Asymptotic Formulas Composite Numbers II

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Abstract

In this article we study the distribution of composite numbers which have in their prime factorization a fixed number of different prime factors, the exponents being fixed. In a previous article [4] we studied a particular case of this problem.

Let us consider the sequence A_n of composite numbers whose prime factorization is of the form $p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$. Where $a_1, a_2, \dots, a_k$ are positive integers fixed and $p_1, p_2, \dots, p_k$ are different primes. Let $A(x)$ be the number of these numbers not exceeding x . In this article we prove

$$A(x) \sim \alpha \frac{x^\beta (\log \log x)^\gamma}{\log x}$$

Where the numbers α, β, γ are defined in this article.

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1 Introduction, notation and lemmas.

Let n be a number such that its prime factorization is of the form

$$n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$$

where $a_i \geq q \geq 2$ ($i = 1, 2, \dots, r$) and $p_1, p_2, \dots, p_r$ ($r \geq 1$) are different primes in the factorization.

These numbers are well known, they are called powerful numbers or q -ful numbers.

There exist various studies on the distribution of these numbers using not elementary methods (see [2]).

Let $C_{n,q}$ be the sequence of these numbers and let $C_q(x)$ be the number of these numbers that do not exceed x . It is well known (see [3] for an elementary proof) that

$$C_{n,q} \sim c_q n^q \quad (1)$$

$$C_q(x) \sim b_q x^{\frac{1}{q}} \quad (2)$$

where b_q and c_q are constants depending of q .

From (1) we can obtain without difficulty the following lemma.

Lemma 1.1 *The following series are convergent ($q \geq 2$)*

$$\sum_{i=1}^{\infty} \frac{1}{(C_{n,q})^{\frac{1}{q-1}}} \quad \sum_{i=1}^{\infty} \frac{\log C_{n,q}}{(C_{n,q})^{\frac{1}{q-1}}}$$

Let us consider the sequence E_n of the numbers whose prime factorization is of the form

$$p_1^{a_1} p_2^{a_2} \dots p_t^{a_t}$$

Where $a_1 \geq a_2 \geq \dots \geq a_t \geq 2$ ($t \geq 1$) are positive integers fixed and the p_j ($j = 1, 2, \dots, t$) are different primes.

For example the sequence of the numbers of the form $p_1^9 p_2^5 p_3^5 p_4^3$ where p_1, p_2, p_3, p_4 are different primes. In this case $a_1 = 9, a_2 = 5, a_3 = 5, a_4 = 3, t = 4$.

We shall denote these numbers in the compact form E .

The number of these numbers not exceeding x we shall denote $E(x)$.

Let us consider the sequence A_n of the numbers whose prime factorization is of the form

$$p_1^{a_1} p_2^{a_2} \dots p_t^{a_t} p_{t+1}^h \dots p_{t+k}^h$$

Where $a_1 \geq a_2 \geq \dots \geq a_t > h \geq 1$ ($t \geq 1$) ($k \geq 1$) are positive integers fixed and the p_j ($j = 1, 2, \dots, t+k$) are different primes.

For example the numbers of the form $p_1^9 p_2^5 p_3^5 p_4^3 p_5 p_6 p_7 p_8$ where $p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8$ are different primes. In this case $a_1 = 9, a_2 = 5, a_3 = 5, a_4 = 3, h = 1, t = 4, k = 4$.

We shall denote these numbers in the compact form $E p_1^h \dots p_k^h$. Where E denotes the numbers of the form $p_1^{a_1} p_2^{a_2} \dots p_t^{a_t}$ (see above), p_1^h denotes $p_{t+1}^h, \dots, p_k^h$ denotes p_{t+k}^h .

Since in this case the E numbers are $(h+1)$ -ful numbers, lemma 1.1 imply the following series are convergent

$$\sum_{n=1}^{\infty} \frac{1}{E_n^{\frac{1}{h}}} = A_{E,h} \quad \sum_{n=1}^{\infty} \frac{\log E_n}{E_n^{\frac{1}{h}}} = B_{E,h} \quad (3)$$

On the other hand (2) imply that from a certain value of x we have

$$E(x) \leq (1 + \epsilon) b_{h+1} x^{\frac{1}{h+1}} \quad (\epsilon > 0) \quad (4)$$

The number of these numbers not exceeding x we shall denote $A(x)$.

