Green’s Relations on the Menger Algebra of n-ary Ordered Preserving Operations

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1 Introduction

For integer $n \geq 1$, let $O^n(A)$ be the set of all $n$-ary operations defined on a set $A$ and let $O(A) := \bigcup_{n \geq 1} O^n(A)$ be the set of all operations defined on $A$.

For $f \in O^n(A)$ and $g_1, \ldots, g_n \in O^m(A)$, we define the superposition operation $S_{m^n} : O^n(A) \times (O^m(A))^n \rightarrow O^m(A)$ by

$$S_{m^n}(f, g_1, \ldots, g_n)(a_1, \ldots, a_m) := f(g_1(a_1, \ldots, a_m), \ldots, g_n(a_1, \ldots, a_m))$$

for all $a_1, \ldots, a_m \in A$. If $n = m$, we will write for short $S^n$ instead of $S_{m^n}$. For each $n \geq 1$ and each $1 \leq j \leq n$, the $n$-ary function $e^n_j : A^n \rightarrow A$ defined on $A$ by $e^n_j(a_1, \ldots, a_n) := a_j$ is called the $j$-th projection mapping of arity $n$.

In [2] and also [3], an algebra $\mathcal{M} := (M; S^n, e_1, \ldots, e_n)$ of type $\tau = (n + 1, 0, \ldots, 0)$ is called a unitary Menger algebra of rank $n$ if it satisfies the following axioms:
\( \tilde{S}\) is an \((n+1)\)-ary operation symbol, \(\lambda_1, \ldots, \lambda_n\) are nullary operation symbols and \(X_0, X_1, \ldots, X_n, Y_1, \ldots, Y_n\) are variables.

In [3], for Menger algebras of rank \(n\), Green’s relations were defined as the followings:

**Definition 1.1** Let \((M; S^n)\) be a Menger algebra of rank \(n\) and let \(a, b \in M\).

(i) \(aLb\) if either \(a = b\) or there are elements \(s_1, \ldots, s_n, t_1, \ldots, t_n \in M\) such that \(S^n(a, s_1, \ldots, s_n) = b\) and \(S^n(t_1, \ldots, t_n) = a\).

(ii) \(aRb\) if either \(a = b\) or there are elements \(s, t \in M\) such that \(S^n(s, a, \ldots, a) = b\) and \(S^n(t, b, b, \ldots, b) = a\).

(iii) \(D = R \circ L = L \circ R\).

(iv) \(H = R \cap L\).

(v) \(aJb\) if either \(a = b\) or there are elements \(s, s_1, \ldots, s_n \in M\) with \(a = S^n(s, s_1, \ldots, s_n)\) such that at least one of the factors is equal to \(b\) and there are elements \(t, t_1, \ldots, t_n \in M\) with \(b = S^n(t, t_1, \ldots, t_n)\) such that at least one of the factors is equal to \(a\).

## 2 The main results

In this section we will study on the Menger algebra \(\left(\text{Pol}_{\leq}^{(n)}(A); S^n\right)\) where \(A\) is a finite chain. For simplification we use \(a\) instead of \((a_1, a_2, \ldots, a_n)\). We also
use $\widehat{a}$ for the $n$-tuple consisting of the same element $a \in A$ i.e. $\widehat{a} = (a, a, ..., a)$. For each $f \in \text{Pol}^{(n)}_\leq (A)$, let

$$\text{Ker}_f := \{(\overline{a}, \overline{b}) \in A^n \times A^n \mid f(\overline{a}) = f(\overline{b})\},$$

and

$$\text{Im}_f := \{x \in A \mid \exists \overline{a} \in A^n, f(\overline{a}) = x\}.$$

Let $x \in \text{Im}_f$ we set $f^{-1}(x) = \{\overline{a} \in A^n \mid f(\overline{a}) = x\}$.

