Connectedness and its Applications

Hiranmay Dasgupta

Department of Mathematics, JIS College of Engineering
Phase-III, Block-A, Kalyani, Nadia
Pin-741235, West Bengal, India
hiranmaydg@yahoo.com

Sucharita Chakrabarti

Guru Nanak Institute of Technology
157/F Nilgunj Road, Panihati
Kolkata-700114, West Bengal, India
sucharitamath@gmail.com

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Abstract

In this paper we introduced and studied the notions of m-connecteness and its applications in spaces with minimal structures. We also explored and studied the concepts of $m_{X}$-components, m-quasi-nodal sets, m-quasi-directed spaces $m_{X}$-ending in this weaker space.

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1. Introduction.

Topological spaces have been generalized in different ways, for example, the abstract space [1], the supra-topological space [7], a space with minimal structure ([6], [9]). In this paper we have explored the notion of connectedness, a familier
and important concept of topological spaces within the framework of minimal structure ([6], [9]). We also introduced the concepts of $m_X$-components, $m$-quasi-nodal sets, $m_X$-ending, $m$-quasi-directed spaces in these weaker spaces and studied some basic properties of these notions. The advantage of considering our study through $m$-structure is that one can get easily the reflection of our exposition in other particular spaces of which $m$-space is a generalization through a single attempt.

For easy understanding of our work provided here we recall some basic definitions and results, for details of which we refer to [6], [9], [3], [4]. Throughout the paper $(X, m_X)$, $(Y, m_Y)$ stand for $m$-spaces with minimal structures $m_X$, $m_Y$ defined on $X$, $Y$ respectively. Also $N$, $R$ are respectively the set of all natural numbers and the set of all reals, $\mathcal{P}(X)$ is the class of all subsets of $X$, $\phi$ being the empty set.

2. Preliminaries

**Definition 2.1** ([6], cf. [9]) Let $X$ be a non-empty set. A subfamily $m_X$ of $\mathcal{P}(X)$ is called a minimal structure (briefly $m$-structure) on $X$ if $\phi, X \in m_X$. $(X, m_X)$ is called a space with minimal structure $m_X$ or simply an $m$-space. Each member of $m_X$ is called an $m_X$-open set and complement of an $m_X$-open set is called an $m_X$-closed set.

**Remark 2.1.** Topological spaces, supra-topological spaces [7], abstract spaces [1] are all spaces with minimal structures, but not conversely as $(X, m_X)$ where $X = [0, 1]$ and $m_X = \{X, \phi, [\frac{1}{8}, \frac{1}{2}], [\frac{4}{7}, \frac{2}{3}]\}$ is a space with minimal structure, which is not any one of the above mentioned spaces.

**Definition 2.2.** [6] For a subset $A$ of $X$ in $(X, m_X)$, the $m_X$-closure of $A$ and the $m_X$-interior of $A$ are defined as follows:

1. $m_X\text{-Cl}(A) = \cap \{F : A \subseteq F, X - F \in m_X\} $
2. $m_X\text{-Int}(A) = \cup \{U : U \subseteq A, U \in m_X\}$

**Lemma 2.1.** [6] For subsets $A$ and $B$ of $X$ in $(X, m_X)$ the following results hold:

1. $m_X\text{-Cl}(X - A) = X - m_X\text{-Int} A$ and $m_X\text{-Int}(X - A) = X - m_X\text{-Cl} A$.
2. $m_X\text{-Cl} A = A$ if $(X - A) \in m_X$ and $m_X\text{-Int} A = A$ if $A \in m_X$.
3. $m_X\text{-Cl}(\phi) = \phi, m_X\text{-Cl}(X) = X, m_X\text{-Int}(\phi) = \phi$ and $m_X\text{-Int}(X) = X$.
4. If $A \subseteq B$, then $m_X\text{-Cl} A \subseteq m_X\text{-Cl} B$ and $m_X\text{-Int} A \subseteq m_X\text{-Int} B$.
5. $A \subseteq m_X\text{-Cl} A$ and $m_X\text{-Int} A \subseteq A$.
6. $m_X\text{-Cl}(m_X\text{-Cl}(A)) = m_X\text{-Cl} A, m_X\text{-Int}(m_X\text{-Int}(A)) = m_X\text{-Int} A$.

**Lemma 2.2.** [9] Let $A$ be a subset of $X$ in $(X, m_X)$. Then $x \in m_X\text{-Cl}(A)$ if and only if $U \cap A \neq \phi$ for every $U \in m_X$ containing $x$.

**Remark 2.2.** In topological space, the converse of Lemma 2.1.(2) is also true. But in $(X, m_X)$ the converse of Lemma 2.1.(2) is not necessarily true as
shown by the following example:

**Example 2.1.** We consider the space \((X, m_X)\), where \(X = \{a, b, c\}, m_X = \{X, \emptyset, \{a\}, \{b\}\}\). Now \(m_X\)-Int\{a, b\} = \{a, b\} and \(m_X\)-Cl\{c\} = \{c\} though \{a, b\} is not \(m_X\)-open and \{c\} is not \(m_X\)-closed.

To overcome this limitation and to make possible further study the following notions are introduced:

**Definition 2.3.**[3] Let \(A \subset X\) in \((X, m_X)\). Then \(A\) is called \(m_X\)-pseudo-open if \(m_X\)-Int\(A\) = \(A\) and \(A\) is called \(m_X\)-pseudo-closed if \(m_X\)-Cl\(A\) = \(A\).

**Remark 2.3.** From Lemma 2.1(1) it follows that complement of an \(m_X\)-pseudo-open set is \(m_X\)-pseudo-closed and conversely. Also from Lemma 2.1(6) it follows that in \((X, m_X)\), for \(A \subset X\), \(m_X\)-Cl\(A\) and \(m_X\)-Int\(A\) are respectively \(m_X\)-pseudo-closed and \(m_X\)-pseudo-open sets.

