Some Results of Weighted Norlund-Euler

Statistical Convergence

[1]Ekrem A. Aljimi

Department of Mathematics, University of Tirana, Tiranë, Albania
ekremhalimii@yahoo.co.uk

[2]Elida Hoxha

Department of Mathematics, University of Tirana, Tiranë, Albania
hoxhaelida@yahoo.com

[3]Valdete Loku

Department of Computer Sciences and Applied Mathematics
College, Vizioni per Arsim, Ahmet Kaciku, Nr=3, Ferizaj, 70000, Kosova

Abstract

In this paper we will define the new weighted statistically summbaility method, known as the weighted Norlund-Euler statistical convergence. We will show some properties of this method and we have proved Korovkin type theorem.

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1. INTRODUCTION

The idea of statistical convergence which is closely related to the concept of natural density or asymptotic density of a subset of the set of natural numbers N, was first introduced by Fast [3]. The weighted statistical convergence is defined by V. Karakaja and T.A.Chishti [2]. And the well defined version of the statistically weighted convergence sequences is given by Mursaleen et al., [15]. The interesting role in our results play the product of summability methods $(N, p, q)(E, 1)$ and we call it Norlund-Euler summability method. The concept of statistical convergence plays an important role in the summability theory and functional analysis. The relationship between the summability theory and statistical convergence has been introduced by Schoenberg [7]. Afterwards, the statistical convergence has been studied as a summability method by many researchers such as Fridy [8], Freedman et al. [9], Kolk [10, 11], Fridy and Miller [12], Fridy and Orhan [13,14], Mursaleen et al., [15] , Savaş [16], Braha [3-5]. Also, some topological properties of statistical convergence sequence spaces have been studied by Salat [17]. Besides in [18, 19], Connor showed the relations between statistical convergence and functional analysis.

In general, statistical convergence of weighted means is studied as a class of regular matrix transformations. In this work, we introduce and study the concept of weighted Norlund-Euler statistical convergence. The relations among $(N, p, q)(E, 1)$ -summability and $S_{(NE)}$ sequence of weighted statistically convergent sequences related to the weight $(N, p, q)(E, 1)$.

Let $K \subseteq N$ and $K_n = \{k \in K : k \le n\}$. Then the natural density of K is defined by

$$\delta(K) = \lim_{n \to \infty} \frac{|K_n|}{n} \text{ if the limit exists, where } |K_n| \text{ denotes the cardinality of } K_n.$$

A sequence $x = (x_k)$ of real numbers is said to be statistically convergent to provided that for every $\varepsilon > 0$ the set $K(\varepsilon) = \{k \in K : |x_n - L| \ge \varepsilon\}$ has natural density zero (Fast [5] and Steinhauss [14]), for each $\varepsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{n} |\{j \le n : |x_j - L| \ge \varepsilon\}| = 0$$

 in this case we write $L = st - \lim x$ or (S-$\lim x = L$). The symbol S denotes the set of all statistically convergent sequences.

Let $\sum\limits_{n=0}^{\infty} x_n$ be a given infinite series with sequence of its nth partial sum $\{S_n\}$.

If $(E,1)$ transform is defined as

$$E_n^1 = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} S_k$$

And we say that this summability method is convergent if $E_n^1 \to S$ as $n \to \infty$. In this case we say the series $\sum\limits_{n=0}^{\infty} x_n$ is $(E,1)$ − summable to a definite number S. (Hardy [19])

And we will write $S_n \to S(E,1)$, as $n \to \infty$.

Let (p_n) and (q_n) be the two sequences of non-zero real constants such that

$$P_n = p_0 + p_1 + ... + p_n \ , \ \ P_{-1} = p_{-1} = 0$$

$$Q_n = q_0 + q_1 + ... + q_n \ , \ \ Q_{-1} = q_{-1} = 0$$

For the given sequences (p_n) and (q_n), convolution $p*q$ is defined by:

$$R_n = p*q = \sum_{k=0}^{n} p_n q_{n-k}$$

The series $\sum\limits_{n=0}^{\infty} x_n$ or the sequence $\{S_n\}$ is summable to S by generalized Norlund method and it is denoted by $S_n \to S(N,p,q)$ if

$$t_n^{p,q} = \frac{1}{R_n} \sum_{v=0}^{n} p_{n-v} q_v S_v$$

tends to S, as $n \to \infty$.