For a subset $B \subseteq A^n$ we define the set $f(B) := \{f(\overline{a}) \in A \mid \overline{a} \in B\}$. For any two subsets $B, B'$ of the chain $A$, we write $B \leq B'$ if $b \leq b'$ for all $b \in B, b' \in B'$.

Let $\triangle_{\text{Im}_f} = \{\widehat{x} \in A^n \mid x \in \text{Im}_f\}$.

In a Menger algebra $(\text{Pol}^{(n)}_\leq (A); S^n)$, we obtain the following theorems.

**Theorem 2.1** Let $f, g \in \text{Pol}^{(n)}_\leq (A)$. Then $f \mathcal{R} g$ if and only if the following conditions are satisfy

(i) $\text{Ker}_f = \text{Ker}_g$,

(ii) for each $x, y \in \text{Im}_f$, $x \leq y$ iff $g(f^{-1}(x)) \leq g(f^{-1}(y))$.

**Proof.** Suppose that $f \mathcal{R} g$. There exist $\alpha, \beta \in \text{Pol}^{(n)}_\leq (A)$ such that $f = S^n(\alpha, g, ..., g)$ and $g = S^n(\beta, f, ..., f)$. Let $(\overline{a}, \overline{b}) \in \text{Ker}_f$. We have $f(\overline{a}) = f(\overline{b})$ and hence $\widehat{f}(\overline{a}) = \widehat{f}(\overline{b})$. Thus $g(\overline{a}) = S^n(\beta, f, ..., f)(\overline{a}) = \beta(\widehat{f}(\overline{a})) = \beta(\widehat{f}(\overline{b})) = S^n(\beta, f, ..., f)(\overline{b}) = g(\overline{b})$. That is $\text{Ker}_f \subseteq \text{Ker}_g$. Similarly, we get $\text{Ker}_g \subseteq \text{Ker}_f$. Thus $\text{Ker}_f = \text{Ker}_g$. Let $x, y \in \text{Im}_f$ such that $x \leq y$. Then there exist $\overline{a}, \overline{b} \in A^n$ such that $f(\overline{a}) = x$ and $f(\overline{b}) = y$ (i.e, $\overline{a} \in f^{-1}(x)$ and $\overline{b} \in f^{-1}(y)$). We have $g(\overline{a}) = S^n(\beta, f, ..., f)(\overline{a}) = \beta(\widehat{f}(\overline{a})) = \beta(\widehat{x}) \leq \beta(\widehat{y}) = \beta(\widehat{f}(\overline{b})) = S^n(\beta, f, ..., f)(\overline{b}) = g(\overline{b})$. Therefore, $g(f^{-1}(x)) \leq g(f^{-1}(y))$. In the same way, for all $x, y \in \text{Im}_f$, if $g(f^{-1}(x)) \leq g(f^{-1}(y))$ we have $x \leq y$.

Conversely, suppose that $\text{Ker}_f = \text{Ker}_g$ and for each $x, y \in \text{Im}_f$, we have $x \leq y$ iff $g(f^{-1}(x)) \leq g(f^{-1}(y))$. For each $x \in \text{Im}_f$, we choose one element $\overline{a}_x \in f^{-1}(x)$. Define an $n$-ary operation $\beta : A^n \rightarrow A$ by $\beta(\widehat{x}) = g(\overline{a}_x)$ for all $\overline{a}_x \in \text{Im}_f$ and $\beta(\overline{c}) = c_0$ for all $\overline{c} \in A^n \triangle_{\text{Im}_f}$ and $c_0$ is a fixed element in $A$. Since $\text{Ker}_f = \text{Ker}_g$, we have $f(\overline{a}) = f(\overline{b})$ iff $g(\overline{a}) = g(\overline{b})$. Thus $\beta$ is well-defined. Let $\overline{a} \in A^n$ such that $f(\overline{a}) = x$. We have $S^n(\beta, f, ..., f)(\overline{a}) = \beta(f(\overline{a})) = \beta(\widehat{x}) = g(\overline{a}_x) = g(\overline{a})$. That is $g = S^n(\beta, f, ..., f)$. Let $x, y \in \text{Im}_f$ such that $x \leq y$. By assumption, we have $g(\overline{a}_x) \leq g(\overline{a}_y)$. Thus $\beta(\widehat{x}) \leq \beta(\widehat{y})$. For all $\overline{a}, \overline{b} \in A^n \triangle_{\text{Im}_f}$ such that $\overline{a} \leq \overline{b}$, we have $\beta(\overline{a}) = c_0 = \beta(\overline{b})$ and hence $\beta \in \text{Pol}^{(n)}_\leq (A)$. Similarly, we can get well-defined an $n$-ary ordered preserving operation $\alpha$ such that $f = S^n(\alpha, g, ..., g)$. Therefore, $f \mathcal{R} g$. 