In [4] was proved that if $k = 1$ then

$$A(x) \sim A_{E,h} \frac{hx^{\frac{1}{h}}}{\log x} \quad (5)$$

In theorem 2.1 we shall prove the general asymptotic formula

$$A(x) \sim A_{E,h} \frac{hx^{\frac{1}{h}} (\log \log x)^{k-1}}{(k-1)! \log x}$$

This formula becomes (5) if $k = 1$.

Let us consider the numbers whose prime factorization is of the form

$$p_1 p_2 \dots p_k$$

where $k \geq 2$ is fixed and $p_1, p_2, \dots, p_k$ are different primes.

Let $B(x)$ be the number of these numbers not exceeding x . We have the following theorem (Landau's theorem) which we shall use as a lemma (see [1]).

Lemma 1.2 *The following asymptotic formula holds*

$$B(x) = \frac{x(\log \log x)^{k-1}}{(k-1)! \log x} + f(x) \frac{x(\log \log x)^{k-1}}{(k-1)! \log x}$$

where $|f(x)| \leq M$ if $x \geq 3$ and $f(x) \rightarrow 0$.

We shall also need the following two lemmas whose proofs are simple.

Lemma 1.3 *The nonnegative function ($x \geq e$) ($k \geq 2$)*

$$f(x) = \frac{(\log \log x)^{k-1}}{\log x}$$

is bounded. That is, there exist $H_k > 0$ such that $f(x) \leq H_k$.

Lemma 1.4 *The function ($c > 1$)*

$$f(x) = \frac{\log \log \left(\frac{x}{c} \right)}{\log \log x}$$

is increasing from a certain value of x .

For sake of completeness we establish the following remark.

Remark. Note that our problem is trivial if all exponents are equal, that is, if the numbers are of the form

$$p_1^h \cdots p_k^h$$

Where $h \geq 1$ and $k \geq 1$. In this case the asymptotic formula which correspond to $A(x)$ is an immediate consequence of the prime number theorem or the Landau's theorem. Namely

$$A(x) \sim \frac{hx^{\frac{1}{h}}(\log \log x)^{k-1}}{(k-1)! \log x}$$

2 Main results

Theorem 2.1 *The following asymptotic formula holds*

$$A(x) \sim A_{E,h} \frac{hx^{\frac{1}{h}}(\log \log x)^{k-1}}{(k-1)! \log x} \quad (6)$$

Proof. We shall use mathematical induction. The theorem is true for $k = 1$ (see (5) and [4]). Suppose the theorem is true for $1, 2, \dots, k-1$ ($k \geq 2$). We shall prove it is also true for k . Let P_k be the product of the first k primes, that is, $P_2 = 2.3 = 6$, $P_3 = 2.3.5 = 30$, etc.

We have

$$\begin{aligned} Ep_1^h p_2^h \cdots p_k^h &\leq x \\ p_1^h p_2^h \cdots p_k^h &\leq \frac{x}{E} \\ E &\leq \frac{x}{p_1^h p_2^h \cdots p_k^h} \leq \frac{x}{P_k^h} \\ \frac{x}{E} &\geq P_k^h \\ \frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}} &\geq P_k \geq 6 \end{aligned}$$

Therefore (lemma 1.2)

$$\begin{aligned}
 A(x) &= \sum_{E \leq \frac{x}{P_k^h}} \sum_{p_1^h \dots p_k^h \leq \frac{x}{E}} 1 - F_k(x) = \sum_{E \leq \frac{x}{P_k^h}} \sum_{p_1 \dots p_k \leq \frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}} 1 - F_k(x) \\
 &= \sum_{E \leq \frac{x}{P_k^h}} B\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right) - F_k(x) = \sum_{E \leq \frac{x}{P_k^h}} \frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}} \frac{\left(\log \log \left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)\right)^{k-1}}{(k-1)! \log \left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)} \\
 &+ \sum_{E \leq \frac{x}{P_k^h}} f\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right) \frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}} \frac{\left(\log \log \left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)\right)^{k-1}}{(k-1)! \log \left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)} - F_k(x) \\
 &= \frac{hx^{\frac{1}{h}} \left(\log \log x^{\frac{1}{h}}\right)^{k-1}}{(k-1)! \log x} \sum_{E \leq \frac{x}{P_k^h}} \frac{1}{E^{\frac{1}{h}}} \frac{\left(\frac{\log \log \left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)}{\log \log x^{\frac{1}{h}}}\right)^{k-1}}{1 - \frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}} \\
 &+ G_k(x) - F_k(x) \tag{7}
 \end{aligned}$$