Let us use in consideration the following method of summability:

$$t_n^{p,q,E} = \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k E_k^1 = \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} S_v \ .$$

If $t_n^{p,q,E} \to S$ as $n \to \infty$, then we say that the series $\sum_{n=0}^{\infty} x_n$ or the sequence $\{S_n\}$ is summable to S by Norlund-Euler method and it is denoted by $S_n \to S\,(N,p,q)(E,1)$

Remark: If $p_k = 1, q_k = 1$, then we get Euler summability method

Let us denote by $|(N,p,q)(E,1)|$ the sequence space all strongly convergent sequences $x = (x_k)$ which are $(N,p,q)(E,1)$ summable to L:

$$|(N,p,q)(E,1)| = \left\{ x = x_n : \lim_{n \to \infty} \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} \,|\, x_v - L \,|\, = 0 \ for\ some\ L \right\}$$

The matrix $A = (a_{n,k})$ in $(N,p,q)(E,1)$ - summability is given by

$$a_{n,k} = \begin{cases} \dfrac{1}{R_n} p_{n-k} q_k \dfrac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v}, if\ k \le n \\ 0,\ if\ k > n \end{cases}$$

Now we are able to give the definition of the weighted statistical convergence related to the $(N,p,q)(E,1)$ – summability method.

Definition 1.1. A sequence $x = (x_k)$ is said to be weighted Norlund-Euler statistical convergent if for every $\varepsilon > 0$.

$$\lim_{n \to \infty} \frac{1}{R_n} \left| \left\{ k \le R_n : p_{n-k} q_k \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{n} \,|\, x_v - L \,|\, \ge \varepsilon \right\} \right| = 0$$

The set of weighted Norlund-Euler statistical convergence sequence is denoted by S_{NE} as follows:

$$S_{NE} = \left\{ x = (x_n) : \lim_{n \to \infty} \frac{1}{R_n} \left\| \left\{ k \le R_n : p_{n-k} q_k \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{n} | x_v - L | \ge \varepsilon \right\} \right\| = 0, \text{ for some } L \right\}$$

If the sequence $x = (x_k)$ is S_{NE} -convergence, then we also use the notation $x_k \to L(S_{NE})$.

Definition 1.2. A sequence $x = (x_k)$ is said to be $[(N, p, q)(E,1)]_r - summable(0 < r < \infty)$

to the limit L if $\lim_{n \to \infty} \frac{1}{R_n} \sum_{k=1}^{n} p_{n-k} q_k \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{n} | x_v - L |^r = 0$

, and we write it as $x_k \to L([(N, p, q)(E,1)]_r)$

In these case L is called the limit $[(N, p, q)(E,1)]_r -$ limit of x.

In these paper, we establish the relation of S_{NE}- convergence with statistical summability $(N, p, q)(E,1)$ and $[(N, p, q)(E,1)]_r$.

2. MAIN RESULTS

In our first theorem we establish the relation between S_{NE} -statistical convergence and statistical summability $(N, p, q)(E,1)$.

Theorem 2.1. Let $p_{n-k} q_k \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} | x_v - L | \le M$ for all $k \in N$. If a sequence $x = (x_k)$ is S_{NE} -statistically convergent to L then it statistically summable $(N, p, q)(E,1)$ to L but not conversely.

Proof: Since $x = (x_k)$ is S_{NE} -statistically convergent to L,

$$\lim_{n \to \infty} \frac{1}{R_n} \left\| \left\{ k \le R_n : p_{n-k} q_k \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} | x_v - L | \ge \varepsilon \right\} \right\| = 0,$$

where $K_\varepsilon = \left\{ k \in N : p_{n-k} q_k \dfrac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} | x_v - L | \geq \varepsilon \right\}$.

