On a Formula of Liouville Type
for the Quadratic Form \( x^2 + 2y^2 + 2z^2 + 4w^2 \)

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Abstract

We generalize the factorization of the classical Lipschitz quaternions to the Lipschitz type quaternions associated with the quaternary quadratic form \( x^2 + 2y^2 + 2z^2 + 4w^2 \). We are able to prove a unique factorization theorem under a suitable model for the Lipschitz type quaternions in question. As a consequence, we obtain a simple and conceptual proof for the number of representations of a positive integer in terms of the above quadratic form, which was first historically stated by Liouville.

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1 Introduction

1.1 A Brief History

Since the proof of Lagrange’s Theorem of Four Squares ([6]), there has been extensive study of other quadratic forms. A positive definite quadratic form is
said to be universal if it represents all positive integers. The study of universality of a given quadratic form and the number of representations for a positive integer in terms of the quadratic form are of particular interest. It was Jacobi who first proved the formula of the number of representations of a positive integer as a sum of four integer squares (see the references of [3]). Among the various different proofs of Jacobi’s result, there were the quaternionic proofs given by Lipschitz ([9]) and Hurwitz ([7]). Based on analogues of Hurwitz quaternions, J. Deutsch gave a quaternionic proof for the universality of some quadratic forms including \( x^2 + 2y^2 + 2z^2 + 4w^2 \) ([4]), and a quaternionic proof for the number of representations of the quadratic form \( x^2 + y^2 + 2z^2 + 2w^2 \) ([5]). In [12], the author presented a proof for Jacobi’s result using Lipschitz quaternions, which is simpler than Lipschitz’s original proof; then this was generalized to the Lipschitz type quaternions associated with the quadratic forms \( x^2 + y^2 + 2z^2 + 2w^2 \) and \( x^2 + y^2 + 3z^2 + 3w^2 \) ([13]). Following the same train of thoughts, we will present a quaternionic proof for the number of representations in terms of the quadratic form \( x^2 + 2y^2 + 2z^2 + 4w^2 \). We remark that this quadratic form has been studied since the days of Liouville ([8], [10], [11], [2], and [1]) and it is one of the 54 universal quaternary quadratic forms mentioned by Ramanujan ([14]).

1.2 Basic Definitions and Notations

The set of Hamilton quaternions \( \mathbb{H}(\mathbb{R}) \) is an associative algebra, which additively is a vector space over \( \mathbb{R} \) with basis 1, \( i, j \) and \( k \), subject to the multiplication rules \( i^2 = j^2 = k^2 = -1, ij = -ji = k \). We define the set \( \mathbb{L} \) of Lipschitz type quaternions associated with \( x^2 + 2y^2 + 2z^2 + 4w^2 \) as

\[
\mathbb{L} = \{ x + y\sqrt{2}i + z\sqrt{2}j + 2wk \mid x, y, z, w \in \mathbb{Z} \}.
\]

Since \( \mathbb{L} \) is a subset of \( \mathbb{H}(\mathbb{R}) \), it retains all the rules for addition and multiplication, and the definition of conjugate, norm, and trace. We adopt a short hand notation \( \left[ x, y, z, w \right] \) for an element \( x + y\sqrt{2}i + z\sqrt{2}j + 2wk \). For \( Q = [x, y, z, w] \), we denote the conjugate of \( Q \) by \( \bar{Q} = [x, -y, -z, -w] \). Similarly, trace and norm of \( Q \) are denoted \( \text{Tr}(Q) = Q + \bar{Q} = 2x \) (always even), \( \text{Nm}(Q) = Q\bar{Q} = \bar{Q}Q = x^2 + 2y^2 + 2z^2 + 4w^2 \) and one has \( Q_1Q_2 = \bar{Q}_2 \bar{Q}_1 \), which implies \( \text{Nm}(Q_1Q_2) = \text{Nm}(Q_1)\text{Nm}(Q_2) \). An element \( \epsilon \) is a unit of \( \mathbb{L} \) if there exists \( \eta \in \mathbb{L} \) such that \( \epsilon\eta = 1 \). It is easy to check that \( \pm 1 \) are the only units of \( \mathbb{L} \), which are the only elements of norm 1. Since multiplication of elements will be required in the proof, we record the formula for the product of two elements \( [a, b, c, d] \) and \( [a', b', c', d'] \) in \( \mathbb{L} \):