Substituting $x = P_k^h E_n$ into

$$\sum_{E \leq \frac{x}{P_k^h}} \frac{1}{E^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}}$$

we obtain the sequence

$$\sum_{i=1}^n \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} \tag{8}$$

Note that if $E_i \leq E_n$ then

$$\frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} \leq \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_i^{\frac{1}{h}}}} = \frac{1}{E_i^{\frac{1}{h}}} \frac{\log P_k + \log E_i^{\frac{1}{h}}}{\log P_k} = \frac{1}{E_i^{\frac{1}{h}}} + \frac{1}{h \log P_k} \frac{\log E_i}{E_i^{\frac{1}{h}}} \tag{9}$$

and if E_i is fixed then

$$\lim_{n \rightarrow \infty} \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} = \frac{1}{E_i^{\frac{1}{h}}} \tag{10}$$

We have (see (8))

$$\sum_{i=1}^n \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} = \sum_{i=1}^j \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} + \sum_{i=j+1}^n \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} \quad (11)$$

where (see (9))

$$\sum_{i=j+1}^n \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} \leq \sum_{i=j+1}^n \frac{1}{E_i^{\frac{1}{h}}} + \frac{1}{h \log P_k} \sum_{i=j+1}^n \frac{\log E_i}{E_i^{\frac{1}{h}}} \quad (12)$$

There exists j such that ($\epsilon > 0$) (see (3))

$$A_{E,h} - \epsilon < \sum_{i=1}^j \frac{1}{E_i^{\frac{1}{h}}} < A_{E,h} \quad (13)$$

$$\frac{1}{h \log P_k} \sum_{i=j+1}^{\infty} \frac{\log E_i}{E_i^{\frac{1}{h}}} < \epsilon \quad (14)$$

If $n \geq j+1$, (12), (13) and (14) give

$$0 \leq \sum_{i=j+1}^n \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} \leq 2\epsilon \quad (15)$$

On the other hand (see (10))

$$\lim_{n \rightarrow \infty} \sum_{i=1}^j \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} = \sum_{i=1}^j \frac{1}{E_i^{\frac{1}{h}}}$$

Consequently there exists $n' > j+1$ such that for all $n \geq n'$ we have

$$\sum_{i=1}^j \frac{1}{E_i^{\frac{1}{h}}} - \epsilon \leq \sum_{i=1}^j \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} \leq \sum_{i=1}^j \frac{1}{E_i^{\frac{1}{h}}} + \epsilon \quad (16)$$

(13) and (16) give

$$A_{E,h} - 2\epsilon \leq \sum_{i=1}^j \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} \leq A_{E,h} + \epsilon \quad (17)$$

Therefore for all $n \geq n'$ we have (see (11), (15) and (17))

$$A_{E,h} - 3\epsilon \leq \sum_{i=1}^n \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} \leq A_{E,h} + 3\epsilon \quad (18)$$

Consequently

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} = A_{E,h} \quad (19)$$

Now , we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} - \sum_{i=1}^{n+1} \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_{n+1}^{\frac{1}{h}}}} \right) \\ &= \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} - \sum_{i=1}^n \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_{n+1}^{\frac{1}{h}}}} - \frac{1}{E_{n+1}^{\frac{1}{h}}} \frac{\log P_k E_{n+1}^{\frac{1}{h}}}{\log P_k} \right) \\ &= 0 \end{aligned} \quad (20)$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{E_{n+1}^{\frac{1}{h}}} \frac{\log P_k E_{n+1}^{\frac{1}{h}}}{\log P_k} = 0 \quad (21)$$

(20) and (21) give

$$\lim_{n \rightarrow \infty} \left(\sum_{i=1}^n \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} - \sum_{i=1}^n \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_{n+1}^{\frac{1}{h}}}} \right) = 0 \quad (22)$$

Therefore (see (19))

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} = A_{E,h} \quad (23)$$

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_{n+1}^{\frac{1}{h}}}} = A_{E,h} \quad (24)$$

The function of x (E_i fixed, $E_i \leq E_n$)

$$\frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}} \quad (25)$$

is decreasing in the interval $[P_k^h E_n, P_k^h E_{n+1})$. Therefore if $x \in [P_k^h E_n, P_k^h E_{n+1})$ we have

$$\sum_{i=1}^n \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k^h E_{n+1}^{\frac{1}{h}}}} \leq \sum_{i=1}^n \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}} \leq \sum_{i=1}^n \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k^h E_n^{\frac{1}{h}}}} \quad (26)$$

Consequently (23), (24) and (26) give

$$\lim_{x \rightarrow \infty} \sum_{E \leq \frac{x}{P_k^h}} \frac{1}{E^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}} = A_{E,h} \quad (27)$$