**Remark 2.4.** Every \(m_X\)-open \((m_X\)-closed\) set is \(m_X\)-pseudo-open \((m_X\)-pseudo-closed\) but the converse is not true as shown by the following example:

**Example 2.2.** We consider the space \((X, m_X)\), where \(X = \{a, b, c\}, m_X = \{X, \emptyset, \{a\}, \{b\}\}\). Now \(m_X\)-Int\(\{a, b\} = \{a\} \cup \{b\} = \{a, b\}\). Thus \{a, b\} is an \(m_X\)-pseudo-open set, though it is not an \(m_X\)-open set. By Remark 2.3, \{c\} is an \(m_X\)-pseudo-closed set, though \{c\} is not an \(m_X\)-closed set.

**Remark 2.5.** If \(m_X\) is a topology on \(X\) then the notion of \(m_X\)-pseudo-open set \((m_X\)-pseudo-closed set\) coincides with the notion of open set \((closed set)\) in the topological space \((X, m_X)\).

**Definition 2.4.**[4] A mapping \(f : (X, m_X) \to (Y, m_Y)\) is called \(M^*\)-continuous if and only if \(f^{-1}(V) \in m_X\) whenever \(V \in m_Y\).

**Proposition 2.1.**[4] Let \(f : (X, m_X) \to (Y, m_Y)\) be \(M^*\)-continuous. Then \(f(m_X\)-Cl\(A\)) \(\subset m_Y\)-Cl\(f(A)\) for every \(A \subset X\).

**Lemma 2.3.** Let \(f : (X, m_X) \to (Y, m_Y)\) be \(M^*\)-continuous. Then \(m_X\)-Cl\(f^{-1}(A)\) \(\subset f^{-1}(m_Y\)-Cl\(f(A)\)) for every \(A \subset Y\).

**Proof.** Replacing \(A\) with \(f^{-1}(A)\) in Proposition 2.1, we get \(f(m_X\)-Cl\(f^{-1}(A)\)) \(\subset m_Y\)-Cl\(f(f^{-1}(A))\). So \(f^{-1}(m_X\)-Cl\(f^{-1}(A))\) \(\subset f^{-1}(m_Y\)-Cl\(f(f^{-1}(A))\)) \(\subset f^{-1}(m_Y\)-Cl\(f(A)\)) \(\subset f^{-1}(f(m_X\)-Cl\(f^{-1}(A)))\). Therefore \(m_X\)-Cl\(f^{-1}(A)\) \(\subset f^{-1}(f(m_X\)-Cl\(f^{-1}(A)))\) \(\subset f^{-1}(m_Y\)-Cl\(f(A)\)). This proves the lemma.

**Definition 2.5.** In \((X, m_X)\) an \(m_X\)-boundary of a subset \(A\) of \(X\), denoted by \(m_X\)-b\(A\), is defined by \(m_X\)-b\(A\) = \(m_X\)-Cl\(A\) \(\cap m_X\)-Int\(X - A\) or equivalently \(m_X\)-b\(A\) = \(m_X\)-Cl\(A\) \(\cap m_X\)-Int\(X - A\). A point \(p\) is called an \(m_X\)-boundary point of \(A\) if \(p \in m_X\)-b\(A\). A point \(q\) is called an external \(m_X\)-boundary point of \(A\) if \(q \in m_X\)-b\(A\), but \(q \notin A\). Thus the collection of all external \(m_X\)-boundary points of \(A\) is called external \(m_X\)-boundary of \(A\) and denoted by ext-\(m_X\)-b\(A\). Hence ext-\(m_X\)-b\(A\) = \(m_X\)-b\(A\) \(\cap A = m_X\)-Cl\(A\) \(\cap A\).

**Example 2.3.** We consider the space \((X, m_X)\), where \(X = [0, 2], m_X = \{X, \emptyset, [0, 1], [\frac{1}{2}, \frac{3}{2}]\}\). We take \(A = [0, 1]\). Then \(m_X\)-Int\(A\) = \([0, 1]\) and \(m_X\)-Cl\(A\) = \(X\). Thus \(m_X\)-b\(A\) = \(m_X\)-Cl\(A\) \(\cap m_X\)-Int\(X - A\) = \(X - [0, 1]\) = \([1, 2]\). Now \(\text{ext-}m_X\)-b\(A\) = \(m_X\)-Cl\(A\) \(\cap A = [1, 2]\).
3. connectedness

**Definition 3.1.** In \((X, m_X)\) an \(m_X\)-neighbourhood of a subset \(A\) of \(X\) is a subset \(U\) of \(X\) such that \(A \subset m_X-\text{Int}U\).

**Definition 3.2.** In \((X, m_X)\) two subsets \(A\) and \(B\) of \(X\) are called \(m_X\)-separated in \((X, m_X)\) (or \(m_X\)-separated in \(X\) or simply \(m_X\)-separated), denoted by \(X = A \setminus B\), if there exist \(m_X\)-neighbourhoods \(U\) of \(A\) and \(V\) of \(B\) in \(X\) such that \(U \cap B = \phi = V \cap A\).

**Example 3.1.** We consider the space \((X, m_X)\), where \(X = [0, 2]\) and \(m_X = \{X, \phi, (0, 1), (1, 2)\}\). We take \(A = (\frac{1}{8}, \frac{1}{4})\) and \(B = (\frac{2}{3}, \frac{5}{6})\). Now, \((0, \frac{1}{2})\) is an \(m_X\)-neighbourhood of \(A\) and \((\frac{1}{2}, \frac{3}{2})\) is an \(m_X\)-neighbourhood of \(B\). Also \(A \cap ((\frac{1}{2}, \frac{3}{2})) = \phi\) and \(B \cap (0, \frac{1}{2}) = \phi\). So, \(A\) and \(B\) are \(m_X\)-separated.

**Theorem 3.1.** In \((X, m_X)\) let \(A, B \subset X\). Then \(A\) and \(B\) are \(m_X\)-separated in \(X\) if and only if \((A \cap m_X-\text{Cl}(B)) \cup (m_X-\text{Cl}(A) \cap B) = \phi\).

