Strong and Weak Convergence Theorems for a New Split Feasibility Problem

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Abstract. Very recently, Moudafi proposed the following new convex feasibility problem in [10,11]:

$$\text{find } x \in C, y \in Q \quad \text{such that} \quad Ax = By,$$

where the two closed convex sets C and Q are the fixed point sets of two firmly quasi-nonexpansive mappings respectively, H_1 , H_2 and H_3 are real Hilbert spaces, $A : H_1 \rightarrow H_3$ and $B : H_2 \rightarrow H_3$ are two bounded linear operators. However, they just obtained weak convergence for such new split feasibility problem. In this paper, we introduce a new algorithm which is more general than the SIM-FPP algorithm presented by Moudafi in [11] and obtain strong and weak convergence theorems for the new split feasibility problem. Our results extend and improve the corresponding result of Moudafi [11].

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1 Introduction

In the process of solving the linear prediction problem, Dirichlet problem, convex smooth function of the minimum problem, as well as the image reconstruction, signal processing in physics and engineering and other issues, we often encountered convex feasibility problems [1].

Let $C_1, C_2, \dots, C_N$ be N nonempty closed convex subsets of a Hilbert space H , the convex feasibility problem is formulated as following

$$\text{find } x^* \in \bigcap_{i=1}^N C_i. \quad (1.1)$$

For modeling inverse problems which arise from phase retrievals and in medical image reconstruction, in 1994 Censor and Elfving [2] introduced the following split feasibility problem:

Let C and Q be the nonempty closed convex subsets of the Hilbert spaces H_1 and H_2 , respectively, $A : H_1 \rightarrow H_2$ be a bounded linear operator. The split feasibility problem (SFP) is formulated as finding a point x^* with the property

$$x^* \in C \quad \text{and} \quad Ax^* \in Q. \quad (1.2)$$

The SFP has been found that it can be used in many areas such as image restoration, computer tomograph, and radiation therapy treatment planning [3-5].

Since convex feasibility problems and split feasibility problems can be widely applied in many areas, some researchers are attracted in constructing some algorithms to solve split feasibility problem, see for instance [6-9].

Recently, Moudafi introduced the following new split feasibility problems [10,11].

Let H_1, H_2, H_3 be real Hilbert spaces, $C \subset H_1, Q \subset H_2$ be two nonempty closed convex sets, $A : H_1 \rightarrow H_3, B : H_2 \rightarrow H_3$ be two bounded linear operators. The new split feasibility problem is to

$$\text{find } x^* \in C, y^* \in Q \quad \text{such that} \quad Ax^* = By^*, \quad (1.3)$$

which allows asymmetric and partial relations between the variables x and y .

Note that, it is easy to see that the problem (1.3) reduces to the problem (1.2) as $H_2 = H_3$ and $B = I$ (I stands for the identity mapping on H_2) in (1.3). Therefore the new split feasibility problem (1.3) proposed by Moudafi is a generalization of the split feasibility problem (1.2).

To solve problem (1.3), Moudafi [11] presented the following simultaneous iterative method and obtained weak convergence theorem:

$$(SIM - FPP) \begin{cases} x_{k+1} = U(x_k - \gamma_k A^*(Ax_k - By_k)); \\ y_{k+1} = T(y_k + \gamma_k B^*(Ax_k - By_k)), \end{cases} \quad (1.4)$$

where H_1, H_2, H_3 are real Hilbert spaces, $U : H_1 \rightarrow H_1, T : H_2 \rightarrow H_2$ are two firmly quasi-nonexpansive mappings, $A : H_1 \rightarrow H_3, B : H_2 \rightarrow H_3$ are two bounded linear operators, A^* and B^* are the adjoints of A and B , respectively, C is the set of fixed points of U and Q is the set of fixed points of T .

However, (SIM-FPP) algorithm (1.4) has only weak convergence theorem for the problem (1.3), in this paper, we introduce a new algorithm as follows

$$\begin{cases} \forall x_1 \in H_1, \quad \forall y_1 \in H_2; \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_1(x_n - \gamma_n A^*(Ax_n - By_n)); \\ y_{n+1} = (1 - \alpha_n)y_n + \alpha_n T_2(y_n + \gamma_n B^*(Ax_n - By_n)), \quad \forall n \geq 1, \end{cases} \quad (1.5)$$

and obtain strong and weak convergence theorems for the problem (1.3) under some mild control conditions in Hilbert spaces. Our results extend and improve the corresponding results of A.Moudafi [11].

2 Preliminaries

We first recall some definitions, notations and lemmas which will be needed in proving our main results.