The function of x (E fixed)

$$0 < \left(\frac{\log \log \left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}} \right)}{\log \log x^{\frac{1}{h}}} \right)^{k-1} < 1$$

is increasing from a certain value of x (see lemma 1.4) and

$$\lim_{x \rightarrow \infty} \left(\frac{\log \log \left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}} \right)}{\log \log x^{\frac{1}{h}}} \right)^{k-1} = 1$$

Let us consider the function

$$\sum_{E \leq \frac{x}{P_k^h}} \frac{1}{E^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}} \left(1 - \left(\frac{\log \log \left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}} \right)}{\log \log x^{\frac{1}{h}}} \right)^{k-1} \right)$$

There exists x_0 such that

$$A_{E,h} - \epsilon \leq \sum_{E \leq \frac{x_0}{P_k^h}} \frac{1}{E^{\frac{1}{h}}} < A_{E,h} + \epsilon \quad (28)$$

Now

$$\lim_{x \rightarrow \infty} \sum_{E \leq \frac{x_0}{P_k^h}} \frac{1}{E^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}} = \sum_{E \leq \frac{x_0}{P_k^h}} \frac{1}{E^{\frac{1}{h}}} \quad (29)$$

Therefore there exist $x_1 > x_0$ such that if $x \geq x_1$ we have (see (28) and (29))

$$A_{E,h} - 2\epsilon \leq \sum_{E \leq \frac{x_0}{P_k^h}} \frac{1}{E^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}} \leq A_{E,h} + 2\epsilon \quad (30)$$

and if $E \leq \frac{x_0}{P_k^h}$ we have

$$0 < 1 - \left(\frac{\log \log \left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}} \right)}{\log \log x^{\frac{1}{h}}} \right)^{k-1} < \epsilon \quad (31)$$

(30) and (31) give

$$0 < \sum_{E \leq \frac{x_0}{P_k^h}} \frac{1}{E^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}} \left(1 - \left(\frac{\log \log \left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}} \right)}{\log \log x^{\frac{1}{h}}} \right)^{k-1} \right) \leq (A_{E,h} + 2\epsilon)\epsilon \quad (32)$$

On the other hand (see (30))

$$\begin{aligned} 0 &< \sum_{\frac{x_0}{P_k^h} < E \leq \frac{x}{P_k^h}} \frac{1}{E^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}} \left(1 - \left(\frac{\log \log \left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}} \right)}{\log \log x^{\frac{1}{h}}} \right)^{k-1} \right) \\ &\leq \sum_{\frac{x_0}{P_k^h} < E \leq \frac{x}{P_k^h}} \frac{1}{E^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}} \leq 4\epsilon \end{aligned} \quad (33)$$

(32) and (33) give

$$\lim_{x \rightarrow \infty} \sum_{E \leq \frac{x}{P_k^h}} \frac{1}{E^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}} \left(1 - \left(\frac{\log \log \left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}} \right)}{\log \log x^{\frac{1}{h}}} \right)^{k-1} \right) = 0 \quad (34)$$

(27) and (34) give

$$\lim_{x \rightarrow \infty} \sum_{E \leq \frac{x}{P_k^h}} \frac{1}{E^{\frac{1}{h}}} \frac{\left(\frac{\log \log \left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}} \right)}{\log \log x^{\frac{1}{h}}} \right)^{k-1}}{1 - \frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}} = A_{E,h} \quad (35)$$

There exists x_0 such that (lemma 1.2)

$$\left| f \left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}} \right) \right| < \epsilon \quad \text{if} \quad \frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}} \geq x_0, \quad \text{that is if} \quad E \leq \frac{x}{x_0^h}$$

$$\left| f\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right) \right| \leq M \quad \text{if} \quad P_k \leq \frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}} < x_0, \quad \text{that is if} \quad \frac{x}{x_0^h} < E \leq \frac{x}{P_k^h}$$

