] Let $f : (X, m_X) \rightarrow (Y, m_Y)$ be M^* -continuous. Then $f(m_X\text{-Cl}(A)) \subset m_Y\text{-Cl}(f(A))$ for every $A \subset X$.

Lemma 2.3. Let $f : (X, m_X) \rightarrow (Y, m_Y)$ be M^* -continuous. Then $m_X\text{-Cl}(f^{-1}(A)) \subset f^{-1}(m_Y\text{-Cl}(A))$ for every $A \subset Y$.

Proof. Replacing A with $f^{-1}(A)$ in Proposition 2.1, we get $f(m_X\text{-Cl}(f^{-1}(A))) \subset m_Y\text{-Cl}(f(f^{-1}(A)))$. So $f^{-1}(f(m_X\text{-Cl}(f^{-1}(A)))) \subset f^{-1}(m_Y\text{-Cl}(f(f^{-1}(A)))) \subset f^{-1}(m_Y\text{-Cl}(A))$. Therefore $m_X\text{-Cl}(f^{-1}(A)) \subset f^{-1}(f(m_X\text{-Cl}(f^{-1}(A)))) \subset f^{-1}(m_Y\text{-Cl}(A))$. This proves the lemma.

Definition 2.5. In (X, m_X) an m_X -boundary of a subset A of X , denoted by $m_X\text{-}b(A)$, is defined by $m_X\text{-}b(A) = m_X\text{-Cl}(A) - m_X\text{-Int}(A)$ or equivalently $m_X\text{-}b(A) = m_X\text{-Cl}(A) \cap m_X\text{-Cl}(X - A)$. A point p is called an m_X -boundary point of A if $p \in m_X\text{-}b(A)$. A point q is called an external m_X -boundary point of A if $q \in m_X\text{-}b(A)$, but $q \notin A$. Thus the collection of all external m_X -boundary points of A is called external m_X -boundary of A and denoted by $\text{ext-}m_X\text{-}b(A)$. Hence $\text{ext-}m_X\text{-}b(A) = m_X\text{-}b(A) - A = m_X\text{-Cl}(A) - A$.

Example 2.3. We consider the space (X, m_X) , where $X = [0, 2]$, $m_X = \{X, \phi, [0, 1], [\frac{1}{2}, \frac{3}{2}]\}$. We take $A = [0, 1]$. Then $m_X\text{-Int}(A) = [0, 1)$ and $m_X\text{-Cl}(A) = X$. Thus $m_X\text{-}b(A) = m_X\text{-Cl}(A) - m_X\text{-Int}(A) = X - [0, 1) = [1, 2]$. Now $\text{ext-}m_X\text{-}b(A) = m_X\text{-Cl}(A) - A = (1, 2]$.

3. connectedness

Definition 3.1. In (X, m_X) an m_X -neighbourhood of a subset A of X is a subset U of X such that $A \subset m_X\text{-Int}U$.

Definition 3.2. In (X, m_X) two subsets A and B of X are called m_X -separated in (X, m_X) (or m_X -separated in X or simply m_X -separated), denoted by $X = A \setminus B$, if there exist m_X -neighbourhoods U of A and V of B in X such that $U \cap B = \phi = V \cap A$.

Example 3.1. We consider the space (X, m_X) , where $X = [0, 2]$ and $m_X = \{X, \phi, (0, \frac{1}{2}), (\frac{1}{2}, \frac{3}{2}), (1, 2)\}$. We take $A = (\frac{1}{8}, \frac{1}{4})$ and $B = (\frac{2}{3}, \frac{5}{6})$. Now, $(0, \frac{1}{2})$ is an m_X -neighbourhood of A and $(\frac{1}{2}, \frac{3}{2})$ is an m_X -neighbourhood of B . Also $A \cap ((\frac{1}{2}, \frac{3}{2})) = \phi$ and $B \cap (0, \frac{1}{2}) = \phi$. So, A and B are m_X -separated.

Theorem 3.1. In (X, m_X) let $A, B \subset X$. Then A and B are m_X -separated in X if and only if $(A \cap m_X\text{-Cl}(B)) \cup (m_X\text{-Cl}(A) \cap B) = \phi$.

Proof. The proof is routine and so omitted.

Theorem 3.2. Let $f : (X, m_X) \rightarrow (Y, m_Y)$ be an M^* -continuous mapping. If A and B are m_Y -separated in (Y, m_Y) , then $f^{-1}(A)$ and $f^{-1}(B)$ are m_X -separated in (X, m_X) .

Proof. The proof is routine and so omitted.

