Green' s Relations on the Menger Algebra of n-ary Ordered Preserving Operations

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1 Introduction

For integer $n \geq 1$, let $O^n(A)$ be the set of all n - ary operations defined on a set A and let $O(A) := \bigcup_{n \geq 1} O^n(A)$ be the set of all operations defined on A .

For $f \in O^n(A)$ and $g_1, \dots, g_n \in O^m(A)$, we define the superposition operation $S_m^n : O^n(A) \times (O^m(A))^n \longrightarrow O^m(A)$ by

$$S_m^n(f, g_1, \dots, g_n)(a_1, \dots, a_m) := f(g_1(a_1, \dots, a_m), \dots, g_n(a_1, \dots, a_m))$$

for all $a_1, \dots, a_m \in A$. If $n = m$, we will write for short S^n instead of S_n^n . For each $n \geq 1$ and each $1 \leq j \leq n$, the n - ary function $e_j^n : A^n \longrightarrow A$ defined on A by $e_j^n(a_1, \dots, a_n) := a_j$ is called the j - th projection mapping of arity n .

In [2] and also [3], an algebra $\mathcal{M} := (M; S^n, e_1, \dots, e_n)$ of type $\tau = (n + 1, 0, \dots, 0)$ is called a unitary Menger algebra of rank n if it satisfies the following axioms:

$$(C1) \quad \tilde{S}^n(X_0, \tilde{S}^n(Y_1, X_1, \dots, X_n), \dots, \tilde{S}^n(Y_n, X_1, \dots, X_n)) \\ \approx \tilde{S}^n(\tilde{S}^n(X_0, Y_1, \dots, Y_n), X_1, \dots, X_n),$$

$$(C2) \quad \tilde{S}^n(\lambda_j, X_1, \dots, X_n) \approx X_j \quad \text{for all } i \leq j \leq n,$$

$$(C3) \quad \tilde{S}^n(X_j, \lambda_1, \dots, \lambda_n) \approx X_j \quad \text{for all } i \leq j \leq n.$$

(Here $\tilde{S}^n$ is an $(n+1)$ -ary operation symbol, $\lambda_1, \dots, \lambda_n$ are nullary operation symbols and $X_0, X_1, \dots, X_n, Y_1, \dots, Y_n$ are variables.)

In [3], an algebra $(M; S^n)$ of type $(n+1)$ which satisfies (C1) is called a Menger algebra of rank n .

Let $(A, \leq)$ be a finite chain. We define the relation $\leq^{(n)}$ on A^n by $(a_1, \dots, a_n) \leq^{(n)} (b_1, \dots, b_n)$ if $a_i \leq b_i$ for all $i = 1, \dots, n$ and define the set

$$Pol_{\leq}^{(n)}(A) := \{f \in O^n(A) \mid f \text{ preserves } \leq\},$$

where f preserves $\leq$ means if $(a_1, \dots, a_n) \leq^{(n)} (b_1, \dots, b_n)$ then $f(a_1, \dots, a_n) \leq f(b_1, \dots, b_n)$. We obtain $(Pol_{\leq}^{(n)}(A); S^n)$ is a Menger algebra of rank n .

In [3], for Menger algebras of rank n , Green's relations were defined as the followings:

Definition 1.1 Let $(M; S^n)$ be a Menger algebra of rank n and let $a, b \in M$.

- (i) $a\mathcal{L}b$ if either $a = b$ or there are elements $s_1, \dots, s_n, t_1, \dots, t_n \in M$ such that $S^n(a, s_1, \dots, s_n) = b$ and $S^n(b, t_1, \dots, t_n) = a$.
- (ii) $a\mathcal{R}b$ if either $a = b$ or there are elements $s, t \in M$ such that $S^n(s, a, \dots, a) = b$ and $S^n(t, b, \dots, b) = a$.
- (iii) $\mathcal{D} = \mathcal{R} \circ \mathcal{L} (= \mathcal{L} \circ \mathcal{R})$.
- (iv) $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$.
- (v) $a\mathcal{J}b$ if either $a = b$ or there are elements $s, s_1, \dots, s_n \in M$ with $a = S^n(s, s_1, \dots, s_n)$ such that at least one of the factors is equal to b and there are elements $t, t_1, \dots, t_n \in M$ with $b = S^n(t, t_1, \dots, t_n)$ such that at least one of the factors is equal to a .