**Proof.** The proof is routine and so omitted.

**Theorem 3.2.** Let \(f : (X, m_X) \rightarrow (Y, m_Y)\) be an \(M^*\)-continuous mapping. If \(A\) and \(B\) are \(m_Y\)-separated in \((Y, m_Y)\), then \(f^{-1}(A)\) and \(f^{-1}(B)\) are \(m_X\)-separated in \((X, m_X)\).

**Proof.** The proof is routine and so omitted.

**Definition 3.3.** A subset \(A\) of \(X\) in \((X, m_X)\) is called \(m\)-connected in \(X\) (or simply \(m\)-connected) if \(A\) is not the union of two nonempty \(m_X\)-separated subsets of \(X\). Thus a nonempty set \(A\) is \(m\)-connected if and only if \(A = A_1 \cup A_2\) with \(A_1, A_2 \subset X\) and \((A_1 \cap m_X-\text{Cl}(A_2)) \cup (m_X-\text{Cl}(A_1) \cap A_2) = \phi\) together imply either \(A_1 = \phi\) or \(A_2 = \phi\).

If \(A\) is not \(m\)-connected in \(X\), then we say \(A\) is \(m\)-disconnected in \(X\). A space \((X, m_X)\) is called \(m\)-connected if the underlying set \(X\) is \(m\)-connected.

**Theorem 3.3.** A space \((X, m_X)\) is \(m\)-connected if and only if \(X\) contains no proper nonempty subset simultaneously \(m_X\)-pseudo-open and \(m_X\)-pseudo-closed.

**Proof.** The proof is routine and so omitted.

**Theorem 3.4.** A space \((X, m_X)\) is \(m\)-connected if and only if \(X\) is not the union of two disjoint, nonempty \(m_X\)-pseudo-closed (or \(m_X\)-pseudo-open) subsets of \(X\).

**Proof.** The proof follows from Theorem 3.3.

**Definition 3.4.** Let \(\phi \neq Y \subset X\) in \((X, m_X)\). Then the space \((Y, m_Y)\) is called an \(m\)-subspace of \((X, m_X)\) if \(m_Y = \{A \cap Y : A \in m_X\}\).

**Lemma 3.1.** Let \((Y, m_Y)\) be an \(m\)-subspace of \((X, m_X)\) and let \(A \subset Y \subset X\). Then \(m_Y-\text{Cl}(A) = m_X-\text{Cl}(A) \cap Y\).

**Proof.** We know that \(m_Y-\text{Cl}(A) = \cap\{F^* : A \subset F^*, Y - F^* \in m_Y\} = \cap\{F \cap Y : A \subset F \cap Y, X - F \in m_X\} = \cap\{F : A \subset F, X - F \in m_X\} \cap Y = m_X-\text{Cl}(A) \cap Y\). This proves the lemma.

**Theorem 3.5.** Let \((Y, m_Y)\) be an \(m\)-subspace of \((X, m_X)\) and let \(A \subset Y \subset X\).
Then $A$ is m-connected in $(Y, m_Y)$ if and only if $A$ is m-connected in $(X, m_X)$.

**Proof.** Let $A$ be m-disconnected in $Y$. Then $A = P \cup Q$, $P \neq \emptyset$, $Q \neq \emptyset$, $P, Q \subset X$, $P \cap m_Y\text{Cl}(Q) = \emptyset = m_Y\text{Cl}(P) \cap Q$. Now $\phi = m_Y\text{Cl}(P) \cap Q = m_X\text{Cl}(P) \cap Q \cap Y \cap Q$, i.e., $(m_X\text{Cl}(P) \cap Q) = \phi$ and similarly $(P \cap m_X\text{Cl}(Q)) = \phi$. So $A$ has an $m_X$-separation in $X$ and hence $A$ is m-disconnected in $X$.

Conversely let $A$ be m-disconnected in $X$. Then $A = P \cup Q$, $P \neq \emptyset$, $Q \neq \emptyset$, $P, Q \subset X$, $P \cap m_X\text{Cl}(Q) = \emptyset = m_X\text{Cl}(P) \cap Q$. Now, $m_Y\text{Cl}(P) \cap Q = m_X\text{Cl}(P) \cap Y \cap Q = (m_X\text{Cl}(P) \cap Q) \cap Y = \phi$. Thus $m_Y\text{Cl}(P) \cap Q = \phi$. Similarly $m_Y\text{Cl}(Q) \cap P = \phi$. So $A$ is m-disconnected in $Y$. This completes the proof.

**Corollary 3.1.** A subset $Y$ of $X$ in $(X, m_X)$ is m-connected if and only if the m-subspace $(Y, m_Y)$ is m-connected.

**Theorem 3.6.** A space $(X, m_X)$ is m-connected if and only if there exists no nonempty proper subset $A$ of $X$ such that $m_X\text{Cl}(A) \cap m_X\text{Cl}(X - A) = \emptyset$, i.e., every subset $A$ of $X$ such that $\phi \neq A \neq X$ satisfies the condition $m_X\text{Cl}(A) \cap m_X\text{Cl}(X - A) \neq \emptyset$.

**Proof.** First let $(X, m_X)$ be m-connected. If possible let there be a subset $A$ of $X$ such that $\phi \neq A \neq X$ and $m_X\text{Cl}(A) \cap m_X\text{Cl}(X - A) = \emptyset$. Let $B = X - A$. So $X = A \cup B$ and $A \cap B = \phi$. Now $(A \cap m_X\text{Cl}(B)) \subset m_X\text{Cl}(A) \cap m_X\text{Cl}(B) = m_X\text{Cl}(A) \cap m_X\text{Cl}(X - A) = \emptyset$. Similarly we can prove that $B \cap m_X\text{Cl}(A) = \emptyset$. Thus $X$ has an $m_X$-separation and so $X$ is m-disconnected, which is a contradiction. Hence $X$ contains no subset $A$ such that $\phi \neq A \neq X$ and $m_X\text{Cl}(A) \cap m_X\text{Cl}(X - A) = \emptyset$.