Definition 3.3. A subset A of X in (X, m_X) is called m -connected in X (or simply m -connected) if A is not the union of two nonempty m_X -separated subsets of X . Thus a nonempty set A is m -connected if and only if $A = A_1 \cup A_2$ with $A_1, A_2 \subset X$ and $(A_1 \cap m_X\text{-Cl}(A_2)) \cup (m_X\text{-Cl}(A_1) \cap A_2) = \phi$ together imply either $A_1 = \phi$ or $A_2 = \phi$.

If A is not m -connected in X , then we say A is m -disconnected in X . A space (X, m_X) is called m -connected if the underlying set X is m -connected.

Theorem 3.3. A space (X, m_X) is m -connected if and only if X contains no proper nonempty subset simultaneously m_X -pseudo-open and m_X -pseudo-closed.

Proof. The proof is routine and so omitted.

Theorem 3.4. A space (X, m_X) is m -connected if and only if X is not the union of two disjoint, nonempty m_X -pseudo-closed (or m_X -pseudo-open) subsets of X .

Proof. The proof follows from Theorem 3.3.

Definition 3.4. Let $\phi \neq Y \subset X$ in (X, m_X) . Then the space (Y, m_Y) is called an m -subspace of (X, m_X) if $m_Y = \{A \cap Y : A \in m_X\}$.

Lemma 3.1. Let (Y, m_Y) be an m -subspace of (X, m_X) and let $A \subset Y \subset X$. Then $m_Y\text{-Cl}(A) = m_X\text{-Cl}(A) \cap Y$.

Proof. We know that $m_Y\text{-Cl}(A) = \bigcap \{F^* : A \subset F^*, Y - F^* \in m_Y\} = \bigcap \{F \cap Y : A \subset F \cap Y, X - F \in m_X\} = \bigcap \{F \cap Y : A \subset F, X - F \in m_X\} = \bigcap \{F : A \subset F, X - F \in m_X\} \cap Y = m_X\text{-Cl}(A) \cap Y$. This proves the lemma.

Theorem 3.5. Let (Y, m_Y) be an m -subspace of (X, m_X) and let $A \subset Y \subset X$.

Then A is m -connected in (Y, m_Y) if and only if A is m -connected in (X, m_X) .

Proof. Let A be m -disconnected in Y . Then $A = P \cup Q$, $P \neq \phi$, $Q \neq \phi$, $P, Q \subset Y$, $P \cap m_Y\text{-Cl}(Q) = \phi = m_Y\text{-Cl}(P) \cap Q$. Now $\phi = m_Y\text{-Cl}(P) \cap Q = m_X\text{-Cl}(P) \cap Y \cap Q$ i.e., $(m_X\text{-Cl}(P) \cap Q) = \phi$ and similarly $(P \cap m_X\text{-Cl}(Q)) = \phi$. So A has an m_X -separation in X and hence A is m -disconnected in X .

Conversely let A be m -disconnected in X . Then $A = P \cup Q$, $P \neq \phi$, $Q \neq \phi$, $P, Q \subset X$, $P \cap m_X\text{-Cl}(Q) = \phi = m_X\text{-Cl}(P) \cap Q$. Now, $m_Y\text{-Cl}(P) \cap Q = m_X\text{-Cl}(P) \cap Y \cap Q = (m_X\text{-Cl}(P) \cap Q) \cap Y = \phi$. Thus $m_Y\text{-Cl}(P) \cap Q = \phi$. Similarly $m_Y\text{-Cl}(Q) \cap P = \phi$. So A is m -disconnected in Y . This completes the proof.

Corollary 3.1. A subset Y of X in (X, m_X) is m -connected if and only if the m -subspace (Y, m_Y) is m -connected.

Theorem 3.6. A space (X, m_X) is m -connected if and only if there exists no nonempty proper subset A of X such that $m_X\text{-Cl}(A) \cap m_X\text{-Cl}(X - A) = \phi$, i.e., every subset A of X such that $\phi \neq A \neq X$ satisfies the condition $m_X\text{-Cl}(A) \cap m_X\text{-Cl}(X - A) \neq \phi$.

Proof. First let (X, m_X) be m -connected. If possible let there be a subset A of X such that $\phi \neq A \neq X$ and $m_X\text{-Cl}(A) \cap m_X\text{-Cl}(X - A) = \phi$. Let $B = X - A$. So $X = A \cup B$ and $A \cap B = \phi$. Now $(A \cap m_X\text{-Cl}(B)) \subset m_X\text{-Cl}(A) \cap m_X\text{-Cl}(B) = m_X\text{-Cl}(A) \cap m_X\text{-Cl}(X - A) = \phi$. Similarly we can prove that $B \cap m_X\text{-Cl}(A) = \phi$. Thus X has an m_X -separation and so X is m -disconnected, which is a contradiction. Hence X contains no subset A such that $\phi \neq A \neq X$ and $m_X\text{-Cl}(A) \cap m_X\text{-Cl}(X - A) = \phi$.