We denote the set of fixed points of a mapping T by $F(T)$ and the solution set of the problem (1.3) by Ω , namely,

$$\Omega = \{(x, y) : Ax = By, x \in C, y \in Q\}.$$

Definition 2.1. Let H be a Hilbert space.

- (1) A single-value mapping $T : H \rightarrow H$ is said to be demi-closed at origin, if for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x^*$, and $\|(I - T)x_n\| \rightarrow 0$, then we have $x^* = Tx^*$.
- (2) A single-value mapping $T : H \rightarrow H$ is said to be semi-compact, if for any bounded sequence $\{x_n\} \subset H$ with $\|(I - T)x_n\| \rightarrow 0$, then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to a $x^* \in H$.

Definition 2.2. Let H be a real Hilbert space.

- (1) A mapping $T : H \rightarrow H$ is said to be firmly nonexpansive if for any $x, y \in H$

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2.$$

- (2) A mapping $T : H \rightarrow H$ is said to be firmly quasi-nonexpansive if for any $x \in H$ and $y \in F(T)$

$$\|Tx - y\|^2 \leq \|x - y\|^2 - \|x - Tx\|^2.$$

Lemma 2.3.(Opial's Lemma) Let H be a Hilbert space and $\{\mu_n\}$ be a sequence in H such that there exists a nonempty set $W \subset H$ satisfying:

- (i) For every $\mu^* \in W$, $\lim_{n \rightarrow \infty} \|\mu_n - \mu^*\|$ exists.
- (ii) Any weak-cluster point of the sequence $\{\mu_n\}$ belongs to W .

Then, there exists $\mu^* \in W$ such that $\{\mu_n\}$ weakly converges to μ^* .

3 Main result

Theorem 3.1. Let H_1, H_2, H_3 be real Hilbert spaces, let $T_1 : H_1 \rightarrow H_1$, $T_2 : H_2 \rightarrow H_2$ be two firmly quasi-nonexpansive mappings, and $A : H_1 \rightarrow H_3$, $B : H_2 \rightarrow H_3$ be two bounded linear operators. Assume that the iteration scheme $\{(x_n, y_n)\}$ is defined as follows:

$$\begin{cases} \forall x_1 \in H_1, \quad \forall y_1 \in H_2, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_1(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n+1} = (1 - \alpha_n)y_n + \alpha_n T_2(y_n + \gamma_n B^*(Ax_n - By_n)), \quad \forall n \geq 1, \end{cases}$$

where λ_A and λ_B stand for the spectral radius of A^*A and B^*B respectively, $\{\gamma_n\}$ is a positive real sequence such that $\gamma_n \in (\varepsilon, 2/(\lambda_A + \lambda_B) - \varepsilon)$ (for ε small enough), and $\{\alpha_n\} \subset [\alpha, 1]$ (for some $\alpha > 0$), $C := F(T_1)$ and $Q := F(T_2)$. If $I - T_1$ and $I - T_2$ are demi-closed at origin and $\Omega \neq \emptyset$, then

- (I) The sequence $\{(x_n, y_n)\}$ converges weakly to a solution of the problem (1.3).
- (II) In addition, if T_1, T_2 are also semi-compact, then $\{(x_n, y_n)\}$ converges strongly to a solution of the problem (1.3).

Proof. Now we prove the conclusion (I).

Let $(x, y) \in \Omega$. Since $\|\cdot\|^2$ is convex and T_1, T_2 are firmly quasi-nonexpansive, we have

$$\begin{aligned} & \|x_{n+1} - x\|^2 \\ &= \|(1 - \alpha_n)x_n + \alpha_n T_1(x_n - \gamma_n A^*(Ax_n - By_n)) - x\|^2 \\ &\leq (1 - \alpha_n)\|x_n - x\|^2 + \alpha_n \|T_1(x_n - \gamma_n A^*(Ax_n - By_n)) - x\|^2 \\ &\leq (1 - \alpha_n)\|x_n - x\|^2 + \alpha_n \|x_n - \gamma_n A^*(Ax_n - By_n) - x\|^2 \\ &\quad - \alpha_n \|T_1(x_n - \gamma_n A^*(Ax_n - By_n)) - (x_n - \gamma_n A^*(Ax_n - By_n))\|^2. \end{aligned} \quad (3.1)$$