For Green’s relation $\mathcal{L}$, we obtain the following result.
Theorem 2.2 Let \( f, g \in Pol_{\leq}(A) \). Then \( f \mathcal{L} g \) if and only if the following conditions are satisfy;

(i) \( \text{Im} f = \text{Im} g \).

(ii) For each \( x, y \in \text{Im} f \) such that \( x \leq y \). Then there exist \( a \in f^{-1}(x), b \in f^{-1}(y) \) such that \( a \leq b \) iff there exist \( a' \in g^{-1}(x), b' \in g^{-1}(y) \) such that \( a' \leq b' \).

Proof. Suppose that \( f \mathcal{L} g \). Then there exist \( s_1, \ldots, s_n, t_1, \ldots, t_n \in Pol_{\leq}(A) \) such that \( f = S^n(g, s_1, \ldots, s_n) \) and \( g = S^n(f, t_1, \ldots, t_n) \). Let \( x \in \text{Im} f \). Then \( f(a) = x \) for some \( a \in A^n \). We have \( x = f(a) = S^n(g, s_1, \ldots, s_n)(a) = g(s_1(a), \ldots, s_n(a)) \in \text{Im} g \). Thus \( \text{Im} f \subseteq \text{Im} g \). Similarly, we have \( \text{Im} g \subseteq \text{Im} f \) and hence \( \text{Im} f = \text{Im} g \). Let \( x, y \in \text{Im} f, x \leq y \). If there exists \( a \in f^{-1}(x), b \in f^{-1}(y) \) such that \( a \leq b \), we have \( x = f(a) = S^n(g, s_1, \ldots, s_n)(a) = g(s_1(a), \ldots, s_n(a)) \).

and \( y = f(b) = S^n(g, s_1, \ldots, s_n)(b) = g(s_1(b), \ldots, s_n(b)) \) where \( a' = (s_1(a), \ldots, s_n(a)) \in g^{-1}(x) \) and \( b' = (s_1(b), \ldots, s_n(b)) \in g^{-1}(y) \). Since \( s_1, \ldots, s_n \in Pol_{\leq}(A) \) and \( a \leq b \), we have \( s_i(a) \leq s_i(b) \) for all \( i = 1, 2, \ldots, n \). Thus \( a' = (s_1(a), \ldots, s_n(a)) \leq (s_1(b), \ldots, s_n(b)) = b' \). Similarly, we get that if there exist \( a' \in g^{-1}(x), b' \in g^{-1}(y) \) such that \( a' \leq b' \) then we have there exist \( a \in f^{-1}(x), b \in f^{-1}(y) \) such that \( a \leq b \).