Conversely let $m_X\text{Cl}(A) \cap m_X\text{Cl}(X - A) \neq \emptyset$, for any subset $A$ of $X$ such that $\phi \neq A \neq X$. We prove that $(X, m_X)$ is m-connected. If possible let $(X, m_X)$ be m-disconnected. Then there exists at least one nonempty subset $P$ of $X$, which is both $m_X$-pseudo-open and $m_X$-pseudo-closed. Let $Q = X - P$. Then $Q$ is also both $m_X$-pseudo-open and $m_X$-pseudo-closed and $P \cap Q = \emptyset$. Therefore $m_X\text{Cl}(P) = P$ and $m_X\text{Cl}(Q) = Q$. So $m_X\text{Cl}(P) \cap m_X\text{Cl}(X - P) = m_X\text{Cl}(P) \cap m_X\text{Cl}(Q) = P \cap Q = \emptyset$, which is a contradiction. Hence $(X, m_X)$ is m-connected.

This completes the proof.

**Remark 3.1.** It follows from Theorem 3.6 that a space $(X, m_X)$ is m-connected if and only if no subset $A$ of $X$, which satisfies the condition $\phi \neq A \neq X$ has an empty $m_X$-boundary.

**Theorem 3.7.** A nonempty subset $C$ of a space $(X, m_X)$ is m-connected if and only if every nonempty proper subset $A$ of $C$ satisfies the condition $C \cap m_X\text{Cl}(A) \cap m_X\text{Cl}(C - A) \neq \emptyset$.

**Proof.** The proof follows from Theorem 3.5 and Theorem 3.6.

**Theorem 3.8.** If $C$ is m-connected and $C \cap A \neq \emptyset$ then $C \cap m_X\text{Cl}(A) \neq \emptyset$.

**Proof.** Let $C$ be m-connected in $(X, m_X)$. Then the subspace $(C, m_C)$ is m-connected in $C$. Then by Theorem 3.6, for every proper nonempty sub-
set $A$ of $C$, $m_C\text{-Cl}(A) \cap m_C\text{-Cl}(X - A) \neq \emptyset$, i.e., $(C \cap m_X\text{-Cl}(A)) \cap (m_X\text{-Cl}(X - A)) \cap C \neq \emptyset$. Also, $C \cap [m_X\text{-Cl}(A) \cap m_X\text{-Cl}(X - A)] \neq \emptyset$. This completes the proof of the theorem.

**Theorem 3.9.** Let $f : (X, m_X) \to (Y, m_Y)$ be $M^*$-continuous and $A \subset X$. Then if $A$ is $m$-connected in $X$ then $f(A)$ is $m$-connected in $Y$.

**Proof.** The proof follows from Theorem 3.2.

**Theorem 3.10.** A subset $C$ of $X$ in $(X, m_X)$ is $m$-connected if and only if the following condition is fulfilled:

If $C$ is contained in the union of two $m_X$-separated sets $A$ and $B$, then either $C \subset A$ or $C \subset B$.

**Proof.** The proof is routine and so omitted.

**Theorem 3.11.** If $E$ is an $m$-connected subset of $X$ in $(X, m_X)$ and $C$ is any subset of $X$ such that $E \subset C \subset m_X\text{-Cl}(E)$, then $C$ is also $m$-connected.

**Proof.** The proof follows from Theorem 3.10.

**Theorem 3.12.** If every two points of a subset $E$ of $X$ in $(X, m_X)$ are contained in some $m$-connected subset of $E$, then $E$ is $m$-connected.

**Proof.** The proof follows from Theorem 3.10.

**Theorem 3.13.** In a space $(X, m_X)$ the union of any family of $m$-connected sets having a nonempty intersection is an $m$-connected set.

**Proof.** The proof is routine and so omitted.

**Theorem 3.14.** In a space $(X, m_X)$ the union of any family of $m$-connected sets with the property that one of the members of the family intersects every other member is $m$-connected.

**Proof.** Let $\{E_\alpha : \alpha \in \Delta\}$ be a family of nonempty $m$-connected subsets with the property that one of the members, say $E_u$ intersects every other member, i.e., $E_u \cap E_\alpha \neq \emptyset$ for $\alpha \in \Delta$, $\alpha \neq u$.

So $A_\alpha = E_u \cup E_\alpha$ is $m$-connected in $X$ for all $\alpha \in \Delta$, by Theorem 3.13. Now $\bigcap_{\alpha \in \Delta} A_\alpha = \bigcap_{\alpha \in \Delta} (E_u \cup E_\alpha) \supset E_u \neq \emptyset$. So by Theorem 3.13, $\bigcup_{\alpha \in \Delta} A_\alpha$ is $m$-connected in $X$. This proves the theorem.

**Theorem 3.15.** Let $\{C_t\}$ be a family of $m$-connected sets in a space $(X, m_X)$ with minimal structure $m_X$. The union $\bigcup_t C_t$ is $m$-connected, provided that there exists such a set $C_0$ in $\{C_t\}$, which is not $m_X$-separated with any other set $C_t$ of the family $\{C_t\}$.

**Proof.** Let $\bigcup_t C_t = M \cup N$, where $M$ and $N$ are $m_X$-separated in $X$. We show that either $M = \emptyset$ or $N = \emptyset$. By theorem 3.10 we assume that $C_0 \subset N$. Now we assert that for any $t$, $C_t \subset N$ because if for any $t$, $C_t \subset M$ then $m_X\text{-Cl}(C_t) \cap C_0 \subset m_X\text{-Cl}(M) \cap N = \emptyset$ and $m_X\text{-Cl}(C_0) \cap C_t \subset m_X\text{-Cl}(N) \cap M = \emptyset$ since $M$ and $N$ are $m_X$-separated in $X$ and so $C_t$ and $C_0$ are $m_X$-separated in $X$, which is a contradiction. Hence $M = \emptyset$. This completes the proof of the
Theorem.  

**Remark 3.2.** The following example shows that a family of m-connected sets in \((X, m_X)\) may be m-connected if none of the conditions in Theorem 3.13, Theorem 3.14 and Theorem 3.15 are satisfied, proving Theorems 3.13, 3.14 and 3.15 only provide a sufficient condition for the m-connectedness of the union of family of m-connected sets.