Conversely let $m_X\text{-Cl}(A) \cap m_X\text{-Cl}(X - A) \neq \phi$, for any subset A of X such that

$\phi \neq A \neq X$. We prove that (X, m_X) is m -connected. If possible let (X, m_X) be m -disconnected. Then there exists at least one nonempty subset P of X , which is both m_X -pseudo-open and m_X -pseudo-closed. Let $Q = X - P$. Then Q is also both m_X -pseudo-open and m_X -pseudo-closed and $P \cap Q = \phi$. Therefore $m_X\text{-Cl}(P) = P$ and $m_X\text{-Cl}(Q) = Q$. So $m_X\text{-Cl}(P) \cap m_X\text{-Cl}(X - P) = m_X\text{-Cl}(P) \cap m_X\text{-Cl}(Q) = P \cap Q = \phi$, which is a contradiction. Hence (X, m_X) is m -connected.

This completes the proof.

Remark 3.1. It follows from Theorem 3.6 that a space (X, m_X) is m -connected if and only if no subset A of X , which satisfies the condition $\phi \neq A \neq X$ has an empty m_X -boundary.

Theorem 3.7. A nonempty subset C of a space (X, m_X) is m -connected if and only if every nonempty proper subset A of C satisfies the condition $C \cap m_X\text{-Cl}(A) \cap m_X\text{-Cl}(C - A) \neq \phi$.

Proof. The proof follows from Theorem 3.5 and Theorem 3.6.

Theorem 3.8. If C is m -connected and $C \cap A \neq \phi \neq C - A$ then $C \cap m_X\text{-Cl}(A) \neq \phi$.

Proof. Let C be m -connected in (X, m_X) . Then the subspace (C, m_C) is m -connected in C . then by Theorem 3.6, for every proper nonempty sub-

set A of C , $m_C\text{-Cl}(A) \cap m_C\text{-Cl}(X - A) \neq \phi$, i.e., $(C \cap m_X\text{-Cl}(A)) \cap (m_X\text{-Cl}(X - A)) \cap C \neq \phi$, i.e., $C \cap [m_X\text{-Cl}(A) \cap m_X\text{-Cl}(X - A)] \neq \phi$, i.e., $m_X\text{-Cl}(A) \cap m_X\text{-Cl}(X - A) \neq \phi$. This completes the proof of the theorem.

Theorem 3.9. Let $f : (X, m_X) \rightarrow (Y, m_Y)$ be M^* -continuous and $A \subset X$. Then if A is m -connected in X then $f(A)$ is m -connected in Y .

Proof. The proof follows from Theorem 3.2.

Theorem 3.10. A subset C of X in (X, m_X) is m -connected if and only if the following condition is fulfilled:

If C is contained in the union of two m_X -separated sets A and B , then either $C \subset A$ or $C \subset B$.

Proof. The proof is routine and so omitted.

Theorem 3.11. If E is an m -connected subset of X in (X, m_X) and C is any subset of X such that $E \subset C \subset m_X\text{-Cl}(E)$, then C is also m -connected.

Proof. The proof follows from Theorem 3.10.

Theorem 3.12. If every two points of a subset E of X in (X, m_X) are contained in some m -connected subset of E , then E is m -connected.

Proof. The proof follows from Theorem 3.10.

Theorem 3.13. In a space (X, m_X) the union of any family of m -connected sets having a nonempty intersection is an m -connected set.

Proof. The proof is routine and so omitted.

Theorem 3.14. In a space (X, m_X) the union of any family of m -connected sets with the property that one of the members of the family intersects every other member is m -connected.

Proof. Let $\{E_\alpha : \alpha \in \Delta\}$ be a family of nonempty m -connected subsets with the property that one of the members, say E_u intersects every other member, i.e., $E_u \cap E_\alpha \neq \phi$ for $\alpha \in \Delta$, $\alpha \neq u$.

So $A_\alpha = E_u \cup E_\alpha$ is m -connected in X for all $\alpha \in \Delta$, by Theorem 3.13. Now $\bigcap_{\alpha \in \Delta} A_\alpha = \bigcap_{\alpha \in \Delta} (E_u \cup E_\alpha) \supset E_u \neq \phi$. So by Theorem 3.13, $\bigcup_{\alpha \in \Delta} A_\alpha$ is m -connected in X . This proves the theorem.

