ϕ-Primary Subtractive Ideals in Semiring

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Abstract

Let $R$ be a commutative semiring with identity $1 \neq 0$. In this paper, we define ϕ-primary ideals in $R$ and prove that for subtractive ideals $I$ and $\varphi(I)$ of $R$, $I$ is a ϕ-primary ideal if and only if for ideals $A$ and $B$ of $R$, $AB \subseteq I - \varphi(I)$ implies that $A \subseteq I$ or $B \subseteq \sqrt{I}$.

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1 Introduction

Let $R$ be a set with "+", "." as binary operations on $R$ named addition and multiplication, respectively. Then $(R, +, .)$ is called asemiring if the following conditions are satisfied:
1) $(R, +)$ is a commutative semigroup.
2) $(R, .)$ is a semigroup.
3) Both operations are connected by the distributive laws $a.(b+c) = a.b + a.c$ and $(a+b).c = ac + bc$ for all $a, b, c \in R$.

A subset $H$ of asemiring $R$ is called asubsemiring provided that $H$ is a semiring under both binary operations on $R$. A non-empty subset $I$ of a semiring $R$ will be called an ideal if $a, b \in I$ and $r \in R$ imply that $a + b \in I$,,
ra ∈ I and ar ∈ I. An ideal I of a semi ring R is called subtractive if a ∈ I, a + b ∈ I and b ∈ R then b ∈ I. An ideal I of a semiring R is called primary, if \(ab \in I\) where \(a, b \in R\) then \(a \in I\) or \(b^n \in I\), for some positive integer \(n\). Easily an ideal of a semiring R is primary if and only if for ideals A and B of \(R, AB \subseteq I\) implies that \(A \subseteq I\) or \(B \subseteq \sqrt{I}\).

Let \(\varphi : I(R) \to I(R) \cup \{\emptyset\}\) be a function where \(I(R)\) is the set of ideals of \(R\). We call a proper ideal \(I\) of \(R\) a \(\varphi\)-primary ideal if \(a, b \in R\) with \(ab \in I - \varphi(I)\) implies \(a \in I\) or \(b^n \in I\) for some positive integer \(n\). Clearly every primary ideal is a \(\varphi\)-primary ideal, but the inverse case is not true. For example, let \(R = \mathbb{Z}_6\). Then 0 is a \(\varphi\)-primary ideal where \(\varphi(I) = 0\). But is not a primary ideal.

**Definition 1.** Let \(I(R)\) be the set of ideals of \(R\) and \(I^*(R)\) the set of proper ideals of \(R\) and \(\varphi : I(R) \to I(R) \cup \{\emptyset\}\) a map. The proper ideal \(I\) of \(R\) is called a \(\varphi\)-primary ideal if for all \(a, b \in R\), with \(ab \in I - \varphi(I)\) implies \(a \in I\) or \(b^n \in I\) for some integer \(n\).

**Note:** Since \(I - \varphi(I) = I - (I \cap \varphi(I))\), there is no loss of generality in assuming that \(\varphi(I) \subseteq I\).

**Lemma 2.** Every primary ideal is \(\varphi\)-primary ideal.

**Lemma 3.** Every \(\varphi\)-prime is \(\varphi\)-primary ideal.

**Lemma 4.** If \(I\) is a \(\varphi\)-primary ideal of \(R\) and \(\varphi(I)\) is a primary ideal, then \(I\) is a primary ideal of \(R\).

**Proof:** Let \(a, b \in R\). If \(ab \notin \varphi(I)\), since \(I\) is a \(\varphi\)-primary ideal, then \(a \in I\) or \(b^n \in I\). And if \(ab \in \varphi(I)\) then \(a \in \varphi(I) \subseteq I\) or \(b^n \in \varphi(I) \subseteq I\).

**Definition 5.** Given two function \(\psi_1, \psi_2 : I(R) \to I(R) \cup \{\emptyset\}\). We define \(\psi_1 \leq \psi_2\) if \(\psi_1(J) \subseteq \psi_2(J)\), for each \(J \in I(R)\).

We maintain notation and terminology used in following Example for the remainder of the article.

**Example:** Let \(R\) be a commutative semiring. Define the following functions \(\varphi_\alpha : I(R) \to I(R) \cup \{\emptyset\}\) and the corresponding \(\varphi_\alpha\)-primary ideals:

- \(\varphi_0 : \varphi(J) = \emptyset\) primary ideal
- \(\varphi_0 : \varphi(J) = 0\) weakly primary ideal
- \(\varphi_2 : \varphi(J) = J^2\) almost primary ideal
- \(\varphi_n : \varphi(J) = J^n\) \(n\) - almost primary ideal
- \(\varphi_\omega : \varphi(J) = \bigcap J^n\) \(\omega\) - primary ideal
- \(\varphi_1 : \varphi(J) = J\) any ideal
Observe that $\varphi_0 \leq \varphi_0 \leq \varphi_\omega \leq \cdots \leq \varphi_{n+1} \leq \varphi_n \leq \cdots \leq \varphi_2 \leq \varphi_1$. Let $A$ be an ideal of $R$.