**Example 3.2.** We consider the space \((X, m_X)\), where \(X = \mathbb{R}\) and \(m_X = \{\mathbb{R}, \emptyset, \mathbb{R} - \{1\}, \mathbb{R} - \{2\}\}\). We consider the family of sets \(\{E_i, i \in K, K \subset \mathbb{N}\}\). We take \(E_i = \{i + 1\}\). Each \(E_i\) is m-connected, does not satisfy the conditions in Theorem 3.13, Theorem 3.14 and Theorem 3.15. But \(\bigcup \{E_i, i \in K, K \subset \mathbb{N}\}\) is m-connected.

**Theorem 3.16.** Let \(\{C_t : t \in \Delta\}\) be a directed family of nonempty m-connected sets (this means that for each pair \(t_1, t_2\) in \(\Delta\) there is \(t_3\) in \(\Delta\) such that \(C_{t_1} \subset C_{t_3}\) and \(C_{t_2} \subset C_{t_3}\)) in a space \((X, m_X)\). Then the union \(S = \bigcup_{t \in \Delta} C_t\) is m-connected.

**Proof.** Let \(S = M \cup N\), where \(M\) and \(N\) are \(m_X\)-separated in \(X\). By Theorem 3.10, we have for each \(t\), \(C_t \subset M\) or \(C_t \subset N\). We may assume that \(C_{t_0} \subset M\). We show that \(S \subset M\), which will complete the proof.

Let \(t\) be an arbitrary index and \(t_1\) be such that \(C_{t_0} \subset C_{t_1}\) and \(C_t \subset C_{t_1}\). The first inclusion yields \(C_{t_1} \not\subset N\). Hence \(C_{t_1} \subset M\) and therefore \(C_t \subset M\). It follows that \(S \subset M\).

**Remark 3.3.** In a space \((X, m_X)\) the intersection of two m-connected sets may not be m-connected, which is shown by the following example:

**Example 3.3.** We consider the space \((X, m_X)\), where \(X = \{a, b, c, d\}, m_X = \{X, \emptyset, \{a, c, d\}, \{a, b, d\}\}\). We take \(A = \{a, b, c\}\) and \(B = \{b, c, d\}\). Here both \(A\) and \(B\) are m-connected and \(A \cap B = \{b, c\}\) is not m-connected.

4. components, quasi-nodal sets and quasi-directed spaces.

**Definition 4.1.** Let \(Y \subset X\) in \((X, m_X)\). Then a set \(C \subset Y\) is called an \(m_X\)-component in \(Y\) if \(C\) is m-connected in \(X\) and if the inclusion \(C \subset C_1 \subset Y\) implies \(C = C_1\) for any m-connected set \(C_1\) in \(X\). \(C\) is an \(m_X\)-component in \((X, m_X)\) if \(C\) is the \(m_X\)-component in the underlying set \(X\). Thus \(m_X\)-components in \((X, m_X)\) are maximal m-connected subsets of \(X\).

**Example 4.1.** We consider the space \((X, m_X)\), where \(X = \{a, b, c, d\}, m_X = \{X, \emptyset, \{c, d\}, \{a, b, d\}\}\). Let \(Y = \{a, b, c\}\). \(Y\) is not m-connected in \(X\) as \(\{c\}/\{a, b\}\) forms an \(m_X\)-separation of \(Y\). Clearly \(\{a, b\}, \{a\}, \{b\}, \{c\}\) are the only nonempty subsets of \(Y\), which are m-connected in \(X\). Thus \(\{a, b\}\) and \(\{c\}\) are the maximal m-connected subsets of \(Y\) in \(X\). Hence \(m_X\)-components in \(Y\) are \(\{a, b\}\) and \(\{c\}\), but \(\{a\}, \{b\}\) are not an \(m_X\)-components in \(Y\). It can be verified that \(X\) is the only \(m_X\)-component in \(X\) as \(X\) is m-connected.
Remark 4.1. The space \((X, m_X)\) is \(m\)-connected if and only if \(X\) is the only \(m\)-component in \(X\).

Definition 4.2. A nonempty subset \(N\) of \(X\) in \((X, m_X)\) is called an \(m\)-quasi-nodal set if the \(m\)-boundary of \(N\) in \(X\) is degenerate (A set is degenerate if it is either empty or singleton).

Example 4.2. We consider the space \((X, m_X)\), where \(X = \{a, b, c, d\}\), \(m_X = \{X, \emptyset, \{a\}, \{b\}, \{c\}\}\). We take \(A = \{a, c\}\). Then \(m_X-\text{Cl}(A) = \{a, c, d\}\) and \(m_X-\text{Int}(A) = \{a, c\}\). Now \(m_X-b(A) = m_X-\text{Cl}(A) - m_X-\text{Int}(A) = \{d\}\). Thus \(A\) is an \(m\)-quasi-nodal set.

Definition 4.3. A point \(p \in X\) in \((X, m_X)\) is called an \(m\)-separating point of an \(m\)-connected subset \(C\) of \(X\) if \(C - \{p\}\) is not \(m\)-connected.

Example 4.3. In Example 4.2, we take \(A = \{a, b, d\}\) which is \(m\)-connected in \(X\). Now \(A - \{d\} = \{a, b\}\), which is not \(m\)-connected in \(X\). Thus \(d\) is an \(m\)-separating point of \(A\) in \(X\), but \(a\) and \(b\) are not \(m\)-separating points of \(A\) in \(X\) as both \(A - \{a\} = \{b, d\}\) and \(A - \{b\} = \{a, d\}\) are \(m\)-connected. Obviously \(c\) is not an \(m\)-separating point of \(A\) in \(X\).

Definition 4.4. A point \(p \in X\) of \((X, m_X)\) is called an \(m\)-end point of \((X, m_X)\) if \(p\) is not an \(m\)-separating point of any \(m\)-connected subset \(C\) of \(X\).

Example 4.4. In Example 4.2, it can be verified that \(b\) is an \(m\)-end point of \((X, m_X)\), but \(d\) is not an \(m\)-end point of \((X, m_X)\).