Theorem 3.15. Let $\{C_t\}$ be a family of m -connected sets in a space (X, m_X) with minimal structure m_X . The union $\bigcup_t C_t$ is m -connected, provided that

there exists such a set C_0 in $\{C_t\}$, which is not m_X -separated with any other set C_t of the family $\{C_t\}$.

Proof. Let $\bigcup_t C_t = M \cup N$, where M and N are m_X -separated in X . We show that either $M = \phi$ or $N = \phi$. By theorem 3.10 we assume that $C_0 \subset N$. Now we assert that for any t , $C_t \subset N$ because if for any t $C_t \subset M$ then $m_X\text{-Cl}(C_t) \cap C_0 \subset m_X\text{-Cl}(M) \cap N = \phi$ and $m_X\text{-Cl}(C_0) \cap C_t \subset m_X\text{-Cl}(N) \cap M = \phi$ since M and N are m_X -separated in X and so C_t and C_0 are m_X -separated in X , which is a contradiction. Hence $M = \phi$. This completes the proof of the

theorem.

Remark 3.2. The following example shows that a family of m -connected sets in (X, m_X) may be m -connected if none of the conditions in Theorem 3.13, Theorem 3.14 and Theorem 3.15 are satisfied, proving Theorems 3.13, 3.14 and 3.15 only provide a sufficient condition for the m -connectedness of the union of family of m -connected sets.

Example 3.2. We consider the space (X, m_X) , where $X = \mathbb{R}$ and $m_X = \{\mathbb{R}, \phi, \mathbb{R} - \{1\}, \mathbb{R} - \{2\}\}$. We consider the family of sets $\{E_i, i \in K, K \subset \mathbb{N}\}$. We take $E_i = \{i + 1\}$. Each E_i is m -connected, does not satisfy the conditions in Theorem 3.13, Theorem 3.14 and Theorem 3.15. But $\bigcup\{E_i, i \in K, K \subset \mathbb{N}\}$ is m -connected.

Theorem 3.16. Let $\{C_t : t \in \Delta\}$ be a directed family of nonempty m -connected sets (this means that for each pair t_1, t_2 in Δ there is t_3 in Δ such that $C_{t_1} \subset C_{t_3}$ and $C_{t_2} \subset C_{t_3}$) in a space (X, m_X) . Then the union $S = \bigcup_{t \in \Delta} C_t$ is m -connected.

Proof. Let $S = M \cup N$, where M and N are m_X -separated in X . By Theorem 3.10, we have for each t , $C_t \subset M$ or $C_t \subset N$. We may assume that $C_{t_0} \subset M$. We show that $S \subset M$, which will complete the proof.

Let t be an arbitrary index and t_1 be such that $C_{t_0} \subset C_{t_1}$ and $C_t \subset C_{t_1}$. The first inclusion yields $C_{t_1} \not\subset N$. Hence $C_{t_1} \subset M$ and therefore $C_t \subset M$. It follows that $S \subset M$.

Remark 3.3. In a space (X, m_X) the intersection of two m -connected sets may not be m -connected, which is shown by the following example:

Example 3.3. We consider the space (X, m_X) , where $X = \{a, b, c, d\}$, $m_X = \{X, \phi, \{a, c, d\}, \{a, b, d\}\}$. We take $A = \{a, b, c\}$ and $B = \{b, c, d\}$. Here both A and B are m -connected and $A \cap B = \{b, c\}$ is not m -connected.

4. components, quasi-nodal sets and quasi-directed spaces.

Definition 4.1. Let $Y \subset X$ in (X, m_X) . Then a set $C \subset Y$ is called an m_X -component in Y if C is m -connected in X and if the inclusion $C \subset C_1 \subset Y$ implies $C = C_1$ for any m -connected set C_1 in X . C is an m_X -component in (X, m_X) if C is the m_X -component in the underlying set X . Thus m_X -components in (X, m_X) are maximal m -connected subsets of X .

Example 4.1. We consider the space (X, m_X) , where $X = \{a, b, c, d\}$, $m_X = \{X, \phi, \{c, d\}, \{a, b, d\}\}$. Let $Y = \{a, b, c\}$. Y is not m -connected in X as $\{c\}/\{a, b\}$ forms an m_X -separation of Y . Clearly $\{a, b\}, \{a\}, \{b\}, \{c\}$ are the only nonempty subsets of Y , which are m -connected in X . Thus $\{a, b\}$ and $\{c\}$ are the maximal m -connected subsets of Y in X . Hence m_X -components in Y are $\{a, b\}$ and $\{c\}$, but $\{a\}, \{b\}$ are not an m_X -components in Y . It can be verified that X is the only m_X -component in X as X is m -connected.