\[
[a, b, c, d][a', b', c', d'] = [aa' - 2bb' - 2cc' - 4dd', ab' + a'b + 2cd' - 2c'd, ac' + a'c - 2bd' + 2b'd, ad' + a'd + bc' - b'c].
\]
Definition 1.1 Let \( Q = [x, y, z, w] \) be in \( \mathbb{L} \). We say \( Q \) is primitive if
\[
\gcd(x, y, z, w) = 1.
\]
We say \( Q \) is \( p \)-primitive if \( p \nmid \gcd(x, y, z, w) \).

Definition 1.2 \( Q \in \mathbb{L} \) is called a prime quaternion if \( \text{Nm}(Q) = p \) is a rational prime.

Definition 1.3 Let \( p \) be a rational prime. We say \( Q \in \mathbb{L} \) is \( p \)-pure if \( \text{Nm}(Q) = p^n \), for some integer \( n \geq 1 \).

Definition 1.4 Let \( Q_1, Q_2 \in \mathbb{L} \). We say that \( Q_1 \) and \( Q_2 \) are equivalent if \( Q_2 = \epsilon Q_1 \) for some unit \( \epsilon \in \mathbb{L} \).

1.3 Main Ideas of the Proof

The formula \( \text{Nm}([x, y, z, w]) = x^2 + 2y^2 + 2z^2 + 4w^2 \) indicates that the number of representations of \( n \) in terms of the quadratic form \( x^2 + 2y^2 + 2z^2 + 4w^2 \) equals the number of quaternions in \( \mathbb{L} \) of norm \( n \). This motivates the study of factorization in \( \mathbb{L} \). The factorization of \( 2 \)-pure \( Q \in \mathbb{L} \) into factors of prime quaternions may not always work, hence we consider only \( 2 \)-pure primitive quaternions as part of the building blocks in the factorization. The proof of the representation formula for the quadratic form \( x^2 + 2y^2 + 2z^2 + 4w^2 \) is based on the unique factorization stated in Theorem 2.16, where we build a factorization by a unit, a representative of \( 2 \)-pure primitive quaternions, and the product of representatives of quaternions of odd prime norm. Since we know how to count the number of equivalence classes of \( 2 \)-pure quaternions, and the number of equivalence classes of \( p \)-pure quaternions (for \( p > 2 \)) based on the Correspondence Theorem (Theorem 2.14), the representation formula (Theorem 3.3) follows easily.

2 Factorization of Lipschitz Type Quaternions

2.1 Primitive Factors of 2-power Norm

Definition 2.1 Let \( m \) be an integer and \( Q \in \mathbb{L} \). We denote \( m|Q \) if \( Q = mR \) for some \( R \in \mathbb{L} \).

Lemma 2.2 Let \( Q \in \mathbb{L} \). If \( 2^5|\text{Nm}(Q) \), then \( 2|Q \).

Proof. Write \( Q = [x, y, z, w] \). If \( 2^5|x^2 + 2y^2 + 2z^2 + 4w^2 = \text{Nm}(Q) \), then \( x \) must be even. Writing \( x = 2x' \), we have
\[
2^4|2x'^2 + y^2 + z^2 + 2w^2.
\]
Now we make the following

**Claim.** If \( 2^3 | 2s^2 + 2t^2 + u^2 + v^2 \), then \( u, v \) must be both even.

**Proof of Claim.** Clearly \( u, v \) have the same parity. If they were both odd, then \( u^2 \equiv v^2 \equiv 1 \pmod{8} \). This would imply \( 2s^2 + 2t^2 + u^2 + v^2 \equiv 2, 4, \) or \( 6 \pmod{8} \), a contradiction.

By the above claim, we see that \( y \) and \( z \) are both even. Writing \( y = 2y' \) and \( z = 2z' \), we then have

\[
2^3 | x'^2 + 2y'^2 + 2z'^2 + w^2.
\]

But applying the claim one more time, we have \( x' \) and \( w \) are both even, hence \( w = 2w' \), and therefore \( [x, y, z, w] = 2[x', y', z', w'] \), i.e. \( 2|Q \). \( \square \)

**Lemma 2.3** In \( \mathbb{L} \), there are 2 equivalence classes of primitive quaternions of norm 2, 3 equivalence classes of primitive quaternions of norm 4, 10 equivalence classes of primitive quaternions of norm 8, and 8 equivalence classes of primitive quaternions of norm 16.