Definition 4.5. A subset \(E\) of \((X, m_X)\) is said to have the \(m\)-ending property (\(m\)-e. p.) in \(X\) if there is no \(m\)-connected subset \(C\) of \(X\) such that \(E\) separates \(C\) (i.e., such that \(C - E\) is not \(m\)-connected).

Example 4.5. In Example 4.2, we take \(E = \{a, b\}\). It can be verified that \(C - E\) is \(m\)-connected for all \(m\)-connected subset \(C\) of \(X\). So \(E\) has \(m\)-e. p., but \(F = \{a, d\}\) does not have \(m\)-e. p. in \(X\).

Theorem 4.1. The union of a finite collection of sets with \(m\)-e. p. in a space \((X, m_X)\) has \(m\)-e. p. in \(X\).

Proof. Let \(A\) and \(B\) be two sets having \(m\)-e. p. in \(X\). We prove that \(A \cup B\) has \(m\)-e. p. in \(X\). Let \(C\) be any \(m\)-connected set in \(X\). Then since \(A\) has \(m\)-e. p. in \(X\), \(C - A\) is \(m\)-connected in \(X\) and so \((C - A) - B = C - (A \cup B)\) is \(m\)-connected in \(X\) as \(B\) has \(m\)-e. p. in \(X\). Thus \(A \cup B\) has \(m\)-e. p. in \(X\). This proves the theorem.

Remark 4.2. The finite intersection of two sets having \(m\)-e. p. in \((X, m_X)\) need not have \(m\)-e. p. in \(X\) as shown by the following example:

Example 4.6. We consider the space \((X, m_X)\), where \(X = \{a, b, c\}, m_X = \{X, \emptyset, \{a\}, \{b\}\}\). Now \(\{a, c\}, \{b, c\}\) have \(m\)-e. p. in \(X\). But \(\{a, c\} \cap \{b, c\} = \{c\}\) does not have \(m\)-e. p. in \(X\).

In the following theorem a sufficient condition is sorted out to enjoy the \(m\)-e. p. property of the intersection of a collection of sets each having \(m\)-e. p. property.

Theorem 4.2. If \(E\) is a nested collection of sets having \(m\)-e. p. in a space
\((X, m_X)\), then \(\cap E\) has \(m_X\)-e. p. (possibly trivially) in \(X\).

**Proof.** Let \(N = \cap E\). We prove that \(N\) has \(m_X\)-e. p. in \(X\). Now if \(N = \emptyset\), then the theorem is trivially true. For \(N \neq \emptyset\), we suppose that there is an \(m\)-connected set \(C\) such that \(C - N\) is the union of two \(m_X\)-separated sets \(A\) and \(B\). Now since \(A \cap N = \emptyset\), \(A\) can not be contained in all the elements of \(E\) and so there is a member \(E_A\) of \(E\) such that \(A - E_A \neq \emptyset\). Similarly there is a member \(E_B\) of \(E\) such that \(B - E_B \neq \emptyset\). Let \(E = E_A \cap E_B\). Since \(E\) is nested, \(E = E_A \cap E_B = E_A\) or \(E_B\) and so \(E \in E\) and \(A - E\) and \(B - E\) both are nonempty. But then \(C - E = (A - E) \cup (B - E)\), an \(m_X\)-separation. This contradicts that \(E\) has \(m_X\)-e. p. This proves the theorem.

**Remark 4.3.** In the Theorem 4.2, nested condition is a sufficient condition as shown by the following example:

**Example 4.7.** We consider the space \((X, m_X)\), where \(X = \mathbb{R}, m_X = \{\mathbb{R}, \emptyset, \mathbb{R} - \{1\}, \mathbb{R} - \{2\}\}\) so that \(m_X\)-closed sets are \(\emptyset, \mathbb{R}, \{1\}, \{2\}\). Now for any non empty subset \(A\) of \(X\)

\[
m_X - \text{Cl}(A) = \begin{cases} \{1\} & \text{if } A = \{1\} \\ \{2\} & \text{if } A = \{2\} \\ X = \mathbb{R}, & \text{otherwise.} \end{cases}
\]

Now \(\{1, 2\}\) is the only nonempty subset of \(X\) which is not \(m\)-connected. We now consider the family \(\{E_i\}\), where \(E_i = \{1, i + 1\}, i \in \mathbb{N}\) each having \(m_X\)-e. p. Then \(\bigcap_{i \in \mathbb{N}} E_i = \{1\}\) also has \(m_X\)-e. p in \(X\), though the collection \(\{E_i\}\) is not nested.

**Theorem 4.3.** In a space \((X, m_X)\) if \(E\) has \(m_X\)-e. p. in \(X\), then the intersection of any \(m\)-connected set with \(X - E\) is \(m\)-connected.

**Proof.** Let \(C\) be any \(m\)-connected set in \(X\). Since \(E\) has \(m_X\)-e. p. in \(X\), so \((C - E)\) is \(m\)-connected. Now \(C \cap (X - E) = X \cap (C - E) = C - E\), which is \(m\)-connected. So \(C \cap (X - E)\) is \(m\)-connected. This proves the theorem.

**Remark 4.4.** In a topological space \(X\), if \(A, B\) are separated sets and \(A, C\) are also separated sets then \(A\) and \(B \cup C\) are also separated sets. This does not happen in \((X, m_X)\) due to the fact that in \((X, m_X)\), \(m_X-\text{Cl}(A) \cup m_X-\text{Cl}(B)\) may be a proper subset of \(m_X-\text{Cl}(A \cup B)\). We therefore introduce the notion of an \(m\)-quasi-directed space (weaker than topological space) that has the same property of a topological space with respect to separation of sets as stated.

**Definition 4.6.** In \((X, m_X)\) let \(A, B, N \subset X\). If \(A, B\) are \(m_X\)-separated in \(X\) and \(A, N\) are \(m_X\)-separated in \(X\) imply \(A, B \cup N\) are \(m_X\)-separated in \(X\), then \((X, m_X)\) is called \(m\)-quasi-directed space (or briefly \(m\)-q-directed space).

**Example 4.8.** The space \((X, m_X)\) in Example 4.2 is clearly a \(m\)-quasi-directed space, which is not a topological space.