Remark 4.1. The space (X, m_X) is m -connected if and only if X is the only m_X -component in X .

Definition 4.2. A nonempty subset N of X in (X, m_X) is called an m_X -quasi-nodal set if the m_X -boundary of N in X is degenerate (A set is degenerate if it is either empty or singleton).

Example 4.2. We consider the space (X, m_X) , where $X = \{a, b, c, d\}$, $m_X = \{X, \phi, \{a\}, \{b\}, \{c\}\}$. We take $A = \{a, c\}$. Then $m_X\text{-Cl}(A) = \{a, c, d\}$ and $m_X\text{-Int}(A) = \{a, c\}$. Now $m_X\text{-b}(A) = m_X\text{-Cl}(A) - m_X\text{-Int}(A) = \{d\}$. Thus A is an m_X -quasi-nodal set.

Definition 4.3. A point $p \in X$ in (X, m_X) is called an m_X -separating point of an m -connected subset C of X if $C - \{p\}$ is not m -connected.

Example 4.3. In Example 4.2, we take $A = \{a, b, d\}$ which is m -connected in X . Now $A - \{d\} = \{a, b\}$, which is not m -connected in X . Thus d is an m_X -separating point of A in X , but a and b are not m_X -separating points of A in X as both $A - \{a\} = \{b, d\}$ and $A - \{b\} = \{a, d\}$ are m -connected. Obviously c is not an m_X -separating point of A in X .

Definition 4.4. A point $p \in X$ of (X, m_X) is called an m_X -end point of (X, m_X) if p is not an m_X -separating point of any m -connected subset C of X .

Example 4.4. In Example 4.2, it can be verified that b is an m_X -end point of (X, m_X) , but d is not an m_X -end point of (X, m_X) .

Definition 4.5. A subset E of X of (X, m_X) is said to have the m_X -ending property (m_X -e. p.) in X if there is no m -connected subset C of X such that E separates C (i.e., such that $C - E$ is not m -connected).

Example 4.5. In Example 4.2, we take $E = \{a, b\}$. It can be verified that $C - E$ is m -connected for all m -connected subset C of X . So E has m_X -e. p., but $F = \{a, d\}$ does not have m_X -e. p. in X .

Theorem 4.1. The union of a finite collection of sets with m_X -e. p. in a space (X, m_X) has m_X -e. p. in X .

Proof. Let A and B be two sets having m_X -e. p. in X . We prove that $A \cup B$ has m_X -e. p. in X . Let C be any m -connected set in X . Then since A has m_X -e. p. in X , $C - A$ is m -connected in X and so $(C - A) - B = C - (A \cup B)$ is m -connected in X as B has m_X -e. p. in X . Thus $A \cup B$ has m_X -e. p. in X . This proves the theorem.

Remark 4.2. The finite intersection of two sets having m_X -e. p. in (X, m_X) need not have m_X -e. p. in X as shown by the following example:

Example 4.6. We consider the space (X, m_X) , where $X = \{a, b, c\}$, $m_X = \{X, \phi, \{a\}, \{b\}\}$. Now $\{a, c\}$, $\{b, c\}$ have m_X -e. p. in X . But $\{a, c\} \cap \{b, c\} = \{c\}$ does not have m_X -e. p. in X .

In the following theorem a sufficient condition is sorted out to enjoy the m_X -e. p. property of the intersection of a collection of sets each having m_X -e. p. property.

Theorem 4.2. If $\mathcal{E}$ is a nested collection of sets having m_X -e. p. in a space

(X, m_X) , then $\bigcap \mathcal{E}$ has m_X -e. p. (possibly trivially) in X .

Proof. Let $N = \bigcap \mathcal{E}$. We prove that N has m_X -e. p. in X . Now if $N = \phi$, then the theorem is trivially true. For $N \neq \phi$, we suppose that there is an m -connected set C such that $C - N$ is the union of two m_X -separated sets A and B . Now since $A \cap N = \phi$, A can not be contained in all the elements of $\mathcal{E}$ and so there is a member E_A of $\mathcal{E}$ such that $A - E_A \neq \phi$. Similarly there is a member E_B of $\mathcal{E}$ such that $B - E_B \neq \phi$. Let $E = E_A \cap E_B$. Since $\mathcal{E}$ is nested, $E = E_A \cap E_B = E_A$ or E_B and so $E \in \mathcal{E}$ and $A - E$ and $B - E$ both are nonempty. But then $C - E = (A - E) \cup (B - E)$, an m_X -separation. This contradicts that E has m_X -e. p. This proves the theorem.