2 The main results

In this section we will study on the Menger algebra $(Pol_{\leq}^{(n)}(A); S^n)$ where A is a finite chain. For simplification we use $\underline{a}$ instead of $(a_1, a_2, \dots, a_n)$. We also

use $\widehat{a}$ for the n -tuple consisting of the same element $a \in A$ i.e. $\widehat{a} = (a, a, \dots, a)$. For each $f \in Pol_{\leq}^{(n)}(A)$, let

$$Ker f := \{(\underline{a}, \underline{b}) \in A^n \times A^n \mid f(\underline{a}) = f(\underline{b})\},$$

and

$$Im f := \{x \in A \mid \exists \underline{a} \in A^n, f(\underline{a}) = x\}.$$

Let $x \in Im f$ we set $f^{-1}(x) = \{\underline{a} \in A^n \mid f(\underline{a}) = x\}$.

For a subset $B \subseteq A^n$ we define the set $f(B) := \{f(\underline{a}) \in A \mid \underline{a} \in B\}$. For any two subsets B, B' of the chain A , we write $B \leq B'$ if $b \leq b'$ for all $b \in B, b' \in B'$.

Let $\Delta_{Im f} = \{\widehat{x} \in A^n \mid x \in Im f\}$.

In a Menger algebra $(Pol_{\leq}^{(n)}(A); S^n)$, we obtain the following theorems.

Theorem 2.1 *Let $f, g \in Pol_{\leq}^{(n)}(A)$. Then $f \mathcal{R} g$ if and only if the following conditions are satisfy*

(i) $Ker f = Ker g$,

(ii) for each $x, y \in Im f$, $x \leq y$ iff $g(f^{-1}(x)) \leq g(f^{-1}(y))$.

Proof. Suppose that $f \mathcal{R} g$. There exist $\alpha, \beta \in Pol_{\leq}^{(n)}(A)$ such that $f = S^n(\alpha, g, \dots, g)$ and $g = S^n(\beta, f, \dots, f)$. Let $(\underline{a}, \underline{b}) \in Ker f$. We have $f(\underline{a}) = f(\underline{b})$ and hence $\widehat{f(\underline{a})} = \widehat{f(\underline{b})}$. Thus $g(\underline{a}) = S^n(\beta, f, \dots, f)(\underline{a}) = \beta(\widehat{f(\underline{a})}) = \beta(\widehat{f(\underline{b})}) = S^n(\beta, f, \dots, f)(\underline{b}) = g(\underline{b})$. That is $Ker f \subseteq Ker g$. Similarly, we get $Ker g \subseteq Ker f$. Thus $Ker f = Ker g$. Let $x, y \in Im f$ such that $x \leq y$. Then there exist $\underline{a}, \underline{b} \in A^n$ such that $f(\underline{a}) = x$ and $f(\underline{b}) = y$ (i.e. $\underline{a} \in f^{-1}(x)$ and $\underline{b} \in f^{-1}(y)$). We have $g(\underline{a}) = S^n(\beta, f, \dots, f)(\underline{a}) = \beta(\widehat{f(\underline{a})}) = \beta(\widehat{x}) \leq \beta(\widehat{y}) = \beta(\widehat{f(\underline{b})}) = S^n(\beta, f, \dots, f)(\underline{b}) = g(\underline{b})$. Therefore, $g(f^{-1}(x)) \leq g(f^{-1}(y))$. In the same way, for all $x, y \in Im f$, if $g(f^{-1}(x)) \leq g(f^{-1}(y))$ we have $x \leq y$.

Conversely, suppose that $Ker f = Ker g$ and for each $x, y \in Im f$, we have $x \leq y$ iff $g(f^{-1}(x)) \leq g(f^{-1}(y))$. For each $x \in Im f$, we choose one element $\underline{a}_x \in f^{-1}(x)$. Define an n -ary operation $\beta : A^n \rightarrow A$ by $\beta(\widehat{x}) = g(\underline{a}_x)$ for all $x \in Im f$ and $\beta(\underline{c}) = c_0$ for all $\underline{c} \in A^n \setminus \Delta_{Im f}$ and c_0 is a fixed element in A . Since $Ker f = Ker g$, we have $f(\underline{a}) = f(\underline{b})$ iff $g(\underline{a}) = g(\underline{b})$. Thus β is well-defined. Let $\underline{a} \in A^n$ such that $f(\underline{a}) = x$. We have $S^n(\beta, f, \dots, f)(\underline{a}) = \beta(\widehat{f(\underline{a})}) = \beta(\widehat{x}) = g(\underline{a}_x) = g(\underline{a})$. That is $g = S^n(\beta, f, \dots, f)$. Let $x, y \in Im f$ such that $x \leq y$. By assumption, we have $g(\underline{a}_x) \leq g(\underline{a}_y)$. Thus $\beta(\widehat{x}) \leq \beta(\widehat{y})$. For all $\underline{a}, \underline{b} \in A^n \setminus \Delta_{Im f}$ such that $\underline{a} \leq \underline{b}$, we have $\beta(\underline{a}) = c_0 = \beta(\underline{b})$ and hence $\beta \in Pol_{\leq}^{(n)}(A)$. Similarly, we can get well-defined an n -ary ordered preserving operation α such that $f = S^n(\alpha, g, \dots, g)$. Therefore, $f \mathcal{R} g$. $\square$

For Green's relation $\mathcal{L}$, we obtain the following result.