**Theorem 4.4.** The \(m_X\)-interior of any \(m_X\)-quasi-nodal set of an \(m\)-quasi-directed space \((X, m_X)\) has \(m_X\)-ending property.
Lemma 4.1. Let $N$ be any $m_X$-quasi-nodal set in $X$. Then, $m_X(b(N) = [m_X-Cl(N) - m_X-Int(N)])$ is either $\phi$ or singleton. We show that $m_X-Int(N)$ has $m_X$-ending property.

Let $C$ be an m-connected set in $X$ and let $m_X(b(N) = \{p\}$. We show that $C - m_X-Int(N)$ is m-connected. We consider the following two cases:

**Case I.** Let $p \in C$. Let $C - m_X-Int(N) = A \cup B$, where $A$ and $B$ are $m_X$-separated sets in $X$. Since $p \notin m_X-Int(N)$, let $p \in B$, so $p \notin A$. Clearly $C = A \cup B \cup (C \cap m_X-Int(N)) = A \cup B \cup Q$, where $Q = C \cap m_X-Int(N)$. Now, $m_X-Cl(A) \cap Q = m_X-Cl(A) \cap (C \cap m_X-Int(N)) = C \cap (m_X-Cl(A) \cap m_X-Int(N))$. We prove that $(m_X-Cl(A) \cap m_X-Int(N)) = \phi$. For if $(m_X-Cl(A) \cap m_X-Int(N)) \neq \phi$, then let $\alpha \in (m_X-Cl(A) \cap m_X-Int(N))$. Then $\alpha \in (m_X-Cl(A), \alpha \in m_X-Int(N))$. Now if $\alpha \in m_X-Int(N)$, then $N$ is an $m_X$-neighbourhood of $\alpha$. We show that $N \cap A \neq \phi$. Now $\alpha \in (m_X-Cl(A)$ and $\alpha \in m_X-Int(N)) = O$, say. If possible let $N \cap A = \phi$. Then $O \cap A = \phi$ as $O \subset N$. Therefore $A \cap X - O = B$, say. Now $m_X-Int(O) = m_X(b(N)) = m_X-Int(N) = O$, by Lemma 2.1(6). Now $m_X-Cl(B) = m_X-Cl(X - O) = X - m_X-Int(O) = X - O = B$. Since $\alpha \in O$, $\alpha \notin B$ and hence $\alpha \notin m_X-Cl(B) = m_X-Cl(B) = B$. Therefore by definition of $m_X-Cl(B)$, there exists an $m_X$-closed set $F$ such that $B \subset F$ and $x \notin F$. But $A \subset B \rightarrow A \subset F$. Hence there exists $x \in X$ such that $\alpha \notin F$, $A \subset F$ and $F$ is $m_X$-closed. This shows that $x \notin m_X-Cl(A)$, which is a contradiction. Hence $N \cap A \neq \phi$. So $(m_X-Int(N) \cup \{p\}) \cap A \neq \phi$, i.e., $(m_X-Int(N)) \cap A \neq \phi$ since $p \notin A$. This contradicts that $C - (m_X-Int(N)) = A \cup B$. Thus $(m_X-Cl(A) \cap m_X-Int(N)) = \phi$. Now $(m_X-Cl(A) \cap Q = (m_X-Cl(A) \cap C \cap (m_X-Int(N) = m_X-Cl(A) \cap m_X-Int(N) \cap C = \phi$.

Again $m_X-Cl(Q) \cap A = m_X-Cl(C \cap m_X-Int(N)) \cap A \subset m_X-Cl(m_X-Int(N)) \cap A \subset m_X-Cl(N) \cap A = A \cap ((m_X-Int(N) \cup \{p\}) = (A \cap (m_X-Int(N)) \cup (A \cap \{p\}) = \phi$. Hence $A$ and $Q$ are $m_X$-separated sets in $X$. Consequently $A$ and $(B \cup Q)$ are $m_X$-separated sets in $X$ as $X$ is an $m$-quasi-directed space and $A$, $B$ are $m_X$-separated sets in $X$. So, $A \cup (B \cup Q)$ has an $m_X$-separation, which contradicts that $C$ is $m$-connected. Thus $C - m_X-Int(N)$ is $m$-connected and so $m_X-Int(N)$ has $m_X$-e. $p$ in $X$.

**Case II.** Let $p \notin C$. The proof is same as of the Case I and hence is omitted. This proves the theorem.

We state the following lemma without proof.

**Lemma 4.1.** In a space $(X, m_X)$ let $A \subset M$ and $M$ and $N$ are $m_X$-separated. Then $A$ and $N$ are $m_X$-separated.

**Theorem 4.5.** Let $P$ and $C$ be $m$-connected sets in an $m$-q-directed space $(X, m_X)$ and $P \cap C \neq \phi$. If $M$ and $N$ are two $m_X$-separated sets such that $P - C = M \cup N$, then the sets $(C \cup M)$ and $(C \cup N)$ are $m$-connected.

**Proof.** Let $C \cup M = A \cup B$, where $A$ and $B$ are two $m_X$-separated sets. So,
$C \subset A \cup B$ and hence by Theorem 3.10, we assume that $C \cap A = \phi$ which gives $A \subset M$ because $A \subset C \cup M$. Now since $M$ and $N$ are two $m_X$-separated, the sets $A$ and $N$ are $m_X$-separated by Lemma 4.1; and therefore $A$ and $N \cup B$ are $m_X$-separated as $A$ and $B$ are $m_X$-separated and $(X, m_X)$ is an m-q-directed space. Now $P \cup C = (P - C) \cup C = M \cup N \cup C = A \cup (B \cup N)$. But, by Theorem 3.13 $P \cup C$ is an m-connected set because $P$ and $C$ are m-connected sets and $P \cap C \neq \phi$. So $A \cup (B \cup N)$ is m-connected, which is a contradiction as $X$ is m-q-directed space. This proves the theorem.

**Theorem 4.6.** Let $(X, m_X)$ be an m-q-directed space. If any set $E$ has $m_X$-e. p. in $X$, then its $m_X$-components also have $m_X$-e. p. in $X$.