Remark 4.3. In the Theorem 4.2, nested condition is a sufficient condition as shown by the following example:

Example 4.7. We consider the space (X, m_X) , where $X = \mathbb{R}$, $m_X = \{\mathbb{R}, \phi, \mathbb{R} - \{1\}, \mathbb{R} - \{2\}\}$ so that m_X -closed sets are $\phi, \mathbb{R}, \{1\}, \{2\}$. Now for any non empty subset A of X

$$\begin{aligned} m_X - Cl(A) &= \{1\} \text{ if } A = \{1\} \\ &= \{2\} \text{ if } A = \{2\} \\ &= X = \mathbb{R}, \text{ otherwise.} \end{aligned}$$

Now $\{1, 2\}$ is the only nonempty subset of X which is not m -connected. We now consider the family $\{E_i\}$, where $E_i = \{1, i + 1\}$, $i \in \mathbb{N}$ each having m_X -e. p. Then $\bigcap_{i \in \mathbb{N}} E_i = \{1\}$ also has m_X -e. p in X , though the collection $\{E_i\}$ is not nested.

Theorem 4.3. In a space (X, m_X) if E has m_X -e. p. in X , then the intersection of any m -connected set with $X - E$ is m -connected.

Proof. Let C be any m -connected set in X . Since E has m_X -e. p. in X , so $(C - E)$ is m -connected. Now $C \cap (X - E) = X \cap (C - E) = C - E$, which is m -connected. So $C \cap (X - E)$ is m -connected. This proves the theorem.

Remark 4.4. In a topological space X , if A, B are separated sets and A, C are also separated sets then A and $B \cup C$ are also separated sets. This does not happen in (X, m_X) due to the fact that in (X, m_X) , $m_X - Cl(A) \cup m_X - Cl(B)$ may be a proper subset of $m_X - Cl(A \cup B)$. We therefore introduce the notion of an m -quasi-directed space (weaker than topological space) that has the same property of a topological space with respect to separation of sets as stated.

Definition 4.6. In (X, m_X) let $A, B, N \subset X$. If A, B are m_X -separated in X and A, N are m_X -separated in X imply $A, B \cup N$ are m_X -separated in X , then (X, m_X) is called m -quasi-directed space (or briefly m -q-directed space).

Example 4.8. The space (X, m_X) in Example 4.2 is clearly a m -quasi-directed space, which is not a topological space.

Theorem 4.4. The m_X -interior of any m_X -quasi-nodal set of an m -quasi-directed space (X, m_X) has m_X -ending property.

Proof. Let N be any m_X -quasi-nodal set in X . Then, $m_X\text{-}b(N) = [m_X\text{-}Cl(N) - m_X\text{-}Int(N)]$ is either ϕ or singleton. We show that $m_X\text{-}Int(N)$ has m_X -ending property.

Let C be an m -connected set in X and let $m_X\text{-}b(N) = \{p\}$. We show that $C - m_X\text{-}Int(N)$ is m -connected. We consider the following two cases:

Case I. Let $p \in C$. Let $C - m_X\text{-}Int(N) = A \cup B$, where A and B are m_X -separated sets in X . Since $p \notin m_X\text{-}Int(N)$, let $p \in B$, so $p \notin A$. Clearly $C = A \cup B \cup (C \cap m_X\text{-}Int(N)) = A \cup B \cup Q$, where $Q = C \cap m_X\text{-}Int(N)$. Now, $m_X\text{-}Cl(A) \cap Q = m_X\text{-}Cl(A) \cap (C \cap m_X\text{-}Int(N)) = C \cap (m_X\text{-}Cl(A) \cap m_X\text{-}Int(N))$. We prove that $(m_X\text{-}Cl(A) \cap m_X\text{-}Int(N)) = \phi$. For if $(m_X\text{-}Cl(A) \cap m_X\text{-}Int(N)) \neq \phi$, then let $\alpha \in (m_X\text{-}Cl(A) \cap m_X\text{-}Int(N))$. Then $\alpha \in (m_X\text{-}Cl(A), \alpha \in m_X\text{-}Int(N))$. Now if $\alpha \in m_X\text{-}Int(N)$, then N is an m_X -neighbourhood of α . We show that $N \cap A \neq \phi$. Now $\alpha \in (m_X\text{-}Cl(A) \cap m_X\text{-}Int(N)) = O$, say. If possible let $N \cap A = \phi$. Then $O \cap A = \phi$ as $O \subset N$. Therefore $A \subset X - O = B$, say.