Write $K_{R_n}(\varepsilon) = \left\{ k \leq R_n : p_{n-k} q_k \dfrac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} | x_v - L | \geq \varepsilon \right\}$ and

$K^c_{R_n}(\varepsilon) = \left\{ k \leq R_n : p_{n-k} q_k \dfrac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} | x_v - L | < \varepsilon \right\}$. Than

$$\left| t_n^{p,q,E} - L \right| = \left| \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} x_v - L \right| = \left| \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} (x_v - L) \right|$$

$$= \left| \frac{1}{R_n} \sum_{\substack{k=0 \\ k \in K_{R_n}(\varepsilon)}}^{n} p_{n-k} q_k \frac{1}{2^k} \sum_{\substack{v=0 \\ k \in K_{R_n}(\varepsilon)}}^{k} \binom{k}{v} (x_v - L) + \frac{1}{R_n} \sum_{\substack{k=0 \\ K^c_{R_n}(\varepsilon)}}^{n} p_{n-k} q_k \frac{1}{2^k} \sum_{\substack{v=0 \\ K^c_{R_n}(\varepsilon)}}^{k} \binom{k}{v} (x_v - L) \right|$$

$$\leq \frac{1}{R_n} \sum_{\substack{k=0 \\ K_{R_n}(\varepsilon)}}^{n} p_{n-k} q_k \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} | (x_k - L) | + \frac{1}{R_n} \sum_{\substack{k=0 \\ K^c_{R_n}(\varepsilon)}}^{n} p_{n-k} q_k \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} | (x_k - L) |$$

$$\leq \frac{1}{R_n} M \left| \left\{ k \leq R_n : p_{n-k} q_k \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{n} | x_v - L | \geq \varepsilon \right\} \right| + \frac{1}{R_n} \sum_{k \in K^c_{R_n}(\varepsilon)} \varepsilon$$

$$= \frac{1}{R_n} M \left| \left\{ k \leq R_n : p_{n-k} q_k \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{n} | x_v - L | \geq \varepsilon \right\} \right| + \varepsilon \frac{| K^c_{R_n}(\varepsilon) |}{R_n} \to 0 + \varepsilon \cdot 1 = \varepsilon$$

as $n \to \infty$ which implies that $t_n^{p,q} = \dfrac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k \dfrac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} x_v \to L$. That is, x is

$(N, p, q)(E,1)$-summable to L and hence statistically summable $(N, p, q)(E,1)$ to L.

Example 1: Let us consider that $p_n = 1$ and $q_n = 1$ for all $n \in \mathbb{N}$. Also we define the following sequence:

$x = (x_k)$ as follows:

$$x_k = \begin{cases} \sqrt{k}, if \ k = n^2 \\ 0, \ if \ k \neq n \end{cases}$$

Then we have $R_n = \sum_{k=0}^{n} 1 = n + 1$ and

$$\frac{1}{R_n}\left|\left\{k \le R_n : \sum_{k=0}^{n} p_{n-k}q_k \frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}|x_v - 0| \ge \varepsilon\right\}\right| = \frac{\sqrt{n}}{n+1} \to 0 \text{, as } n \to \infty.$$

On the other hand,

$$\frac{1}{R_n}\sum_{k=0}^{n} p_{n-k}q_k \frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}|x_v - 0| = \frac{1}{n+1}\sum_{k=0}^{n} 1\frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}x_{v^2} = \frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}v$$

$$=$$

$$\frac{1}{n+1}\sum_{k=0}^{n}\frac{k}{2} = \frac{n}{4} \to \infty \text{, as } n \to \infty.$$

These examples proves that converse is not true.

Next theorem gives the relation between S_{NE}-statistically convergent and $[(N,p,q)(E,1)]_r$

Theorem 2.2.

a. Let a sequence $x = (x_k)$ is $[(N,p,q)(E,1)]_r$ - summable to L. If the following are valid , then $x = (x_k)$ is S_{NE}-statistically convergent to L:

   Case 1: $0 < r < 1$ and $0 \le |x_k - L| < 1$

   Case 2: $1 \le r < \infty$ and $1 \le |x_k - L| < \infty$.

b. Let $x = (x_k)$ is S_{NE}-statistically convergent to L and $p_{n-k}q_k\frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}|x_v - L| \le M$ (k=1,2,3,...). If the following are valid, then $x_k \to L[(N,p,q)(E,1)]_r$:

   Case 1: $0 < r < 1$ and $1 \le M < \infty$

   Case 2: $1 \le r < \infty$ and $0 \le M < 1$.