**Proposition 6.** (1) Let $R$ be a commutative semiring and $J$ a proper ideal of $R$. Let $\psi_1, \psi_2 : I(R) \to I(R) \cup \{\emptyset\}$ be functions with $\psi_1 \leq \psi_2$. Then, if $J$ is $\psi_1$-primary ideal, then $J$ is $\psi_2$-primary ideal.

**Proof:** Suppose that $ab \in J - \psi_2(J)$ where; $a, b \in R$. Since $\psi_1(J) \subseteq \psi_2(J)$, hence we have $ab \in J - \psi_1(J)$ and therefore $a \in J$ or $b \in J^n$ for some positive integer $n$. Thus $J$ is $\psi_2$-primary.

Let $R$ be a semiring and let $I$ be a proper ideal of $R$. our mean of $R_{n \times n}$ is the set of Matrixes with entiers of $R$, also $\varphi_{n \times n}(I_{n \times n})$ is the set of Matrixes in $\varphi(I)$.

**Theorem 7.** Let $R$ be a commutative semiring and $I$ a proper ideal of $R$. If $I_{n \times n}$ is a $\varphi_{n \times n}$-primary ideal of $R_{n \times n}$, then $I$ is a $\varphi$-primary ideal of $R$.

**Proof:** Let $ab \in I - \varphi(I)$ where $a, b \in R$. Then $aE_{11}bE_{11} \in I_{n \times n} - \varphi_{n \times n}(I_{n \times n})$. Hence $aE_{11} \in I_{n \times n}$ or $bE_{11} = (bE_{11})^m \in I_{n \times n}$ for some integer $n$. Now $a \in I$ or $b^n \in I$.

**Lemma 8.** Let $I$ be a $\varphi$-primary subtractive ideal of a semiring $R$. Then $I^2 \subseteq \varphi(I)$ or $I$ is a primary ideal.

**Proof:** Suppose that $I^2 \not\subseteq \varphi(I)$. Let $ab \in I$ where $a, b \in R$. If $ab \not\in \varphi(I)$, then $a \in I$ or $b^n \in I$ for some positive integer $n$. So assume that $ab \in \varphi(I)$. If $aI \not\subseteq \varphi(I)$, then there exist $x \in I$ such that $ax \not\in \varphi(I)$. Now $a(x + b) \in I - \varphi(I)$. Hence $a \in I$ or $(b + x)^n \in I$. Since $(b + x)^n = \sum_{k=0}^{n} b^k x^{n-k} \in I$ and since $x \in I$, then $b^n \in I$. So $a \in I$ or $b^n \in I$. Now assume that $aI \subseteq \varphi(I)$. If $bI \not\subseteq \varphi(I)$, then there exist $y \in I$ such that $by \not\in \varphi(I)$. Now $b(a + y) \in I - \varphi(I)$, so $a \in I$ or $b^n \in I$. Hence we can assume that $bI \subseteq \varphi(I)$. Since $I^2 \not\subseteq \varphi(I)$, there exist $r, s \in I$ such that $rs \not\in \varphi(I)$. Then $(a + r)(b + s) \in I - \varphi(I)$. So $a \in I$ or $b^n \in I$.

**Definition 9.** A semiring $R$ is called $\varphi$-semiprime if $I^n \subseteq \varphi(I)$ for each ideal $I$ of $R$ and $n \in N$, implies $I \subseteq \varphi(I)$.

**Corollary 10.** Let $R$ be a $\varphi$-semiprime semiring. Then a subtractive ideal $I$ of $R$ is $\varphi$-primary if and only if $I = \varphi(I)$ or $I$ is a primary ideal.

**Proof:** This follows by Lemma 8 and definition 9.

**Corollary 11.** Let $I$ be a $\varphi$-primary ideal where $\varphi \leq \varphi_3$. Then $I$ is a $\varphi_\omega$-primary.

**Proof:** If $I$ is a primary, then it is $\varphi$-primary ideal for each $\varphi$. Suppose that $I$ is not primary. By Lemma 8, $I^2 \subseteq \varphi(I) \subseteq I^3$. Hence $\varphi(I) = I^n$ for each $n \geq 2$, so $I$ is $\varphi_n$- primary for each $n \geq 3$ and thus $\varphi_\omega$-primary.
Let $I$ be an ideal of a semiring $R$ and let $\sqrt{I} = \{x \in R : x^n \in I \text{ for some } n \geq 1\}$. Clearly $\sqrt{I}$ is an ideal of $R$ and $I \subseteq \sqrt{I}$.

**Lemma 12.** Let $I$ be a primary ideal of a semiring $R$, Then $\sqrt{I}$ is a prime ideal of $R$.

**Proof:** Let $ab \in \sqrt{I}$, where $a, b \in R$. Then there are some integer $n$ such that $(ab)^n = a^n b^n \in I$. Since $I$ is a primary ideal we have $a^n \in I$ or $(b^n)^m \in I$ for some integer $m$. Then $a \in \sqrt{I}$ or $b \in \sqrt{I}$.