**Proof.** By direct computation, we list all the representatives of the equivalence classes of primitive quaternions of the required norms (since the only units in \( \mathbb{L} \) are \( \pm 1 \), every equivalence class consists of two elements):

(a) Norm 2: \([0, 1, 0, 0], [0, 0, 1, 0] \) (2 classes).

(b) Norm 4: \([0, 0, 0, 1], [0, 1, \pm 1, 0] \) (3 classes).

(c) Norm 8: \([2, 0, 0, \pm 1], [0, 1, \pm 1, \pm 1], [2, \pm 1, \pm 1, 0] \) (10 classes).

(d) Norm 16: \([2, 0, \pm 2, \pm 1], [2, \pm 2, 0, \pm 1] \) (8 classes). \( \square \)

**Corollary 2.4** In \( \mathbb{L} \), there is one equivalence class of quaternions of norm 1. There are 2 equivalence classes of quaternions of norm 2, and 4 equivalence classes of quaternions of norm 4. Furthermore, there are 12 equivalence classes of quaternions of norm \( 2^n \) for \( n \geq 3 \).

**Proof.** Clearly the quaternions in \( \mathbb{L} \) of norm 1 is represented by the unique class \( Q_0 = 1 = [1, 0, 0, 0] \). In general, let \( Q_1, Q_2, Q_3, \) and \( Q_4 \) be any representative of quaternions in \( \mathbb{L} \) of norm 2, 4, 8, and 16. We refer to (a), (b), (c) and (d) in the proof of Lemma 2.3. By (a), there are 2 equivalence classes of norm 2. Any quaternion in \( \mathbb{L} \) of norm 4 is equivalent to \( 2Q_0 \) or \( Q_2 \), so by (b) there are
in total $1 + 3 = 4$ equivalence classes of quaternions of norm 4. Furthermore, since any quaternion in $\mathbb{L}$ of norm $2^{2k+1}$ with $k \geq 1$ is equivalent to $2^k Q_1$, or $2^{k-1} Q_3$, it follows from (a) and (c) that there are $2 + 10 = 12$ equivalence classes of quaternions of norm $2^{2k+1}$ with $k \geq 1$. Finally, any quaternion in $\mathbb{L}$ of norm $2^{2k}$ with $k \geq 2$ is equivalent to $2^k Q_0$, or $2^{k-1} Q_3$, or $2^{k-2} Q_4$, so by (b) and (d) there are in total $1 + 3 + 8 = 12$ equivalence classes of quaternions of norm $2^{2k}$ with $k \geq 2$.

\[ \square \]

**Definition 2.5** We say $Q \in \mathbb{L}$ is even if $2 \mid \text{Nm}(Q)$, otherwise it is odd.

**Lemma 2.6** Let $Q \in \mathbb{L}$ be even. Then $Q$ can be factored as $Q = Q_0 Q_1$, where $Q_0$ is 2-pure and $Q_1$ is odd.

*Proof.* Clearly we just need to handle the case when $Q$ is primitive, which we assume. Let $\text{Nm}(Q) = 2^r m$, where $m$ is odd. Then by Lemma 2.2, $1 \leq r \leq 4$. If suffices to check the result for the congruence classes $Q'$ mod $2^r$ such that $2^r \mid \text{Nm}(Q')$. More precisely, if $Q = Q' + 2^r R$, then one has $2^r \mid \text{Nm}(Q')$, and if $Q' = Q_0 Q'$ with $\text{Nm}(Q_0) = 2^r$, then $Q = Q_0 Q' + 2^r R = Q_0 (Q' + Q_0) R$, so we may take $Q_1 = (Q' + Q_0) R$ to conclude. The computation by hand may be tedious, but it’s easily done by a computer algebra system. \[ \square \]

### 2.2 Factorization of Factors of Odd Norm

**Lemma 2.7** Let $p > 2$ be a rational prime. Let $Q \in \mathbb{L}$ be $p$-primitive such that $p \mid \text{Nm}(Q)$. Then there exists $Z \in \mathbb{L}$ with $p \nmid \text{Nm}(Z)$ such that $X := ZQ$ is of the form whose $k$ component is zero mod $p$.

