Multiparameter K-Mittag-Leffler Function

Kuldeep Singh Gehlot

Government Bangur P.G. College, Pali
 Pali-Marwar, Rajasthan, India-306401
drksgehlol@rediffmail.com

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Abstract

In this paper author introduce Multiparameter K-Mittag-Leffler Function defined as,

$${}_pK_{q,k}^{(\beta,\eta)_m}[z] = {}_pK_{q,k}^{(\beta,\eta)_m}[a_1, \dots, a_p; b_1, \dots, b_q, (\beta_1, \eta_1), \dots, (\beta_m, \eta_m); z],$$

$${}_pK_{q,k}^{(\beta,\eta)_m}[z] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k} z^n}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n + \beta_i)},$$

where $k \in R_+ = (0, \infty)$; $a_j, b_r, \beta_i \in C$; $\eta_i \in R$ ($j = 1, 2, \dots, p; r = 1, 2, \dots, q; i = 1, 2, \dots, m$).

Certain relations that exist between ${}_pK_{q,k}^{(\beta,\eta)_m}[z]$ function and Riemann-Liouville fractional integral and derivatives has been evaluated. It has been shown that the fractional integration and differentiation of ${}_pK_{q,k}^{(\beta,\eta)_m}[z]$ function with power multipliers into the function of the same form.

Also deduce Mittag-Leffler functions introduced by [1],[3],[4],[5],[6],[7],[8],[13],[14],[15] and [16] are particular cases of Multiparameter K-Mittag-Leffler function for some particular values of parameters and deduce some particular cases.

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1 Introduction

In [9] the author introduce the generalized K-Gamma Function $\Gamma_k(x)$ as

$$\Gamma_k(x) = \lim_{n \rightarrow \infty} \frac{n! k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, k > 0, x \in C \setminus kZ^-, \quad (1)$$

where $(x)_{n,k}$ is the k-Pochhammer symbol and is given by

$$(x)_{n,k} = x(x+k)(x+2k)\dots(x+(n-1)k), x \in C, k \in R, n \in N^+. \quad (2)$$

K-Gamma function is given by,

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t}{k}} dt, x \in C, k \in R, Re(x) > 0, \quad (3)$$

and it follows easily that

$$\Gamma_k(x) = k^{\frac{x}{k}-1} \Gamma\left(\frac{x}{k}\right) \text{ and } (\gamma)_{nq,k} = (k)^{nq} \left(\frac{\gamma}{k}\right)_{nq}. \quad (4)$$

The Fractional Integral operators for $\alpha > 0$ ([10], Definition 2.1, Page 33) are defined as,

$$(I_{0+}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, (x > 0), \quad (5)$$

$$(I_-^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty \frac{f(t)}{(t-x)^{1-\alpha}} dt, (x > 0), \quad (6)$$

The Fractional Derivative for $\alpha > 0$ ([10], Definition 2.2, Page 35) are defined as,

$$(D_{0+}^\alpha f)(x) = \left(\frac{d}{dx}\right)^{[\alpha]+1} (I_{0+}^{1-\{\alpha\}} f)(x) = \frac{1}{\Gamma(1-\{\alpha\})} \left(\frac{d}{dx}\right)^{[\alpha]+1} \int_0^x \frac{f(t)}{(x-t)^{\{\alpha\}}} dt, (x > 0), \quad (7)$$

$$(D_-^\alpha f)(x) = \left(\frac{d}{dx}\right)^{[\alpha]+1} (I_-^{1-\{\alpha\}} f)(x) = \frac{1}{\Gamma(1-\{\alpha\})} \left(-\frac{d}{dx}\right)^{[\alpha]+1} \int_x^\infty \frac{f(t)}{(t-x)^{\{\alpha\}}} dt, (x > 0). \quad (8)$$

where $[\alpha]$ means the maximal integer not extending α and $\{\alpha\}$ is the fractional part of α .

The next assertion is well known; see ([10], (2.44) and table 9.3, formula 1).

Let $\alpha \in C$, $(Re(\alpha) > 0)$ and $\gamma \in C$.

$$\text{if } Re(\gamma) > 0, \text{ then } (I_{0+}^\alpha t^{\gamma-1})(x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} x^{\alpha+\gamma-1} \quad (9)$$

$$\text{if } Re(\gamma) > Re(\alpha) > 0, \text{ then } (I_-^\alpha t^{-\gamma})(x) = \frac{\Gamma(\gamma-\alpha)}{\Gamma(\gamma)} x^{\alpha-\gamma} \quad (10)$$

2 Main result

In this section we introduce Multiparameter K-Mittag-Leffler function and for particular values of parameter we can deduce Multiparameter K-Mittag-Leffler function in to eariler known Mittag-Leffler functions. Finally we calculate Fractional integral and derivative of Multiparameter K-Mittag-Leffler function.