Theorem 2.2 *Let $f, g \in Pol_{\leq}^{(n)}(A)$. Then $f \mathcal{L} g$ if and only if the following conditions are satisfy;*

(i) $Imf = Img$.

(ii) *For each $x, y \in Imf$ such that $x \leq y$. Then there exist $\underline{a} \in f^{-1}(x), \underline{b} \in f^{-1}(y)$ such that $\underline{a} \leq \underline{b}$ iff there exist $\underline{a}' \in g^{-1}(x), \underline{b}' \in g^{-1}(y)$ such that $\underline{a}' \leq \underline{b}'$.*

Proof. Suppose that $f \mathcal{L} g$. Then there exist $s_1, \dots, s_n, t_1, \dots, t_n \in Pol_{\leq}^{(n)}(A)$ such that $f = S^n(g, s_1, \dots, s_n)$ and $g = S^n(f, t_1, \dots, t_n)$. Let $x \in Imf$. Then $f(\underline{a}) = x$ for some $\underline{a} \in A^n$. We have $x = f(\underline{a}) = S^n(g, s_1, \dots, s_n)(\underline{a}) = g(s_1(\underline{a}), \dots, s_n(\underline{a})) \in Img$. Thus $Imf \subseteq Img$. Similarly, we have $Img \subseteq Imf$ and hence $Imf = Img$. Let $x, y \in Imf, x \leq y$. If there exists $\underline{a} \in f^{-1}(x), \underline{b} \in f^{-1}(y)$ such that $\underline{a} \leq \underline{b}$, we have $x = f(\underline{a}) = S^n(g, s_1, \dots, s_n)(\underline{a}) = g(s_1(\underline{a}), \dots, s_n(\underline{a})) = g(\underline{a}')$, and $y = f(\underline{b}) = S^n(g, s_1, \dots, s_n)(\underline{b}) = g(s_1(\underline{b}), \dots, s_n(\underline{b})) = g(\underline{b}')$ where $\underline{a}' = (s_1(\underline{a}), \dots, s_n(\underline{a})) \in g^{-1}(x)$ and $\underline{b}' = (s_1(\underline{b}), \dots, s_n(\underline{b})) \in g^{-1}(y)$. Since $s_1, \dots, s_n \in Pol_{\leq}^n(A)$ and $\underline{a} \leq \underline{b}$, we have $s_i(\underline{a}) \leq s_i(\underline{b})$ for all $i = 1, 2, \dots, n$. Thus $\underline{a}' = (s_1(\underline{a}), \dots, s_n(\underline{a})) \leq (s_1(\underline{b}), \dots, s_n(\underline{b})) = \underline{b}'$. Similarly, we get that if there exist $\underline{a}' \in g^{-1}(x), \underline{b}' \in g^{-1}(y)$ such that $\underline{a}' \leq \underline{b}'$ then we have there exist $\underline{a} \in f^{-1}(x), \underline{b} \in f^{-1}(y)$ such that $\underline{a} \leq \underline{b}$.

Conversely, suppose that the conditions (i) and (ii) are hold. For each $x, y \in Imf$ such that $x \leq y$, we choose one element $\underline{a}_x^* \in g^{-1}(x)$ satisfies the following conditions;

–if no elements $\underline{a} \in f^{-1}(x), \underline{b} \in f^{-1}(y)$ such that $\underline{a} \leq \underline{b}$, we choose $\underline{a}_x^* = (a_{x1}^*, \dots, a_{xn}^*) \in g^{-1}(x)$, and $\underline{a}_y^* = (a_{y1}^*, \dots, a_{yn}^*) \in g^{-1}(y)$
–if there exist elements $\underline{a} \in f^{-1}(x), \underline{b} \in f^{-1}(y)$ such that $\underline{a} \leq \underline{b}$, there exist $\underline{a}' \in g^{-1}(x), \underline{b}' \in g^{-1}(y)$ such that $\underline{a}' \leq \underline{b}'$, we choose $\underline{a}_x^* = \underline{a}' = (a_{x1}^*, \dots, a_{xn}^*) \in g^{-1}(x)$ and $\underline{a}_y^* = \underline{b}' = (a_{y1}^*, \dots, a_{yn}^*) \in g^{-1}(y)$. Since $A^n = \bigcup_{x \in Imf} f^{-1}(x)$ is a dis-

joint union, then for all $j = 1, \dots, n$, we define n -ary operations $s_j : A^n \longrightarrow A$ by $s_j(\underline{a}) = a_{xj}^*$ for all $\underline{a} \in f^{-1}(x)$ and for all $x \in Imf$.

