A Note on Finite Groups in which C-Normality is a Transitive Relation

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Abstract

A subgroup H of G is c -normal in G if there exists a normal subgroup N of G such that $HN = G$ and $H \cap N \leq H_G$. A group G is called

CT -group if c -normality is transitive relation in G . In this paper, a number of new characterization of finite solvable CT -groups are given.

Keywords: permutable, subnormal, c -normal subgroup, CT -group

1 INTRODUCTION

All groups considered in this paper are finite. Notations and terminology not explained can be found in Doerk and Hawkes [4], Robinson [1] and Wang [7]. This paper will focus mainly on characterizing several classes of solvable groups. More specifically, we will consider the class of solvable groups in which c -normality is a transitive relation. First a few definitions and some known results should be discussed.

A subgroup H of a group G is said to permute with a subgroup K if HK is a subgroup of G . H is said to be permutable in G if it permutes with all the subgroups of G . We write $H \text{ per } G$ to denote H is permutable in G . Among the first studies on permutable subgroups was done by Ore in 1939. Ore used the term quasinormal subgroups to represent permutable subgroups. Kegel showed that such subgroups are necessarily subnormal. Actually, a result stronger than permutable subgroups are subnormal subgroups. For a subgroup H of G , it is enough to know that H permutes with all of its own conjugate to deduce that H is subnormal [2]. On the other hand subnormality does not imply permutability. As an example, one might consider the alternating group of order 12. It has a subnormal subgroup of order 2 that is not permutable. A subgroup H is called S -permutable in G . This concept was introduced by Kegel in 1962 [3], who showed such subgroups are necessarily subnormal.

A group G is said to be a T -group (resp. PT -group, PST -group), if normality (resp. permutability, S -permutability) is a transitive relation in G . That is, G is a T -group (resp. PT -group, PST -group) if for all subgroups H and K where $H \trianglelefteq K \trianglelefteq G$ (resp. $H \text{ per } K \text{ per } G, H \text{ s-per } K \text{ s-per } G$) we have $H \trianglelefteq K$ (resp. $H \text{ per } G, H \text{ s-per } G$). It is easy to see that T -groups are those groups in which normality and subnormality coincide. It is a direct consequence of the subnormality of permutable (S -permutable) subgroups that PT -groups (PST -groups) are precisely those groups in which subnormality and permutability (S -permutability) coincide.

Let us consider the following definitions.

Definition 1 [7] Let H be a subgroup of a finite group G . The core of H in G , denoted as H_G , is defined to be the largest normal subgroup of G contained in H , (or equivalently to $H_G = \cap \{H^g : g \in G\}$).

Definition 2 [7] Let G be a group. We will call a subgroup H of G c -normal in G if there exists a normal subgroup N of G such that $HN = G$ and $H \cap N \leq H_G$. H c -norm G denotes H is normal in G .

While it is clear that normal subgroups are c -normal, the converse is not true. For example the Sylow 2-subgroup H of the symmetric group S_3 is c -normal in S_3 but H is not even subnormal in S_3 .

Stimulated by the theory of T -groups (resp. PT -groups, PST -group), we introduce the class of CT -groups..

Definition 3 A group G is called a CT -group if c -normality is a transitive relation in G . That is, G is a CT -group if for all subgroups H and K where H c -norm K and K c -norm G , we have H c -norm G .

Notations:

1. If G is a CT -group then $\phi(G)$ is normal in G .
2. If G is a nilpotent CT -group then every subgroup of G is c -normal in G .
3. A solvable CT -group need not be a T -group and a T -group need not be a CT -group.

The following example shows that a solvable CT -group need not be a T -group.

Let G be the group of order 18 which is the direct product of symmetric group S_3 by cyclic group of order 3 ,

$$S_3 \times Z_3 = \{(e, o), ((12), 0), ((13), 0), ((23), 0), ((132), 0), ((123), 0), (e, 1), ((12), 1), ((13), 1), ((23), 1), ((132), 1), ((123), 1), (e, 2), ((12), 2), ((13), 2), ((23), 2), ((132), 2), ((123), 2)\}$$

Now every subgroup of G_{18} is c -normal in G_{18} .

But if we take $H = \langle ((123), 1) \rangle$, then $H \neq H^{(12)}$. Hence G is a CT -group that is not a T -group.

2 PRELIMINARIES

In the years 1953, 1964, and 1975, Gaschutz, Zacher, and Agrawal respectively, proved the following definitive results on solvable T -groups, PT -groups, and PST -groups.

Theorem 1 [3] [6] *A solvable PT -group (resp. PST -group) is supersolvable.*

Theorem 2 [3] [6] *Let G be a group with L the nilpotent residual of G . Then G is a solvable PT -group (resp. PST -group), if and only if the following condition hold:*

- (i) L is normal abelian Hall subgroup of G with odd order;
- (ii) G/L is Dedekind (nilpotent) group;
- (iii) G acts by conjugation as power automorphism on L .

The following known results about c -normal subgroups will be used in this paper several times.

Lemma 3 [7] *Let G be a group. Then*

- (i) if H is normal in G , then H is c -normal in G ;
- (ii) G is c -simple if and only if G is simple;
- (iii) if H is c -normal in G , $H \leq K \leq G$, then H is c -normal in K ;
- (iv) let $K \trianglelefteq G$ and $K \leq H$, then H is c -normal in G if and only if H/K is c -normal in G/K .