Therefore

$$\begin{aligned} |G_k(x)| &= \left| \sum_{E \leq \frac{x}{P_k^h}} f\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right) \frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}} \frac{\left(\log \log \left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)\right)^{k-1}}{(k-1)! \log \left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)} \right| \\ &\leq \sum_{E \leq \frac{x}{P_k^h}} \left| f\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right) \right| \frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}} \frac{\left(\log \log \left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)\right)^{k-1}}{(k-1)! \log \left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)} \\ &\leq \epsilon \sum_{E \leq \frac{x}{x_0^h}} \frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}} \frac{\left(\log \log \left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)\right)^{k-1}}{(k-1)! \log \left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)} + Mx_0 \sum_{\frac{x}{x_0^h} < E \leq \frac{x}{P_k^h}} \frac{\left(\log \log \left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)\right)^{k-1}}{(k-1)! \log \left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)} \\ &= \frac{\epsilon h x^{\frac{1}{h}} (\log \log x^{\frac{1}{h}})^{k-1}}{(k-1)! \log x} \sum_{E \leq \frac{x}{x_0^h}} \frac{1}{E^{\frac{1}{h}}} \frac{\left(\frac{\log \log \left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)}{\log \log x^{\frac{1}{h}}}\right)^{k-1}}{1 - \frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}} \\ &\quad + Mx_0 \sum_{\frac{x}{x_0^h} < E \leq \frac{x}{P_k^h}} \frac{\left(\log \log \left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)\right)^{k-1}}{(k-1)! \log \left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)} \end{aligned} \quad (36)$$

Now (see lemma 1.3 and (4))

$$\begin{aligned} Mx_0 \sum_{\frac{x}{x_0^h} < E \leq \frac{x}{P_k^h}} \frac{\left(\log \log \left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)\right)^{k-1}}{(k-1)! \log \left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)} &\leq \frac{Mx_0 H}{(k-1)!} \sum_{E \leq x} 1 \\ &\leq \frac{Mx_0 H}{(k-1)!} (1 + \epsilon) b_{h+1} x^{\frac{1}{h+1}} \end{aligned} \quad (37)$$

and (see (35))

$$\frac{\epsilon h x^{\frac{1}{h}} (\log \log x^{\frac{1}{h}})^{k-1}}{(k-1)! \log x} \sum_{E \leq \frac{x}{x_0^h}} \frac{1}{E^{\frac{1}{h}}} \frac{\left(\frac{\log \log \left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)}{\log \log x^{\frac{1}{h}}}\right)^{k-1}}{1 - \frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}}$$

$$\begin{aligned}
&\leq \frac{\epsilon h x^{\frac{1}{h}} (\log \log x^{\frac{1}{h}})^{k-1}}{(k-1)! \log x} \sum_{E \leq \frac{x}{P_k^h}} \frac{1}{E^{\frac{1}{h}}} \frac{\left(\frac{\log \log \left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}} \right)}{\log \log x^{\frac{1}{h}}} \right)^{k-1}}{1 - \frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}} \\
&\leq \frac{\epsilon h x^{\frac{1}{h}} (\log \log x^{\frac{1}{h}})^{k-1}}{(k-1)! \log x} (A_{E,h} + \epsilon)
\end{aligned} \tag{38}$$

Consequently (36), (37) and (38) give

$$G_k(x) = o \left(\frac{x^{\frac{1}{h}} (\log \log x)^{k-1}}{\log x} \right) \tag{39}$$

On the other hand from the inductive hypothesis and (4) (if it is necessary) we can obtain without difficulty that

$$F_k(x) = o \left(\frac{x^{\frac{1}{h}} (\log \log x)^{k-1}}{\log x} \right) \tag{40}$$

Finally (7), (35), (39) and (40) give (6). The theorem is proved.

Corollary 2.2 *The following asymptotic formula holds*

$$A_n \sim \frac{((k-1)!)^h n^h \log^h n}{A_{E,h}^h (\log \log n)^{h(k-1)}} \tag{41}$$

Proof. From (6) we obtain

$$\log A(x) \sim \frac{1}{h} \log x \tag{42}$$

(42) gives

$$\log \log A(x) \sim \log \log x \tag{43}$$

Therefore substituting (42) and (43) into (6) we find that

$$A(x) \sim A_{E,h} \frac{x^{\frac{1}{h}} (\log \log A(x))^{k-1}}{(k-1)! \log A(x)} \tag{44}$$

Finally, substituting $x = A_n$ into (44) we obtain

$$n \sim A_{E,h} \frac{A_n^{\frac{1}{h}} (\log \log n)^{k-1}}{(k-1)! \log n}$$

That is (41). The corollary is proved.

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References

- [1] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Oxford, 1960.
- [2] A. Ivic, *The Riemann zeta-function*, Dover, 2003.
- [3] R. Jakimczuk, On the distribution of certain composite numbers, *International Journal of Contemporary Mathematical Sciences*, **3** (2008), 1245 - 1254.
- [4] R. Jakimczuk, Asymptotic formulas. Composite numbers, *International Journal of Contemporary Mathematical Sciences*, **7** (2012), 171- 178.

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