Conversely, suppose that the conditions (i) and (ii) are hold. For each \( x, y \in \text{Im} f \) such that \( x \leq y \), we choose one element \( a^* \in g^{-1}(x) \) satisfies the following conditions;

–if no elements \( a \in f^{-1}(x), b \in f^{-1}(y) \) such that \( a \leq b \), we choose \( a^* = (a_{x_1}, \ldots, a_{x_n}) \in g^{-1}(x) \), and \( a^*_y = (a_{y_1}, \ldots, a_{y_n}) \in g^{-1}(y) \)

–if there exist elements \( a \in f^{-1}(x), b \in f^{-1}(y) \) such that \( a \leq b \), there exist \( a' \in g^{-1}(x), b' \in g^{-1}(y) \) such that \( a' \leq b' \), we choose \( a^*_x = a' = (a_{x_1}, \ldots, a_{x_n}) \in g^{-1}(x) \) and \( a^*_y = b' = (a_{y_1}, \ldots, a_{y_n}) \in g^{-1}(y) \). Since \( A^n = \bigcup_{x \in \text{Im} f} f^{-1}(x) \) is a disjont union, then for all \( j = 1, \ldots, n \), we define \( n-ary \) operations \( s_j : A^n \rightarrow A \) by \( s_j(a) = a_{x}^j \) for all \( a \in f^{-1}(x) \) and for all \( x \in \text{Im} f \).

Let \( a, b \in A^n \) such that \( a \leq b \). We have

–if \( a, b \in f^{-1}(x) \) then \( s_j(a) = a_{x}^j = s_j(b) \),

–if \( a \in f^{-1}(x), b \in f^{-1}(y) \) such that \( a \leq b \) then \( s_j(a) = a_{x}^j \leq a_{y}^j = s_j(b) \) and hence \( s_j \in Pol_{\leq}(A) \). Let \( a \in A^n \) such that \( f(a) = x \). We have \( S^n(g, s_1, \ldots, s_n)(a) = g(s_1(a), \ldots, s_n(a)) = g(a_{x_1}^*, \ldots, a_{x_n}^*) = g(a^*_x) = x = f(a) \). Thus \( f = S^n(g, s_1, \ldots, s_n) \). Similarly, we can get well-defined \( n-ary \) operations \( t_k : A^n \rightarrow A \) for all \( k = 1, \ldots, n \) such that \( g = S^n(f, t_1, \ldots, t_n) \).

Therefore, \( f \mathcal{L} g \). \( \square \)

For Green’s relation \( \mathcal{H} \), we obtain the following:
Theorem 2.3 Let \( f, g \in \text{Poi}_{\leq}^{(n)}(A) \). Then \( fHg \) if and only if \( f = g \).

**Proof.** If \( f = g \), we have \( fRg \) and \( fLg \). Hence, \( fHg \).

Suppose that \( fHg \), then \( fRG \) and \( fLG \). Thus \( Kef = Ker \) and \( Img = Img \). If \( |Img| = 1 \), we have \( f = g \). We assume that \( |Img| \geq 2 \) and \( f \neq g \). Then there exist \( x \in Img \) such that \( f^{-1}(x) \neq g^{-1}(x) \). Say that \( x_0 \) is a smallest element of \( Img \) such that \( f^{-1}(x_0) \neq g^{-1}(x_0) \). Then there exist \( y, z \in Img \) such that \( x_0 \leq y, x_0 \leq z \) and \( g^{-1}(x_0) = f^{-1}(y), f^{-1}(x_0) = g^{-1}(z) \). Since \( fRG \), we have \( g(f^{-1}(x_0)) \leq g(f^{-1}(y)) \). But \( g(f^{-1}(x_0)) = \{z\} \geq \{x_0\} = g(f^{-1}(y)) \) gives a contradiction. Thus \( f^{-1}(x) = g^{-1}(x) \) for all \( x \in Img = Img \). Hence, \( f = g \). \( \square \)

For Green’s relation \( D \), we obtain the following:

**Theorem 2.4** Let \( f, g \in \text{Poi}_{\leq}^{(n)}(A) \). Then \( fDg \) if and only if the following conditions are satisfy:

(i) \( |Img| = |Img| \).