Proof:

a. Since $p_{n-k}q_k \dfrac{1}{2^k}\sum\limits_{v=0}^{k}\binom{k}{v}\mid x_v - L\mid^r \geq p_{n-k}q_k \dfrac{1}{2^k}\sum\limits_{v=0}^{k}\binom{k}{v}\mid x_v - L\mid$

for case 1. and case2., then as $n \to \infty$

$$0 \leftarrow \frac{1}{R_n}\sum_{k=0}^{n} p_{n-k}q_k \frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}\mid x_v - L\mid^r$$

$$\geq \frac{1}{R_n}\sum_{k=0}^{n} p_{n-k}q_k \frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}\mid x_v - L\mid \geq \frac{1}{R_n}\sum_{\substack{k=0 \\ K_{R_n}(\varepsilon)}}^{n} p_{n-k}q_k \frac{1}{2^k}\sum_{\substack{v=0 \\ K_{R_n}(\varepsilon)}}^{k}\binom{k}{v}\mid x_v - L\mid \geq$$

$$\frac{1}{R_n}\sum_{\substack{k=1 \\ K_{R_n}(\varepsilon)}}^{n}\varepsilon = \varepsilon\frac{\mid K_{R_n}(\varepsilon)\mid}{R_n}$$

$$\lim_{n\to\infty}\frac{\mid K_{R_n}(\varepsilon)\mid}{R_n} = 0 \text{ where}$$

$$K_{R_n}(\varepsilon) = \left\{ k \leq R_n : p_{n-k}q_k \frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}\mid x_v - L\mid \geq \varepsilon \right\}$$

Hence S_{NE}-statistically convergent to *L*.

b. Suppose that $x = (x_k)$ is S_{NE}-statistically convergent to *L*. Then for $\varepsilon > 0$, we have $\delta_{NE}(K_\varepsilon) = \delta_{NE}(\left\{ k \in N : p_{n-k}q_k \frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}\mid x_v - L\mid \geq \varepsilon \right\}) = 0$. Since

$$p_{n-k}q_k \frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}\mid x_v - L\mid \leq M \text{ (k=1,2,...), we have}$$

$$\frac{1}{R_n}\sum_{\substack{k=0 \\ K_{R_n}(\varepsilon)}}^{n} p_{n-k}q_k \frac{1}{2^k}\sum_{\substack{v=0 \\ K_{R_n}(\varepsilon)}}^{k}\binom{k}{v}\mid x_v - L\mid^r =$$

$$\frac{1}{R_n}\sum_{\substack{k=0 \\ k\notin K_{R_n}(\varepsilon)}}^{n} p_{n-k}q_k \frac{1}{2^k}\sum_{\substack{v=0 \\ k\notin K_{R_n}(\varepsilon)}}^{k}\binom{k}{v}\mid x_v - L\mid^r + \frac{1}{R_n}\sum_{\substack{k=0 \\ k\in K_{R_n}(\varepsilon)}}^{n} p_{n-k}q_k \frac{1}{2^k}\sum_{\substack{v=0 \\ k\in K_{R_n}(\varepsilon)}}^{k}\binom{k}{v}\mid x_v - L\mid^r = S_1(n) + S_2(n)$$

where

$$S_1(n) = \frac{1}{R_n} \sum_{\substack{k=0 \\ k \notin K_{R_n}(\varepsilon)}}^{n} p_{n-k} q_k \frac{1}{2^k} \sum_{\substack{v=0 \\ k \notin K_{R_n}(\varepsilon)}}^{k} \binom{k}{v} |x_v - L|^r$$

$$S_2(n) = \frac{1}{R_n} \sum_{\substack{k=0 \\ k \in K_{R_n}(\varepsilon)}}^{n} p_{n-k} q_k \frac{1}{2^k} \sum_{\substack{v=0 \\ k \in K_{R_n}(\varepsilon)}}^{k} \binom{k}{v} |x_v - L|^r .$$

Now if $k \notin K_{R_n}(\varepsilon)$ then $S_1(n) = \dfrac{1}{R_n} \sum\limits_{\substack{k=0 \\ k \notin K_{R_n}(\varepsilon)}}^{n} p_{n-k} q_k \dfrac{1}{2^k} \sum\limits_{\substack{v=0 \\ k \notin K_{R_n}(\varepsilon)}}^{k} \binom{k}{v} |x_v - L|^r$

$$\leq \frac{1}{R_n} \sum_{\substack{k=0 \\ k \notin K_{R_n}(\varepsilon)}}^{n} p_{n-k} q_k \frac{1}{2^k} \sum_{\substack{v=0 \\ k \notin K_{R_n}(\varepsilon)}}^{k} \binom{k}{v} |x_v - L| = \varepsilon \frac{1}{R_n} |K_{R_n}^c| = \varepsilon$$