Let $\underline{a}, \underline{b} \in A^n$ such that $\underline{a} \leq \underline{b}$. We have

–if $\underline{a}, \underline{b} \in f^{-1}(x)$ then $s_j(\underline{a}) = a_{xj}^* = s_j(\underline{b})$,
–if $\underline{a} \in f^{-1}(x), \underline{b} \in f^{-1}(y)$ such that $\underline{a} \leq \underline{b}$ then $s_j(\underline{a}) = a_{xj}^* \leq a_{yj}^* = s_j(\underline{b})$ and hence $s_j \in Pol_{\leq}^n(A)$. Let $\underline{a} \in A^n$ such that $f(\underline{a}) = x$. We have $S^n(g, s_1, \dots, s_n)(\underline{a}) = g(s_1(\underline{a}), \dots, s_n(\underline{a})) = g(a_{x1}^*, \dots, a_{xn}^*) = g(\underline{a}_x^*) = x = f(\underline{a})$. Thus $f = S^n(g, s_1, \dots, s_n)$. Similarly, we can get well-defined n -ary operations $t_k : A^n \longrightarrow A$ for all $k = 1, \dots, n$ such that $g = S^n(f, t_1, \dots, t_n)$.

Therefore, $f \mathcal{L} g$. □

For Green's relation $\mathcal{H}$, we obtain the following :

Theorem 2.3 *Let $f, g \in \text{Pol}_{\leq}^{(n)}(A)$. Then $f\mathcal{H}g$ if and only if $f = g$.*

Proof. If $f = g$, we have $f\mathcal{R}g$ and $f\mathcal{L}g$. Hence, $f\mathcal{H}g$.

Suppose that $f\mathcal{H}g$, then $f\mathcal{R}g$ and $f\mathcal{L}g$. Thus $\text{Ker}f = \text{Ker}g$ and $\text{Im}f = \text{Im}g$. If $|\text{Im}f| = 1$, we have $f = g$. We assume that $|\text{Im}f| \geq 2$ and $f \neq g$. Then there exist $x \in \text{Im}f$ such that $f^{-1}(x) \neq g^{-1}(x)$. Say that x_0 is a smallest element of $\text{Im}f$ such that $f^{-1}(x_0) \neq g^{-1}(x_0)$. Then there exist $y, z \in \text{Im}f$ such that $x_0 \leq y, x_0 \leq z$ and $g^{-1}(x_0) = f^{-1}(y), f^{-1}(x_0) = g^{-1}(z)$. Since $f\mathcal{R}g$, we have $g(f^{-1}(x_0)) \leq g(f^{-1}(y))$. But $g(f^{-1}(x_0)) = \{z\} \geq \{x_0\} = g(f^{-1}(y))$ gives a contradiction. Thus $f^{-1}(x) = g^{-1}(x)$ for all $x \in \text{Im}f = \text{Im}g$. Hence, $f = g$. $\square$

For Green's relation $\mathcal{D}$, we obtain the following:

Theorem 2.4 *Let $f, g \in \text{Pol}_{\leq}^{(n)}(A)$. Then $f\mathcal{D}g$ if and only if the following conditions are satisfy:*

- (i) $|\text{Im}f| = |\text{Im}g|$.
- (ii) For $\text{Im}f = \{x_1, x_2, \dots, x_r\}, \text{Im}g = \{y_1, y_2, \dots, y_r\}$ where $x_1 < x_2 < \dots < x_r$ and $y_1 < y_2 < \dots < y_r$, we have for each x_i, x_j there exist $\underline{a} \in f^{-1}(x_i), \underline{b} \in f^{-1}(x_j)$ such that $\underline{a} \leq \underline{b}$ iff there exist $\underline{a}' \in g^{-1}(y_i), \underline{b}' \in g^{-1}(y_j)$ such that $\underline{a}' \leq \underline{b}'$.