*Proof.* Let $Q = [s, u, v, w]$ be as given. By assumption, $s^2 + 2u^2 + 2v^2 + 4w^2 = 0 \mod p$. If $w = 0 \mod p$, we may take $Z = [1, 0, 0, 0]$ and the result is clear. So we may assume that $w \neq 0 \mod p$.

**Claim.** At least one of the following expressions is nonzero mod $p$: $u^2 + v^2$, $s^2 + 2u^2$, and $s^2 + 2v^2$.

*Proof of Claim.* If all of these are zero mod $p$, then we have $2(u^2 + v^2) + (s^2 + 2u^2) + (s^2 + 2v^2) = 0 \mod p$, which implies $2(s^2 + 2u^2 + 2v^2 + 4w^2) - 8w^2 = 0 \mod p$, i.e. $8w^2 = 0 \mod p$, which is impossible by our assumption. \[ \square \]

Now we construct a $Z$ depending on each of the above situations:
If \( u^2 + v^2 \neq 0 \mod p \), we let \( Z = [0, u, v, 0] \). Then
\[
X = [0, u, v, 0][s, u, v, w] = [-2u^2 - 2v^2, su + 2vw, sv - 2uw, 0]
\]
with \( \text{Nm}(Z) = 2(u^2 + v^2) \).

If \( s^2 + 2u^2 \neq 0 \mod p \), we let \( Z = [0, 0, s, u] \). Then
\[
X = [0, 0, s, u][s, u, v, w] = [-2sv - 4uw, 2sw - 2uv, s^2 + 2u^2, 0]
\]
with \( \text{Nm}(Z) = 2(s^2 + 2u^2) \).

If \( s^2 + 2v^2 \neq 0 \mod p \), we let \( Z = [0, s, 0, -v] \). Then
\[
X = [0, s, 0, -v][s, u, v, w] = [-2su + 4vw, s^2 + 2v^2, -2sw - 2uv, 0]
\]
with \( \text{Nm}(Z) = 2(s^2 + 2v^2) \).
\[ \square \]

**Corollary 2.8** With assumptions as in Lemma 2.7, there exists \( Z \in \mathbb{L} \) with \( p \nmid \text{Norm}(Z) \) such that \( ZQ \) is congruent to some \( X' \in \mathbb{L} \) mod \( p \) such that \( \text{Nm}(X') = mp \) with \( m < p \).

**Proof.** Case 1. If in Lemma 2.7, \( ZQ \equiv X' = [x, y, 0, 0] \mod p \), then we may choose \( x, y \) such that \( |x| \leq \frac{p-1}{2} \) and \( |y| \leq \frac{p-1}{2} \). Clearly \( p|\text{Nm}(X') \) and \( \text{Nm}(X') < \left( \frac{p-1}{2} \right)^2 + 2 \left( \frac{p-1}{2} \right)^2 < p^2 \). Therefore \( ZQ \) is congruent to \( X' \) such that \( \text{Nm}(X') = mp \) with \( m < p \).

Case 2. If \( ZQ = X = [x, y, z, 0] \mod p \), where \( z \neq 0 \mod p \). Letting \( d \) be a multiplicative inverse of \( z \mod p \), we have \( (dZ)Q \equiv X' = [x', y', 1, 0] \mod p \). Choosing congruence classes of \( x', y' \) with \( |x'| \leq \frac{p-1}{2} \) and \( |y'| \leq \frac{p-1}{2} \), we then have \( p|\text{Nm}(X') \) and \( \text{Nm}(X') \leq \left( \frac{p-1}{2} \right)^2 + 2 \left( \frac{p-1}{2} \right)^2 < p^2 \). Rewriting \( dZ \) with \( Z \), we have again \( ZQ \equiv X' \mod p \), where \( \text{Nm}(X') = mp \) with \( m < p \).
\[ \square \]

**Corollary 2.9** Let \( P \in \mathbb{L} \) be a prime quaternion of norm \( p > 2 \). Then there exists \( Z \in \mathbb{L} \) with \( p \nmid \text{Nm}(Z) \) such that \( X := ZP \) has zero \( k \) component, where every prime factor of \( \text{Nm}(Z) \) is less than \( p \).