For $\qquad\qquad k \in K_{R_n}(\varepsilon),$ $\qquad\qquad$ we $\qquad\qquad$ have

$$S_2(n) = \frac{1}{R_n} \sum_{\substack{k=0 \\ k \in K_{R_n}(\varepsilon)}}^{n} p_{n-k} q_k \frac{1}{2^k} \sum_{\substack{v=0 \\ k \in K_{R_n}(\varepsilon)}}^{k} \binom{k}{v} |x_v - L|^r$$

$$\leq \frac{1}{R_n} \sum_{\substack{k=0 \\ k \in K_{R_n}(\varepsilon)}}^{n} p_{n-k} q_k \frac{1}{2^k} \sum_{\substack{v=0 \\ k \in K_{R_n}(\varepsilon)}}^{k} \binom{k}{v} |x_v - L| \leq (\sup_k \sum_{\substack{k=0 \\ k \in K_{R_n}(\varepsilon)}}^{n} p_{n-k} q_k \frac{1}{2^k} \sum_{\substack{v=0 \\ k \in K_{R_n}(\varepsilon)}}^{k} \binom{k}{v} |x_v - L|)(\frac{K_{R_n}(\varepsilon)}{R_n})$$

$$\leq M \frac{|K_{R_n}(\varepsilon)|}{R_n} \to 0, \text{ as } n \to \infty. \text{ Since}$$

$$\delta_{NE}(K_\varepsilon) = \delta_{NE}(\left\{ k \in N : p_{n-k} q_k \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} |x_v - L| \geq \varepsilon \right\}) = 0.$$

Hence $x_k \to L[(N, p, q)(E, 1)]_r$.

3. Application to approximation theorem

Let $C[a,b]$ be the space of all functions f continuous on $[a,b]$. We know that $C[a,b]$ is Banach space with norm

$$\|f\|_{\infty} = \sup_{x \in [a,b]} |f(x)|, f \in C[a,b]$$

The classical Korovkin approximation theorem states as follows [26]:

Let T_n be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$. Then $\lim_{n}\|T_n(f,x) - f(x)\|_{\infty} = 0$,   for   all $f \in C[a,b]$ if   only   if $\lim_{n}\|T_n(f_i,x) - f_i(x)\|_{\infty} = 0$,   for $i = 0,1,2$ where $f_0(x) = 1$, $f_1(x) = x$ dhe $f_2(x) = x^2$.

Its statistical version was given by Gadjiev and Orhan [25]. Such type of approximation theorems are proved by using the concept of almost convergence [2],[6], [21-27], λ-statistical convergence [29,31] and statistical lacunary summability [30]. Boyanov and Veselinov [23] have proved the Korovkin theorem on $C[0,\infty)$ by using the test functions $1, e^{-x}, e^{-2x}$. In addition, some related papers on this topic can be found in [33–35]. In this paper, we generalize the result of Boyanov and Veselinov by using the notion of statistical summability $(N, p, q)(E,1)$ and the same test functions $1, e^{-x}, e^{-2x}$. We also give an example to justify that our result is stronger than that of Boyanov and Veselinov [23].

Let $C(I)$ be the Banach space with the uniform norm $\|,\|_{\infty}$ of all real-valued two dimensional continuous functions on $I = [0,\infty)$; provided that $\lim_{x \to \infty} f(x)$ is finite. Suppose that $L_n : C(I) \to C(I)$. We write $L_n(f;x)$ for $L_n(f(s);x)$; and we say that L is a positive operator if $L_n(f;x) \geq 0$ for all $f(x) \geq 0$.

The following statistical version of Boyanov and Veselinov's result can be obtained as a special case of [24].