Lemma 4 [7] *Let G be a finite group. Then G is solvable if and only if every maximal subgroup of G is c -normal in G .*

The following Lemma is a direct of consequence of Lemma 4 and definition of CT -groups.

Lemma 5 *Every Maximal subgroup of a solvable CT -group is a CT -group.*

Lemma 6 *If N is a minimal normal solvable subgroup of a CT -group. Then N is cyclic.*

Proof. Let N be a minimal normal solvable subgroup of G , then $|N| = p^n$ for some prime p . Hence every subgroup of N is c -normal in G . Let P be a Sylow p -subgroup of G . Then N is a normal subgroup of P and $N \cap Z(P) \neq 1$. Let x be a nonidentity element $N \cap Z(P)$. Assume that $\langle x \rangle \neq N$. Since $\langle x \rangle$ is c -normal in G then $\langle x \rangle$ has a normal complement K in G such that $K \cap \langle x \rangle = 1$. So $G \simeq \langle x \rangle \times K$. Therefore $\langle x \rangle$ is a normal subgroup of G . Hence $N = \langle x \rangle$ is a cyclic subgroup of G . ■

Lemma 7 *Let G be a solvable CT -group. Then:*

- (i) every subgroup of G is a CT -group;
- (ii) if $N \trianglelefteq G$ then G/N is a CT -group.

Proof. For (i) let H be subgroup of a CT -group G . We consider two cases for H :

Case 1. If H is maximal in G then by Lemma 5, H is a CT -group.

Case 2. Suppose H is not maximal in G . Let M be a maximal subgroup of G containing H . By Lemma 5 M is a CT -group. By induction, every subgroup of M is a CT -group. Hence H is a CT -group.

(ii) follows from Lemma 3 ■

3 MAIN RESULTS

The following theorems state our main results, and show evidence that CT -groups are quite close to T -groups, PT -groups, and PST -groups.

Theorem 8 *A solvable CT -group is supersolvable.*

Proof. Suppose that the statement is false and let G be a counter example of minimal order. Let N be a minimal normal subgroup of G . By Lemma 6, N is cyclic. By part (ii) of Lemma 7, G/N is a CT -group. Therefore G/N is a supersolvable group by induction. Hence G is supersolvable. ■

Theorem 9 *If G_1 and G_2 are two CT -groups and $(|G_1|, |G_2|) = 1$, then $G = G_1 \times G_2$ is also a CT -group*

Proof. Let H and K be subgroups of G such that H is c -normal in K and K is c -normal in G . To prove the theorem we must show that H is c -normal in G . Since K is c -normal in G then there exists a normal subgroup N of G such that $G = KN$ and $K \cap N \leq K_G$. From $(|G_1|, |G_2|) = 1$ we get $N \simeq N_1 \times N_2$ where $N_i = (G_i \cap N)$ and $K_G = K_{1G} \times K_{2G}$ with $K_{iG} = (K_G \cap N)$ for each $i \in \{1, 2\}$. It is clear now that c -normality of K in G is equivalent to the c -normality of K_i in G_i . Arguing in a similar way, we get $H \simeq H_1 \times H_2$ with H_i c -normal in K_i for each $i \in \{1, 2\}$. Therefore H is c -normal in G . ■

Theorem 10 *Let G be a group with L the nilpotent residual of G . If G is a solvable CT -group, then the following conditions hold:*

- (i) L is a normal abelian Hall subgroup of G with odd order;
- (ii) Every subgroup of the group G/L is c -normal in G/L ;
- (iii) G acts by conjugation as power automorphism on L .

Proof. Let G be a solvable CT -group. By Theorem 8, G is supersolvable. Hence $L \leq G^\eta \leq G'$, and L is nilpotent subgroup of odd order. Since a normal subgroup is a c -normal subgroup, it follows that every subgroup of L is subnormal in G .

Let a be any element of L , then $\langle a \rangle$ is c -normal in G . Therefore there exists a normal subgroup N of G such that $G = \langle a \rangle N$ and $\langle a \rangle \cap N = \langle a \rangle_G$. Now since $G/N = \langle a \rangle N/N$ by isomorphism theorem $\langle a \rangle N/N \simeq \langle a \rangle / \langle a \rangle \cap N$ so $G/N \simeq \langle a \rangle / \langle a \rangle \cap N$ which is cyclic (the factor group of cyclic group is a cyclic) hence G/N is abelian. Hence $L \leq G^\eta \leq G' \leq N$. It follows that $\langle a \rangle \leq N$ and $\langle a \rangle \trianglelefteq G$. So (iii) holds.

By (iii), we have that L is a Dedekind group of odd order. Hence L is abelian. So (i) holds.

G/L is nilpotent and so every subgroup of G/L is subnormal, we have H/L is a subgroup such that

$$H/L \triangleleft \triangleleft G/L,$$

$$H_1/L \triangleleft H_2/L \triangleleft \dots \triangleleft H_n/L = G/L$$

where $H_1 \triangleleft H_2 \triangleleft \dots \triangleleft H_n = G$. But G is a CT -group, so H_i is c -normal, hence H_i/L is c -normal in G/L .

So every subgroup of G/L is c -normal. Therefore (ii) holds. ■

4 CONCLUSION

New characterizations of finite solvable CT -groups have been stated and proven.

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