Proof. Suppose that $f\mathcal{D}g$, then there exist $\lambda \in \text{Pol}_{\leq}^{(n)}(A)$ such that $f\mathcal{R}\lambda$ and $\lambda\mathcal{L}g$. We have $\text{Ker}f = \text{Ker}\lambda$ and $\text{Im}\lambda = \text{Im}g$ and so $|\text{Im}f| = |\text{Im}\lambda| = |\text{Im}g|$. Let $\text{Im}f = \{x_1, x_2, \dots, x_r\}, \text{Im}g = \{y_1, y_2, \dots, y_r\}$ where $x_1 < x_2 < \dots < x_r$ and $y_1 < y_2 < \dots < y_r$. Since $\lambda\mathcal{L}g$, we have for all $y_i \leq y_j$ there exist $\underline{a} \in \lambda^{-1}(y_i), \underline{b} \in \lambda^{-1}(y_j)$ such that $\underline{a} \leq \underline{b}$ iff there exist $\underline{a}' \in g^{-1}(y_i), \underline{b}' \in g^{-1}(y_j)$ such that $\underline{a}' \leq \underline{b}'$. Since $f\mathcal{R}\lambda$, we have $f^{-1}(x_i) = \lambda^{-1}(y_i)$ for all $i = 1, \dots, r$. Therefore, for all $x_i \leq x_j$ there exist $\underline{a} \in f^{-1}(x_i), \underline{b} \in f^{-1}(x_j)$ such that $\underline{a} \leq \underline{b}$ iff there exist $\underline{a}' \in g^{-1}(y_i), \underline{b}' \in g^{-1}(y_j)$ such that $\underline{a}' \leq \underline{b}'$.

Conversely, suppose that the conditions (i) and (ii) are hold. Let $\text{Im}f = \{x_1, x_2, \dots, x_r\}, \text{Im}g = \{y_1, y_2, \dots, y_r\}$ where $x_1 < x_2 < \dots < x_r$ and $y_1 < y_2 < \dots < y_r$. Since $A^n = \bigcup_{x \in \text{Im}f} f^{-1}(x)$ is a disjoint union, then for each $x_i \in \text{Im}f$,

we defined an n -ary operation $\lambda : A^n \rightarrow A$ by $\lambda(\underline{a}) = y_i$ for all $\underline{a} \in f^{-1}(x_i)$. We have $\text{Ker}f = \text{Ker}\lambda$ and $\lambda(f^{-1}(x_i)) = \{y_i\} \leq \{y_j\} = \lambda(f^{-1}(x_j))$ iff $x_i \leq x_j$. That is $f\mathcal{R}\lambda$. Since $|\text{Im}f| = |\text{Im}g|$, we have $\text{Im}g = \text{Im}\lambda$. By assumption (ii) we have, for all $x_i \leq x_j$ there exist $\underline{a} \in \lambda^{-1}(y_i) = f^{-1}(x_i), \underline{b} \in \lambda^{-1}(y_j) = f^{-1}(x_j)$ such that $\underline{a} \leq \underline{b}$ iff there exist $\underline{a}' \in g^{-1}(y_i), \underline{b}' \in g^{-1}(y_j)$ such that $\underline{a}' \leq \underline{b}'$. That is $\lambda\mathcal{L}g$. Therefore, $f\mathcal{D}g$. $\square$

For Green's relation $\mathcal{J}$, we obtain the following:

Theorem 2.5 *Let $f, g \in Pol_{\leq}^{(n)}(A)$ such that $\mathcal{J} = Pol_{\leq}^{(n)}(A) \times Pol_{\leq}^{(n)}(A)$.*

Proof. Since the projections $e_j^n \in Pol_{\leq}^{(n)}(A)$ for all $j = 1, 2, \dots, n$, then $f = S^n(e_1^n, f, g, \dots, g)$ and $g = S^n(e_1^n, g, f, \dots, f)$. This means that $f\mathcal{J}g$ for all $f, g \in Pol_{\leq}^{(n)}(A)$. Hence, $\mathcal{J} = Pol_{\leq}^{(n)}(A) \times Pol_{\leq}^{(n)}(A)$. $\square$

Acknowledgment

This research is supported by the Centre of Excellence in Mathematics, the commission on Higher Education, Ministry of Education, Thailand.

References

- [1] J.M. Howie, *Fundamentals of Semigroup Theory*, Oxford Science Publication, Clarendon Press, Oxford 1995.
- [2] B. Schein and V.S. Trochimenko, *Algebras of multiplace functions*, Semigroup Forum Vol. 17 (1979). 1-64.
- [3] K. Denecke, P. Jampachon, *Regular Elements and Green's relations in Menger Algebras of terms*, Discussiones Mathematicae, General Algebra and Applications, 26(2006), 85-109.

Received: August 5, 2013