Teorem A. Let (T_k) be a sequence of positive linear operators from $C(I)$ into $C(I)$. Than for all $f \in C(I)$.

$$st - \lim_{k \to \infty}\|T_k(f;x) - f(x)\|_{\infty} = 0$$

if only if

$$st - \lim_{k \to \infty}\|T_k(1;x) - f(1)\|_{\infty} = 0$$

$$st - \lim_{k \to \infty} \left\| T_k(e^{-s}; x) - e^{-x} \right\|_\infty = 0$$

$$st - \lim_{k \to \infty} \left\| T_k(e^{-2s}; x) - e^{-2x} \right\|_\infty = 0$$

Similarly one can prove the S_{NE}-statistical version. Now we prove the following stronger version by using the notion of statistical summability $(N, p, q)(E,1)$.

Teorem 3.1. Let (T_k) be a sequence of positive linear operators from $C(I)$ into $C(I)$. Than for all $f \in C(I)$.

$$NE(st) - \lim_{k \to \infty} \left\| T_k(f; x) - f(x) \right\|_\infty = 0 \tag{3.1.0}$$

if only if

$$NE(st) - \lim_{k \to \infty} \left\| T_k(1; x) - f(1) \right\|_\infty = 0 \tag{3.1.1}$$

$$NE(st) - \lim_{k \to \infty} \left\| T_k(e^{-s}; x) - e^{-x} \right\|_\infty = 0 \tag{3.1.2}$$

$$NE(st) - \lim_{k \to \infty} \left\| T_k(e^{-2s}; x) - e^{-2x} \right\|_\infty = 0 \tag{3.1.3}$$

Proof. Since each $1, e^{-x}, e^{-2x}$ belongs to $C(I)$, conditions (3.1.1)-(3.1.3) follow immediately from (3.1.0). Let $f \in C(I)$. Then there exists a constant $M > 0$ such that $|f(x)| \le M$ for $x \in I$. Therefore,

$$|f(s) - f(x)| \le 2M, \quad -\infty < s, x < +\infty \tag{3.1.4}$$

It is easy to prove that for a given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(s) - f(x)| < \varepsilon \tag{3.1.5}$$

whenever $\left| e^{-s} - e^{-x} \right| < \delta$ for all $x \in I$.

Using (3.1.4) and (3.1.5), putting $\psi_1 = \psi_1(s, x) = (e^{-s} - e^{-x})^2$, we get

$$|f(s) - f(x)| < \varepsilon + \frac{2M}{\delta^2}(\psi_1), \ \forall |s - x| < \delta.$$

This is,

$$-\varepsilon - \frac{2M}{\delta^2}(\psi_1) < f(s) - f(x) < \varepsilon + \frac{2M}{\delta^2}(\psi_1).$$

Now, operating $T_k(1;x)$ to this inequality since $T_k(f;x)$ is monotone and linear. We obtain

$$T_k(1;x)\left(-\varepsilon - \frac{2M}{\delta^2}(\psi_1)\right) < T_k(1;x)(f(s) - f(x)) < T_k(1;x)\left(\varepsilon + \frac{2M}{\delta^2}(\psi_1)\right).$$

Note that x is fixed and so $f(x)$ is constant number. Therefore,

$$-\varepsilon T_k(1;x) - \frac{2M}{\delta^2}(\psi_1)T_k(1;x) < (f(s) - f(x))T_k(1;x) < \varepsilon T_k(1;x) + \frac{2M}{\delta^2}(\psi_1)T_k(1;x)$$

$$-\varepsilon T_k(1;x) - \frac{2M}{\delta^2}T_k(\psi_1;x) < T_k(f;x) - f(x)T_k(1;x) < \varepsilon T_k(1;x) + \frac{2M}{\delta^2}T_k(\psi_1;x)$$
(3.1.6)

Also,

$$T_k(f;x) - f(x) = T_k(f;x) - f(x)T_k(1;x) + f(x)T_k(1;x) - f(x)$$
$$= T_k(f;x) - f(x)T_k(1;x) + f(x)[T_k(1;x) - 1]$$
$$\qquad (3.1.7)$$

If follows (3.1.6) and (3.1.7) that

$$T_k(f;x) - f(x) < \varepsilon T_k(1;x) + \frac{2M}{\delta^2}T_k(\psi_1;x) + f(x)[T_k(1;x) - 1]$$
$$\qquad (3.1.8)$$