(ii) For \( Img = \{x_1, x_2, \ldots, x_r\} \), \( Img = \{y_1, y_2, \ldots, y_r\} \) where \( x_1 < x_2 < \cdots < x_r \) and \( y_1 < y_2 < \cdots < y_r \). We have for each \( x_i, x_j \) there exist \( a \in f^{-1}(x_i), b \in f^{-1}(x_j) \) such that \( a < b \) iff there exist \( a' \in g^{-1}(y_i), b' \in g^{-1}(y_j) \) such that \( a' \leq b' \).

**Proof.** Suppose that \( fDg \), then there exist \( \lambda \in \text{Poi}_{\leq}^{(n)}(A) \) such that \( fRL \) and \( \lambda LG \). We have \( Kerf = Ker \) and \( Imf = Img \) and so \( |Img| = |Img| \).

Let \( Img = \{x_1, x_2, \ldots, x_r\} \), \( Img = \{y_1, y_2, \ldots, y_r\} \) where \( x_1 < x_2 < \cdots < x_r \) and \( y_1 < y_2 < \cdots < y_r \). Since \( \lambda LG \), we have for all \( y_i \leq y_j \) there exist \( a \in \lambda^{-1}(y_i), b \in \lambda^{-1}(y_j) \) such that \( a \leq b \) iff there exist \( a' \in g^{-1}(y_i), b' \in g^{-1}(y_j) \) such that \( a' \leq b' \). Since \( fRL \), we have \( f^{-1}(x_i) = \lambda^{-1}(y_i) \) for all \( i = 1, \ldots, r \). Therefore, for all \( x_i < x_j \) there exist \( a \in f^{-1}(x_i), b \in f^{-1}(x_j) \) such that \( a < b \) iff there exist \( a' \in g^{-1}(y_i), b' \in g^{-1}(y_j) \) such that \( a' \leq b' \).

Conversely, suppose that the conditions (i) and (ii) are hold. Let \( Img = \{x_1, x_2, \ldots, x_r\} \), \( Img = \{y_1, y_2, \ldots, y_r\} \) where \( x_1 < x_2 < \cdots < x_r \) and \( y_1 < y_2 < \cdots < y_r \). Since \( A^n = \bigcup_{x \in Img} f^{-1}(x) \) is a disjoint union, then for each \( x_i \in Img \), we defined an \( n \)-ary operation \( \lambda : A^n \rightarrow A \) by \( \lambda(a) = y_i \) for all \( a \in f^{-1}(x_i) \).

We have \( Kerf = Ker \) and \( \lambda(f^{-1}(x_i)) = \{y_i\} \leq \{y_j\} = \lambda(f^{-1}(x_j)) \) iff \( x_i \leq x_j \). That is \( fRL \). Since \( |Img| = |Img| \), we have \( Img = Img \). By assumption (ii) we have, for all \( x_i \leq x_j \) there exist \( a \in \lambda^{-1}(y_i) = f^{-1}(x_i), b \in \lambda^{-1}(y_j) = f^{-1}(x_j) \) such that \( a < b \) iff there exist \( a' \in g^{-1}(y_i), b' \in g^{-1}(y_j) \) such that \( a' \leq b' \). That is \( \lambda LG \). Therefore, \( fDg \). \( \square \)

For Green’s relation \( J \), we obtain the following:
Theorem 2.5 Let $f, g \in \text{Pol}_{\leq}^n(A)$ such that $\mathcal{J} = \text{Pol}_{\leq}^n(A) \times \text{Pol}_{\leq}^n(A)$.

Proof. Since the projections $e^n_j \in \text{Pol}_{\leq}^n(A)$ for all $j = 1, 2, ..., n$, then $f = S^n(e^n_1, f, g, ..., g)$ and $g = S^n(e^n_1, g, f, ..., f)$. This means that $f \mathcal{J} g$ for all $f, g \in \text{Pol}_{\leq}^n(A)$. Hence, $\mathcal{J} = \text{Pol}_{\leq}^n(A) \times \text{Pol}_{\leq}^n(A)$. \hfill \Box

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