Now $m_X\text{-}Int(O) = m_X\text{-}Int(m_X\text{-}Int(N)) = m_X\text{-}Int(N) = O$, by Lemma 2.1(6). Now $m_X\text{-}Cl(B) = m_X\text{-}Cl(X - O) = X - m_X\text{-}Int(O) = X - O = B$. Since $\alpha \in O$, $\alpha \notin B$ and hence $\alpha \notin m_X\text{-}Cl(B)$ as $m_X\text{-}Cl(B) = B$. Therefore by definition of $m_X\text{-}Cl(B)$, there exists an m_X -closed set F such that $B \subset F$ and $\alpha \notin F$. But $A \subset B \rightarrow A \subset F$. Hence there exists $x \in X$ such that $\alpha \notin F$, $A \subset F$ and F is m_X -closed. This shows that $x \notin m_X\text{-}Cl(A)$, which is a contradiction. Hence $N \cap A \neq \phi$. So $(m_X\text{-}Int(N) \cup \{p\}) \cap A \neq \phi$, i.e., $(m_X\text{-}Int(N) \cap A) \neq \phi$ since $p \notin A$. This contradicts that $C - (m_X\text{-}Int(N)) = A \cup B$. Thus $(m_X\text{-}Cl(A) \cap m_X\text{-}Int(N)) = \phi$.

Now $(m_X\text{-}Cl(A) \cap Q) = (m_X\text{-}Cl(A) \cap C \cap (m_X\text{-}Int(N))) = m_X\text{-}Cl(A) \cap m_X\text{-}Int(N) \cap C = \phi$.

Again $m_X\text{-}Cl(Q) \cap A = m_X\text{-}Cl(C \cap m_X\text{-}Int(N)) \cap A \subset m_X\text{-}Cl(m_X\text{-}Int(N)) \cap A \subset m_X\text{-}Cl(N) \cap A = A \cap ((m_X\text{-}Int(N) \cup \{p\})) = (A \cap (m_X\text{-}Int(N))) \cup (A \cap \{p\}) = \phi$. Hence A and Q are m_X -separated sets in X . Consequently A and $(B \cup Q)$ are m_X -separated sets in X as X is an m -quasi-directed space and A, B are m_X -separated sets in X . So, $A \cup (B \cup Q)$ has an m_X -separation, which contradicts that C is m -connected. Thus $C - m_X\text{-}Int(N)$ is m -connected and so $m_X\text{-}Int(N)$ has m_X -e. p in X .

Case II. Let $p \notin C$. The proof is same as of the Case I and hence is omitted. This proves the theorem.

We state the following lemma without proof.

Lemma 4.1. In a space (X, m_X) let $A \subset M$ and M and N are m_X -separated. Then A and N are m_X -separated.

Theorem 4.5. Let P and C be m -connected sets in an m -q-directed space (X, m_X) and $P \cap C \neq \phi$. If M and N are two m_X -separated sets such that $P - C = M \cup N$, then the sets $(C \cup M)$ and $(C \cup N)$ are m -connected.

Proof. Let $C \cup M = A \cup B$, where A and B are two m_X -separated sets. So,

$C \subset A \cup B$ and hence by Theorem 3.10, we assume that $C \cap A = \phi$ which gives $A \subset M$ because $A \subset C \cup M$. Now since M and N are two m_X -separated, the sets A and N are m_X -separated by Lemma 4.1; and therefore A and $N \cup B$ are m_X -separated as A and B are m_X -separated and (X, m_X) is an m-q-directed space. Now $P \cup C = (P - C) \cup C = M \cup N \cup C = A \cup (B \cup N)$. But, by Theorem 3.13 $P \cup C$ is an m-connected set because P and C are m-connected sets and $P \cap C \neq \phi$. So $A \cup (B \cup N)$ is m-connected, which is a contradiction as X is m-q-directed space. This proves the theorem.

Theorem 4.6. Let (X, m_X) be an m-q-directed space. If any set E has m_X -e. p. in X , then its m_X -components also have m_X -e. p. in X .

Proof. Let P be an m-connected set in X and let C be any m_X -component of E . If $P \cap C = \phi$, then clearly $P - C = P$ is m-connected. Now let $P \cap C \neq \phi$. If possible let $P - C$ be not m-connected. Then $P - C = M \cup N$, where M and N are two m_X -separated sets. Now $P - E \subset P - C = M \cup N$, so by Theorem 3.10 it can be assumed that $(P - E) \cap M = \phi$ as $(P - E)$ is m-connected since E has m_X -e. p. in X . Hence $(P - E) \cap (C \cup M) = \phi$ because $P - E \subset P - C$. Consequently, $C \cup M \subset E$. for if $C \cup M \not\subset E$, then there exists a $p \in C \cup M$ such that $p \notin E$. Clearly $p \notin C$ as $C \subset E$. So $p \in M$ and $p \notin E$. Therefore $p \in P - E$ as $M \subset P$. Thus $p \in (P - E) \cap (C \cup M)$ which contradicts that $(P - E) \cap (C \cup M) = \phi$. Now by Theorem 4.5 we get, $C \cup M$ is an m-connected set contained in E . But C is an m_X -component of E , so $C \subset C \cup M$ implies $M = \phi$. So $P - C$ is m-connected. This proves the theorem.