Now

$$T_k(\psi_1;x) = T_k\left((e^{-s} - e^{-x})^2;x\right) = T_k(e^{-2s} - 2e^{-s}e^{-x} + e^{-2x};x)$$
$$= T_k(e^{-2s};x) - 2e^{-x}T_k(e^{-s};x) + e^{-2x}T_k(1;x)$$
$$= [T_k(e^{-2s};x) - e^{-2x}] - 2e^{-x}[T_k(e^{-s};x) - e^{-x}] + e^{-2x}[T_k(1;x) - 1]$$

Using (3.1.8) , we obtain

$$T_k(f;x) - f(x) < \varepsilon T_k(1;x) + \frac{2M}{\delta^2}\{T_k(e^{-2s};x) - e^{-2x}] - 2e^{-x}[T_k(e^{-s};x) - e^{-x}] + e^{-2x}[T_k(1;x) - 1]\} +$$
$$+ f(x)[T_k(1;x) - 1]$$

There for

$$|T_k(f;x) - f(x)| \le \varepsilon + (\varepsilon + M)|T_k(1;x) - 1| + \frac{2M}{\delta^2}|e^{-2x}||T_k(1;x) - 1| + \frac{2M}{\delta^2}|T_k(e^{-2x};x) - e^{-2x}| +$$
$$\frac{4M}{\delta^2}|e^{-x}||T_k(e^{-s};x) - e^{-x}| \le \varepsilon + (\varepsilon + M + \frac{4M}{\delta^2})|T_k(1;x) - 1| + \frac{2M}{\delta^2}|e^{-2x}||T_k(1;x) - 1| + \frac{2M}{\delta^2}|T_k(e^{-2x};x) - e^{-2x}| +$$
$$\frac{4M}{\delta^2}|T_k(e^{-s};x) - e^{-x}|.$$

Since $|e^{-x}| \le 1$ for all $x \in I$ .Now, taking $\sup_{x \in I}$, we get

$$\|T_k(f;x) - f(x)\|_\infty \le \varepsilon + K(\|T_k(1;x) - 1\|_\infty + \|T_k(e^{-s};x) - e^{-x}\|_\infty + \|T_k(e^{-2s};x) - e^{-2x}\|_\infty),$$

where $K = \max\left\{\varepsilon + M + \frac{4M}{\delta^2}, \frac{2M}{\delta^2}\right\}$. Hence

$$\|T_k(f;x) - f(x)\|_\infty \le$$
$$\varepsilon + K(\|T_k(1;x) - 1\|_\infty + \|T_k(e^{-s};x) - e^{-x}\|_\infty + \|T_k(e^{-2s};x) - e^{-2x}\|_\infty),$$
$$(3.1.6)$$

Now replacing $T_k(\cdot,x)p_{n-k}q_k\frac{1}{2^k}\sum_{v=1}^{k}\binom{k}{v}$ by $\frac{1}{R_m}\sum_{k=0}^{m}T_k(\cdot,x)p_{n-k}q_k\frac{1}{2^k}\sum_{v=1}^{k}\binom{k}{v}$ and

then by $B_m(\cdot,x)p_{n-k}q_k\frac{1}{2^k}\sum_{v=1}^{k}\binom{k}{v}$ in (3.1.6) on both sides. For a given

$r > 0$ choose $\varepsilon' > 0$ such that $\varepsilon' > r$. Define the following sets

$$D = \left\{m \le n : \|B_m(f,x) - f(x)\|_\infty \ge r\right\},$$

$$D_1 = \left\{m \le n : \|B_m(1,x) - 1\|_\infty \ge \frac{r - \varepsilon'}{4K}\right\},$$

$$D_2 = \left\{ m \leq n : \left\| B_m(t,x) - e^{-x} \right\|_\infty \geq \frac{r - \varepsilon'}{4K} \right\},$$

$$D_3 = \left\{ m \leq n : \left\| B_m(t^2,x) - e^{-2x} \right\|_\infty \geq \frac{r - \varepsilon'}{4K} \right\},$$

Then $D \subset D_1 \cup D_2 \cup D_3$, and so $\delta(D) \leq \delta(D_1) + \delta(D_2) + \delta(D_3)$. Therefore, using conditions (3.1.1)–(3.1.3), we get

$$NE(st) - \lim_{k \to \infty} \left\| T_k(f;x) - f(x) \right\|_\infty = 0 \qquad \text{This completes the proof of the theorem.}$$

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