**Proof.** Let $P$ be an m-connected set in $X$ and let $C$ be any $m_X$-component of $E$. If $P \cap C = \phi$, then clearly $P - C = P$ is m-connected. Now let $P \cap C \neq \phi$. If possible let $P - C$ be not m-connected. Then $P - C = M \cup N$, where $M$ and $N$ are two $m_X$-separated sets. Now $P - E \subset P - C = M \cup N$, so by Theorem 3.10 it can be assumed that $(P - E) \cap M = \phi$ as $(P - E)$ is m-connected since $E$ has $m_X$-e. p. in $X$. Hence $(P - E) \cap (C \cup M) = \phi$ because $P - E \subset P - C$. Consequently, $C \cup M \subset E$. for if $C \cup M \notin E$, then there exists a $p \in C \cup M$ such that $p \notin E$. Clearly $p \notin C$ as $C \subset E$. So $p \in M$ and $p \notin E$. Therefore $p \in P - E$ as $M \subset P$. Thus $p \in (P - E) \cap (C \cup M)$ which contradicts that $(P - E) \cap (C \cup M) = \phi$. Now by Theorem 4.5 we get, $C \cup M$ is an m-connected set contained in $E$. But $C$ is an $m_X$-component of $E$, so $C \subset C \cup M$ implies $M = \phi$. So $P - C$ is m-connected. This proves the theorem.

5. ending and boundaries of ending

**Definition 5.1.** A nonempty m-connected set having $m_X$-ending property in $(X, m_X)$ is called an $m_X$-ending of $X$.

**Example 5.1.** In Example 4.2, we take $E = \{c\}$, which is nonempty m-connected and has $m_X$-ending property. So $E$ is an $m_X$-ending set of $X$, but $F = \{a, d\}$ is a nonempty m-connected set which does not have $m_X$-e. p. So $F$ is not an $m_X$-ending set of $X$.

**Theorem 5.1.** Let $E$ be an $m_X$-ending of a space $(X, m_X)$ and let $B$ be the external $m_X$-boundary of $E$. Then $B$ can not contain two mutually $m_X$-separated sets.

**Proof.** If possible let $A$ and $C$ be two mutually $m_X$-separated sets such that $A \cup C \subset B$. Now $E \subset E \cup A \cup C \subset E \cup B = m_X\text{Cl}(E)$. So by Theorem 3.11 $E \cup A \cup C$ is m-connected. But since $E \cap (A \cup C) = \phi$, $(E \cup A \cup C) - E = A \cup C$, which is not m-connected. This is a contradiction as $E$ has $m_X$-e. p. in $X$. This proves the theorem.

**Remark 5.1.** The result in Theorem 5.1 may not be true if $B$ is $m_X$-boundary of $E$, which is shown by the following example:

**Example 5.2.** We consider the space $(X, m_X)$, where $X = \{a, b, c, d\}$ and $m_x = \{X, \phi, \{a\}\}$. We take $E = \{b, c, d\}$, which is m-connected and clearly $E$
is an $m_X$-ending of $(X, m_X)$. Now $B = m_X-b(E) = \{b, c, d\} - \phi = \{b, c, d\}$, which is not $m_X$-seperated.

**Corollary 5.1.** The external $m_X$-boundary of any $m_X$-ending of a space $(X, m_X)$ is $m$-connected.

**Proof.** Let $E$ be an $m_X$-ending of the space $(X, m_X)$ with minimal structure $m_X$ and $B$ be the $m_X$-external boundary of $E$.

If possible let $B$ be not $m$-connected. Then $B = P \cup Q$, where $P$ and $Q$ are $m_X$-seperated sets. Then $E \subseteq E \cup B = m_X-Cl(E)$ as $E \cap B = \phi$. Since $E$ is $m$-connected $E \cup B$ is also $m$-connected. Therefore since $E$ is an $m_X$-ending $E$ has the $m_X$-e. p. property. So $E \cup B - E = B$ is $m$-connected.

**Example 5.3.** We consider the space $(X, m_X)$, where $X = \{a, b, c, d\}$ and $m_x = \{X, \phi, \{a, b, c\}, \{b, c, d\}\}$. We take $E = \{b, c\}$. It can be verified that $E$ does not have $m_X$-e. p. in $X$. But $\text{ext-}m_X-b(E) = m_X-Cl(E) - E = \{a, d\}$, which is not $m$-connected.

**Remark 5.2.** The following example shows that for a subset $E$ in $(X, m_X)$, $m_X-b(E)$ may be $m$-connected, though it does not have $m_X$-e. p. in $X$.

**Example 5.4.** In Example 5.3, we take $E = \{b, c\}$. Clearly $E$ does not have $m_X$-e. p. in $X$.

But $m_X-b(E) = m_X-Cl(E) - m_X-Int(E) = X$, which is $m$-connected.

**Definition 5.2.** Let $A \subseteq X$ in $(X, m_X)$. Then a point $x \in X$ is called an $m_X$-cluster point of $A$ if and only if $x \in m_X-Cl(A)$.

**Definition 5.3.** The space $(X, m_X)$ is called $m$-$T_1$ if every singleton set in $X$ is $m_X$-pseudo-closed, i.e., if and only if $m_X-Cl(\{x\}) = \{x\}$.

**Corollary 5.2.** For any two points of an external $m_X$-boundary $B$ of any $m_X$-ending of a space $(X, m_X)$, one is an $m_X$-cluster point of the other. Hence if $X$ is $m$-$T_1$ then $B$ is degenerate.

**Proof.** Let $\alpha, \beta \in B$. If possible, let $m_X-Cl(\{\alpha\}) \cap \{\beta\} = \phi$ and $m_X-Cl(\{\beta\}) \cap \{\alpha\} = \phi$. Then $\{\alpha\}$ and $\{\beta\}$ are two mutually $m_X$-seperated sets contained in $B$, a contradiction by Theorem 5.1. Hence for any two points of $B$ of any $m_X$-ending of $X$, atleast one is an $m_X$-cluster point of the other.

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**References**


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