Since

$$\begin{aligned} \|x_n - \gamma_n A^*(Ax_n - By_n) - x\|^2 &= \|x_n - x\|^2 + \|\gamma_n A^*(Ax_n - By_n)\|^2 \\ &\quad - 2\gamma_n \langle x_n - x, A^*(Ax_n - By_n) \rangle \\ &= \|x_n - x\|^2 + \|\gamma_n A^*(Ax_n - By_n)\|^2 \\ &\quad - 2\gamma_n \langle Ax_n - Ax, Ax_n - By_n \rangle, \end{aligned} \quad (3.2)$$

and

$$\begin{aligned}
\|\gamma_n A^*(Ax_n - By_n)\|^2 &= \gamma_n^2 \langle A^*(Ax_n - By_n), A^*(Ax_n - By_n) \rangle \\
&= \gamma_n^2 \langle Ax_n - By_n, AA^*(Ax_n - By_n) \rangle \\
&\leq \lambda_A \gamma_n^2 \langle Ax_n - By_n, Ax_n - By_n \rangle \\
&= \lambda_A \gamma_n^2 \|Ax_n - By_n\|^2,
\end{aligned} \tag{3.3}$$

combine with (3.1), (3.2) and (3.3), then we have

$$\begin{aligned}
&\|x_{n+1} - x\|^2 \\
&\leq (1 - \alpha_n) \|x_n - x\|^2 + \alpha_n \|x_n - x\|^2 + \alpha_n \lambda_A \gamma_n^2 \|Ax_n - By_n\|^2 \\
&\quad - 2\gamma_n \alpha_n \langle Ax_n - Ax, Ax_n - By_n \rangle \\
&\quad - \alpha_n \|T_1(x_n - \gamma_n A^*(Ax_n - By_n)) - (x_n - \gamma_n A^*(Ax_n - By_n))\|^2 \\
&= \|x_n - x\|^2 - 2\gamma_n \alpha_n \langle Ax_n - Ax, Ax_n - By_n \rangle + \alpha_n \lambda_A \gamma_n^2 \|Ax_n - By_n\|^2 \\
&\quad - \alpha_n \|T_1(x_n - \gamma_n A^*(Ax_n - By_n)) - (x_n - \gamma_n A^*(Ax_n - By_n))\|^2.
\end{aligned} \tag{3.4}$$

Similarly, from the second equality of algorithm we can get

$$\begin{aligned}
&\|y_{n+1} - y\|^2 \\
&\leq \|y_n - y\|^2 + 2\gamma_n \alpha_n \langle By_n - By, Ax_n - By_n \rangle + \alpha_n \lambda_B \gamma_n^2 \|Ax_n - By_n\|^2 \\
&\quad - \alpha_n \|T_2(y_n + \gamma_n B^*(Ax_n - By_n)) - (y_n + \gamma_n B^*(Ax_n - By_n))\|^2.
\end{aligned} \tag{3.5}$$

Since $(x, y) \in \Omega$ so we have the fact that $Ax = By$, and finally we have

$$\begin{aligned}
&\|x_{n+1} - x\|^2 + \|y_{n+1} - y\|^2 \\
&\leq \|x_n - x\|^2 + \|y_n - y\|^2 + \alpha_n \gamma_n^2 (\lambda_A + \lambda_B) \|Ax_n - By_n\|^2 \\
&\quad + 2\gamma_n \alpha_n \langle By_n - Ax_n, Ax_n - By_n \rangle \\
&\quad - \alpha_n \|T_1(x_n - \gamma_n A^*(Ax_n - By_n)) - (x_n - \gamma_n A^*(Ax_n - By_n))\|^2 \\
&\quad - \alpha_n \|T_2(y_n + \gamma_n B^*(Ax_n - By_n)) - (y_n + \gamma_n B^*(Ax_n - By_n))\|^2 \\
&= \|x_n - x\|^2 + \|y_n - y\|^2 - \alpha_n \gamma_n (2 - \gamma_n (\lambda_A + \lambda_B)) \|Ax_n - By_n\|^2 \\
&\quad - \alpha_n \|T_1(x_n - \gamma_n A^*(Ax_n - By_n)) - (x_n - \gamma_n A^*(Ax_n - By_n))\|^2 \\
&\quad - \alpha_n \|T_2(y_n + \gamma_n B^*(Ax_n - By_n)) - (y_n + \gamma_n B^*(Ax_n - By_n))\|^2.
\end{aligned} \tag{3.6}$$

Let $\Omega_n(x, y) := \|x_n - x\|^2 + \|y_n - y\|^2$ then we have

$$\begin{aligned}
\Omega_{n+1}(x, y) &\leq \Omega_n(x, y) - \alpha_n \gamma_n (2 - \gamma_n (\lambda_A + \lambda_B)) \|Ax_n - By_n\|^2 \\
&\quad - \alpha_n \|T_1(x_n - \gamma_n A^*(Ax_n - By_n)) - (x_n - \gamma_n A^*(Ax_n - By_n))\|^2 \\
&\quad - \alpha_n \|T_2(y_n + \gamma_n B^*(Ax_n - By_n)) - (y_n + \gamma_n B^*(Ax_n - By_n))\|^2.
\end{aligned} \tag{3.7}$$