5. ending and boundaries of ending

Definition 5.1. A nonempty m-connected set having m_X -ending property in (X, m_X) is called an m_X -ending of X .

Example 5.1. In Example 4.2, we take $E = \{c\}$, which is nonempty m-connected and has m_X -ending property. So E is an m_X -ending set of X , but $F = \{a, d\}$ is a nonempty m-connected set which does not have m_X -e. p. So F is not an m_X -ending set of X .

Theorem 5.1. Let E be an m_X -ending of a space (X, m_X) and let B be the external m_X -boundary of E . Then B can not contain two mutually m_X -separated sets.

Proof. If possible let A and C be two mutually m_X -separated sets such that $A \cup C \subset B$. Now $E \subset E \cup A \cup C \subset E \cup B = m_X\text{-Cl}(E)$. So by Theorem 3.11 $E \cup A \cup C$ is m-connected. But since $E \cap (A \cup C) = \phi$, $(E \cup A \cup C) - E = A \cup C$, which is not m-connected. This is a contradiction as E has m_X -e. p. in X . This proves the theorem.

Remark 5.1. The result in Theorem 5.1 may not be true if B is m_X -boundary of E , which is shown by the following example:

Example 5.2. We consider the space (X, m_X) , where $X = \{a, b, c, d\}$ and $m_x = \{X, \phi, \{a\}\}$. We take $E = \{b, c, d\}$, which is m-connected and clearly E

is an m_X -ending of (X, m_X) . Now $B = m_X-b(E) = \{b, c, d\} - \phi = \{b, c, d\}$, which is not m_X -separated.

Corollary 5.1. The external m_X -boundary of any m_X -ending of a space (X, m_X) is m-connected.

Proof. Let E be an m_X -ending of the space (X, m_X) with minimal structure m_X and B be the m_X -external boundary of E .

If possible let B be not m-connected. Then $B = P \cup Q$, where P and Q are m_X -separated sets. Then $E \subset E \cup B = m_X-Cl(E)$ as $E \cap B = \phi$. Since E is m-connected $E \cup B$ is also m-connected. Therefore since E is an m_X -ending E has the m_X -e. p. property. So $E \cup B - E = B$ is m-connected.

Example 5.3. We consider the space (X, m_X) , where $X = \{a, b, c, d\}$ and $m_x = \{X, \phi, \{a, b, c\}, \{b, c, d\}\}$. We take $E = \{b, c\}$. It can be verified that E does not have m_X -e. p. in X . But $\text{ext-}m_X-b(E) = m_X-Cl(E) - E = \{a, d\}$, which is not m-connected.

Remark 5.2. The following example shows that for a subset E in (X, m_X) , $m_X-b(E)$ may be m-connected, though it does not have m_X -e. p. in X .

Example 5.4. In Example 5.3, we take $E = \{b, c\}$. Clearly E does not have m_X -e. p. in X .

But $m_X-b(E) = m_X-Cl(E) - m_X-Int(E) = X$, which is m-connected.

Definition 5.2. Let $A \subset X$ in (X, m_X) . Then a point $x \in X$ is called an m_X -cluster point of A if and only if $x \in m_X-Cl(A)$.

Definition 5.3. The space (X, m_X) is called m- T_1 if every singleton set in X is m_X -pseudo-closed, i.e., if and only if $m_X-Cl(\{x\}) = \{x\}$.

Corollary 5.2. For any two points of an external m_X -boundary B of any m_X -ending of a space (X, m_X) , one is an m_X -cluster point of the other. Hence if X is m- T_1 then B is degenerate.

Proof. Let $\alpha, \beta \in B$. If possible, let $m_X-Cl(\{\alpha\}) \cap \{\beta\} = \phi$ and $m_X-Cl(\{\beta\}) \cap \{\alpha\} = \phi$. Then $\{\alpha\}$ and $\{\beta\}$ are two mutually m_X -separated sets contained in B , a contradiction by Theorem 5.1. Hence for any two points of B of any m_X -ending of X , atleast one is an m_X -cluster point of the other.

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