**Proof.** Clear from the proof of Lemma 2.7.
\[ \square \]

**Lemma 2.10** Let \( p > 2 \) be a prime. Let \( X \) be a \( p \)-primitive quaternion of norm divisible by \( p \). Then there exists a prime quaternion \( P \) (resp. \( P' \)) of norm \( p \) and a quaternion \( Y \) (resp. \( Y' \)) such that \( X = YP \) (resp. \( X = P'Y' \)).
Proof. This is done using Lemma 2.6, Corollary 2.8 and by induction. We refer to Lemma 2.15 (or Lemma 2.16) of [13] for analogous arguments.

Lemma 2.11 Let $p > 2$ be a rational prime. Let $X$ be $p$-primitive and $X = YP = Y'P'$, where $\text{Nm}(P) = \text{Nm}(P') = p$. Then $P$ and $P'$ are equivalent, i.e. $P' = \epsilon P$, where $\epsilon = \pm 1$.

Proof. This is the same as Lemma 2.18 of [13].

Definition 2.12 Let $p > 2$. Let $X \in \mathbb{L}$ be $p$-primitive such that $p \mid \text{Nm}(X)$. We define $\text{GCD}_R(X,p)$ to be the equivalence class of $P$, denoted $[P]$, if $X$ can be factored as $X = YP$. By Lemma 2.10 and Lemma 2.11, this is well-defined.

2.3 Correspondence Theorem

Lemma 2.13 Let $p \neq 2$ be a prime. There are precisely $p + 1$ projective solutions for

$$x^2 + 2y^2 + 2z^2 = 0 \text{ over } \mathbb{F}_p.$$  

We will lift the solutions to $\mathbb{Z}$ and represent them in the form of $p$-primitive quaternion $X = [x,y,z,0]$ (i.e. $x + y\sqrt{2}i + z\sqrt{2}j$, $x, y, z \in \mathbb{Z}$, not all zero mod $p$).

Proof. Similar to Lemma 2.20 of [13].

Let $p > 2$. Let $S$ be the set of projective solutions of $x^2 + 2y^2 + 2z^2 = 0$ over $\mathbb{F}_p$ and $T$ be the set of equivalence classes of prime quaternions of norm $p$. We define a mapping $\Phi : S \to T$ by $\Phi(\xi) = \text{GCD}_R(X,p)$, where $X$ is any lifting of $\xi \in S$. For any two liftings $X_1$ and $X_2$ of $\xi$, it is clear that there exists $d$ with $p \nmid d$ such that $X_2 \equiv dX_1 \mod p$, i.e. $X_2 = dX_1 + pR$. Now if $X_1 = YP$, then $X_2 = dX_1 + pR = dYP + R\overline{P}P = (dY + R\overline{P})P$, thus $\text{GCD}_R(X_1,p) = \text{GCD}_R(X_2,p) = [P]$. Therefore $\Phi$ is well-defined.

Theorem 2.14 (Correspondence Theorem) Let $p > 2$ and $\Phi : S \to T$ be defined above. Then $\Phi$ is a bijection.

Proof. Let $[P] \in T$ and $X = ZP$ be given by Corollary 2.9. If $\xi$ be a solution associated with $X$, then clearly $\Phi(\xi) = \text{GCD}_R(X,p) = [P]$. This shows that $\Phi$ is surjective.

We will show that the $\Phi$ is also injective. For this, let $\Phi(\xi_1) = \Phi(\xi_2) = [P]$. Let $X_1 = Z_1P = [a,b,c,0]$ and $X_2 = Z_2P = [a',b',c',0]$ be the liftings of two points $\xi_1$ and $\xi_2$. Then

$$X_1X_2 = Z_1P = Z_2 = 0 \mod p.$$
In terms of components, this says
\[ [a, b, c, 0][a', -b', -c', 0] = 0 \] over \( \mathbb{F}_p \),
i.e. \([aa' + 2bb' + 2cc', a'b - ab', a'c - ac', b'c - bc'] = [0, 0, 0, 0]\), whence \( X_1 \) and \( X_2 \) are proportional. It follows that \( \xi_1 = \xi_2 \) and therefore \( \Phi \) is injective.