Obviously the sequence $\{\Omega_n(x, y)\}$ is decreasing and has lower bounded, so it converges to some finite limit $\omega(x, y)$. This means that the first condition of Lemma 2.3(Opial's lemma) is satisfied with $W = \Omega$, $\mu_n := (x_n, y_n)$, $\mu^* := (x, y)$. And by

passing to limit in (3.7), we obtain that

$$\lim_{n \rightarrow \infty} \|Ax_n - By_n\| = 0,$$

and

$$\lim_{n \rightarrow \infty} \|T_1(x_n - \gamma_n A^*(Ax_n - By_n)) - (x_n - \gamma_n A^*(Ax_n - By_n))\| = 0, \quad (3.8)$$

$$\lim_{n \rightarrow \infty} \|T_2(y_n + \gamma_n B^*(Ax_n - By_n)) - (y_n + \gamma_n B^*(Ax_n - By_n))\| = 0. \quad (3.9)$$

Since $\{\Omega_n(x, y)\}$ is convergent for any $(x, y) \in \Omega$, we know that $\{x_n\}$ and $\{y_n\}$ are bounded. So we may assume that $\{x_n\}$ converges weakly to x^* and $\{y_n\}$ converges weakly to y^* , respectively. Further, $\{x_n - \gamma_n A^*(Ax_n - By_n)\}$ also converges weakly to x^* , $\{y_n + \gamma_n B^*(Ax_n - By_n)\}$ converges weakly to y^* . Due to (3.8), (3.9) and $I - T_1$ and $I - T_2$ are demi-closed at origin, we know that $x^* \in F(T_1)$ and $y^* \in F(T_2)$.

On the other hand, since the squared norm is weakly lower semicontinuous, we have

$$\|Ax^* - By^*\|^2 \leq \liminf_{n \rightarrow \infty} \|Ax_n - By_n\|^2 = 0.$$

Therefore $Ax^* = By^*$. This implies that $(x^*, y^*) \in \Omega$. Thus from Lemma 2.3, we know that $\{(x_n, y_n)\}$ converges weakly to (x^*, y^*) . The proof of conclusion(I) is completed.

Next, we prove the conclusion(II)

Since $\lim_{n \rightarrow \infty} \|T_1(x_n - \gamma_n A^*(Ax_n - By_n)) - (x_n - \gamma_n A^*(Ax_n - By_n))\| = 0$ and T_1 is semi-compact, there exists subsequence $\{x_{n_j} - \gamma_{n_j} A^*(Ax_{n_j} - By_{n_j})\}$ of $\{x_n - \gamma_n A^*(Ax_n - By_n)\}$ such that $\{x_{n_j} - \gamma_{n_j} A^*(Ax_{n_j} - By_{n_j})\}$ converges strongly to u^* . So from the facts that $\lim_{n \rightarrow \infty} \|Ax_n - By_n\| = 0$ and $\{x_n\}$ converges weakly to x^* , we know that $u^* = x^*$. Set $u_n = x_n - \gamma_n A^*(Ax_n - By_n)$, then $\{u_{n_j}\}$ converges strongly to x^* . In addition, due to

$$\begin{aligned} \|x_{n_j} - x^*\| &= \|x_{n_j} - u_{n_j} + u_{n_j} - x^*\| \\ &\leq \|\gamma_{n_j} A^*(Ax_{n_j} - By_{n_j})\| + \|u_{n_j} - x^*\|, \end{aligned}$$

we obtain that $\lim_{n \rightarrow \infty} \|x_{n_j} - x^*\| = 0$. Likewise, we also can obtain that $\lim_{n \rightarrow \infty} \|y_{n_j} - y^*\| = 0$.

On the other hand, since $\Omega_n(x, y) = \|x_n - x\|^2 + \|y_n - y\|^2$ for any $(x, y) \in \Omega$, we know that $\lim_{j \rightarrow \infty} \Omega_{n_j}(x^*, y^*) = 0$. From Conclusion (I), we have $\lim_{n \rightarrow \infty} \Omega_n(x^*, y^*)$ exists, therefore $\lim_{n \rightarrow \infty} \Omega_n(x^*, y^*) = 0$. Further, we can obtain that $\lim_{n \rightarrow \infty} \|x_n - x^*\| = 0$ and $\lim_{n \rightarrow \infty} \|y_n - y^*\| = 0$. This completes the proof of the Conclusion (II).

Remark 3.2. Our Theorem 3.1 reduces to the Theorem 2.2 appeared in [11] as $\alpha_n = 1$ ($n = 1, 2, \dots$).

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