**Lemma 13.** Let $I$ be a $\varphi$-primary ideal of a semiring $R$ with $\sqrt{\varphi(I)} = \varphi(\sqrt{I})$. Then $\sqrt{I}$ is a $\varphi$-primary ideal of $R$.

**Proof:** Let $ab \in \sqrt{I} - \varphi(\sqrt{I})$ and $a \notin \sqrt{I}$, where there exist a positive integer $n$ with $a^n b^n \in I$. If $(ab)^n \in \varphi(I)$, then $ab \in \sqrt{\varphi(I)} = \varphi(\sqrt{I})$ that is a contradiction. Since $I$ is $\varphi$-primary, it follows from $a^n b^n \in I - \varphi(I)$ that $b \in \sqrt{I}$.

**Lemma 14.** Let $I$ and $J$ be subtractive ideals of a semiring $R$. Then $I \cup J = I$ or $I \cup J = J$.

**Lemma 15.** Let $R$ be a semiring and $I$ be an ideal of $R$ and $x \notin I$.

(i) If $I$ is subtractive, then $(I : x)$ is also subtractive.

(ii) If $I$ is a prime ideal of $R$ and $(I : x)$ is subtractive, then $I$ is subtractive.

**Lemma 16.** Let $I$ be a subtractive ideal of a semiring $R$. Then the following statement are equivalent:

(i) $I$ is $\varphi$-primary ideal.

(ii) If $x \notin \sqrt{I}$, then $(I : x) = I \cup (\varphi(I) : x)$

(iii) If $x \notin \sqrt{I}$, then $(I : x) = I$ or $(\varphi(I) : x)$.

**Proof:** $(i \Rightarrow ii)$: Let $y \in (I : x)$. Then $xy \in I$. If $xy \in \varphi(I)$, then $y \in (\varphi(I) : x) \subseteq I \cup (\varphi(I) : x)$. If $xy \notin \varphi(I)$, then $x^n \in I$ or $y \in I \cup (\varphi(I) : x)$.

On the other way let $y \in I \cup (\varphi(I) : x)$. Then $yx \in \varphi(I) \subseteq I$. Now $y \in (I : x)$.

$(ii \Rightarrow iii)$: It is follows by Lemma 14 and 15.

$(iii \Rightarrow i)$: Let $xy \in I - \varphi(I)$ and $x \notin \sqrt{I}$, then $y \notin (\varphi(I) : x)$ and $y \in (I : x)$ So $y \in I$.

**Lemma 17.** Let $I$ be a $\varphi$-primary subtractive ideal of a semiring $R$ that is not a primary. Then $\sqrt{\varphi(I)} = \sqrt{I}$.

**Proof:** Let $I$ is a $\varphi$-primary ideal that is not a primary ideal, then by Lemma 8, $I^2 \subseteq \varphi(I)$. So $I \subseteq \sqrt{\varphi(I)}$, hence $\sqrt{I} \subseteq \sqrt{\varphi\varphi(I)}$. It is clear that $\sqrt{\varphi(I)} \subseteq \sqrt{I}$. There for $\sqrt{\varphi(I)} = \sqrt{I}$.
Theorem 18. Let $I$ and $\varphi(I)$ be subtractive ideals of a semiring $R$. Then $I$ is a $\varphi$-primary ideal if and only if for ideals $A, B$ of $R$, $AB \subseteq I - \varphi(I)$ implies that $A \subseteq I$ or $B \subseteq \sqrt{I}$.

Proof: Suppose that $I$ is a $\varphi$-primary ideal of $R$. Let $A, B$ be ideals of $R$ such that $AB \subseteq I - \varphi(I)$, $A \not\subseteq I$ and $B \not\subseteq \sqrt{I}$. We will show that $AB \subseteq \varphi(I)$. Let $b \in B$. We have two cases $b \in \sqrt{I}$ or $b \notin \sqrt{I}$. If $b \notin \sqrt{I}$, then either $(I : b) = I$ or $(I : b) = (\varphi(I) : b)$. Now from $AB \subseteq AB \subseteq I$, we have $A \subseteq (I : b)$. Choose $a \in A - I$. Then from $a \in (I : b) - I$ we get $A \not\subseteq I$ and $(I : b) = (\varphi(I) : b)$. Therefore $A \subseteq (\varphi(I) : b)$ and $AB \subseteq \varphi(I)$. Now suppose that $b \in \sqrt{I}$, then $b \in I$ and $b \in \sqrt{I} \cap B$. Choose $b' \in B - \sqrt{I}$, then $(b + b') \in B - \sqrt{I}$ and hence we have $Ab' \subseteq (I)$ and $A(b + b') \subseteq \varphi(I)$, so $Ab \subseteq \varphi(I)$. Therefore $AB \subseteq \varphi(I)$ and this is a contradiction. Other way if $ab \in I - \varphi(I)$ where $a, b \in R$ then $a \not\subseteq I$ or $b \not\subseteq \sqrt{I}$. Hence $a \not\subseteq I$ or $b \not\subseteq \sqrt{I}$. So $a \in I$ or $b \in \sqrt{I}$.

References


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