Corollary 2.15 Let \( p > 2 \). Then there are \( p + 1 \) equivalence classes of quaternions in \( \mathbb{L} \) of norm \( p \).

Proof. Immediate from Lemma 2.13 and Theorem 2.14.

2.4 Unique Factorization

Let \( N \) be a positive integer and \( N = 2^{s_0}p_1^{s_1} \cdots p_k^{s_k} \) be a standard factorization, where \( s_i \geq 0 \) (if \( s_i = 0 \), we agree that the factor is void) and \( p_1 < \cdots < p_k \).

We call this the standard model. Now let \( Q \) be a primitive quaternion of norm \( N \). By successively applying Lemma 2.10, we obtain a factorization \( Q = Q_0Q_1 \cdots Q_k \) under the standard model, where \( \text{Nm}(Q_0) = 2^{s_0} \), \( \text{Nm}(Q_1) = p_1^{s_1} \) and each \( Q_i \) is a product of \( s_i \)'s prime quaternions of norm \( p_i \). By choosing a representative for each equivalence class of 2-pure primitive quaternions, and for each equivalence class of prime quaternions of norm \( p \) for \( p > 2 \), we arrive at the following

Theorem 2.16 (a) Any primitive quaternion \( Q \) of norm \( 2^{s_0}p_1^{s_1} \cdots p_k^{s_k} \) can be factored uniquely under the standard model, namely
\[ Q = \epsilon Q_0Q_1 \cdots Q_k, \]
where \( \epsilon \) is a unit, \( Q_0 = 1 \) (if \( s_0 = 0 \) ) or one of the representatives of the primitive quaternions of norm \( 2^{s_0} \), and for \( 1 \leq i \leq k \), \( Q_i \) is a product of \( s_i \)'s prime quaternions from the set of representatives of prime quaternions of norm \( p_i \).

(b) Any non-primitive quaternion \( Q' = mQ \) (with \( m > 1 \) and \( Q \) primitive of norm given as above) can be factored uniquely in the form
\[ Q' = \epsilon(2^{t_0}Q_0)(p_1^{t_1}Q_1) \cdots (p_k^{t_k}Q_k) \]
under the model \( 2^{t_0}p_1^{t_1} \cdots p_k^{t_k} \) (still called standard), where \( r_i = 2t_i + s_i, 0 \leq i \leq k \) and \( m = 2^{t_0}p_1^{t_1} \cdots p_k^{t_k} \), and \( Q_i 0 \leq i \leq k \) is as described in (a).

Proof. The result (b) follows easily from (a). The existence of factorization was explained above. The uniqueness follows from the well-definedness of \( \text{GCD}_R(X, p) \) (see Definition 2.12).

□
3 Number of Representations of \( n \) in terms of the quadratic form \( x^2 + 2y^2 + 2z^2 + 4w^2 \)

**Lemma 3.1** Let \( p > 2 \) be a rational prime. There are precisely \( p^{n-1}(p + 1) \) equivalence classes of primitive quaternions in \( \mathbb{L} \) of norm \( p^n \), \( n \geq 1 \).

**Proof.** See Corollary 2.25 of [13].

**Lemma 3.2** Let \( p > 2 \) be a rational prime. There are precisely \( \sigma(p^n) \) equivalence classes of quaternions in \( \mathbb{L} \) of norm \( p^n \), \( n \geq 0 \).

**Proof.** See Lemma 2.26 of [13].

**Theorem 3.3** Let \( n = 2^\alpha N \). Then the number \( S \) of representations of \( n \) in terms of the quadratic form \( x^2 + 2y^2 + 2z^2 + 4w^2 \) is given by

\[
S = \begin{cases}
2\sigma(N) & \text{if } \alpha = 0, \\
4\sigma(N) & \text{if } \alpha = 1, \\
8\sigma(N) & \text{if } \alpha = 2, \\
24\sigma(N) & \text{if } \alpha \geq 3.
\end{cases}
\]

**Proof.** As mentioned in 1.3 of Introduction, we need to count the number of quaternions of norm \( n \); by Theorem 2.16, this equals the number of factorizations under the standard model which is the standard factorization of \( n \). We recall that there are 2 units in \( \mathbb{L} \). Then the result follows by the Fundamental Counting Principle from (b) of Theorem 2.16, Corollary 2.4 and Lemma 3.2.

\[ \square \]

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