2.1 Multiparameter K-Mittag-Leffer Function

Definition: Let $k \in R_+ = (0, \infty)$; $a_j, b_r, \beta_i \in C$; $\eta_i \in R$ ($j = 1, 2, \dots, p$; $r = 1, 2, \dots, q$; $i = 1, 2, \dots, m$). Then the Multiparameter K-Mittag-Leffler function definded as,

$${}_pK_{q,k}^{(\beta,\eta)^m}[a_1, \dots, a_p; b_1, \dots, b_q, (\beta_1, \eta_1), \dots, (\beta_m, \eta_m); z] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k} z^n}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n + \beta_i)}. \quad (11)$$

Where $\Gamma_k(x)$ is the K-Gamma function given by (1) and $(\gamma)_{n,k}$ is the K-Pochhammer symbol given by (2).

The series (11) is definded when none of the parameter b_r ($r = 1, 2, \dots, q$) is negative integer or zero. If any parameter a_j ($j = 1, 2, \dots, p$) in (11) is zero or negative, the series terminates into polynomial in z .

Convergent conditions for the series (11) are given by Ratio test,

(i) If $p < q + \sum_{i=1}^m (\frac{\eta_i}{k})$, then the power series on the right of (11) is absolutely convergent for all $z \in C$.

(ii) If $p = q + \sum_{i=1}^m (\frac{\eta_i}{k})$, then the power series on the right of (11) is absolutely convergent for all $|k^{p-q-\sum_{i=1}^m (\frac{\eta_i}{k})} z| < \prod_{i=1}^m (|\frac{\eta_i}{k}|)^{\frac{\eta_i}{k}}$ and $|k^{p-q-\sum_{i=1}^m (\frac{\eta_i}{k})} z| = \prod_{i=1}^m (|\frac{\eta_i}{k}|)^{\frac{\eta_i}{k}}$, $Re(\sum_{r=1}^q (\frac{b_r}{k}) + \sum_{i=1}^m (\frac{\beta_i}{k}) - \sum_{j=1}^p (\frac{a_j}{k})) > \frac{2+q+m-p}{2}$.

2.1.1 Particular cases

By particularizing the parameters in (11), we obtain following known Mittag-leffler functions.

(a) If we set $k = 1$ in (11), we have

$${}_pK_{q,1}^{(\beta,\eta)^m}[a_1, \dots, a_p; b_1, \dots, b_q, (\beta_1, \eta_1), \dots, (\beta_m, \eta_m); z] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_n z^n}{\prod_{r=1}^q (b_r)_n \prod_{i=1}^m \Gamma(\eta_i n + \beta_i)},$$

$${}_pK_{q,1}^{(\beta,\eta)^m}[a_1, \dots, a_p; b_1, \dots, b_q, (\beta_1, \eta_1), \dots, (\beta_m, \eta_m); z] = {}_pK_q^{(\beta,\eta)^m}[z].$$

Which is K-Series defined by [3,4].

(b) If we set $k = 1, p = q = m$ and $b_1 = b_2 = \dots = b_m = 1$ in (11), we have

$${}_m K_{m,1}^{(\beta,\eta)^m} [a_1, \dots, a_m; 1, 1, \dots, 1, (\beta_1, \eta_1), \dots, (\beta_m, \eta_m); z] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^m (a_i)_n z^n}{\prod_{i=1}^m \Gamma(\eta_i n + \beta_i) (n!)^m},$$

$${}_m K_{m,1}^{(\beta,\eta)^m} [a_1, \dots, a_m; 1, 1, \dots, 1, (\beta_1, \eta_1), \dots, (\beta_m, \eta_m); z] = E_{(\eta_i), (\beta_i)}^{(a_i), m} [z].$$

Which is the 3M-Parameter Multi-Index Mittag-Leffler function defined by [5].

(c) If we set $k = 1, p = q = 1, a_1 = \rho, b_1 = 1$ in (11), then we obtain

$${}_1 K_{1,1}^{(\beta,\eta)^m} [\rho; 1, (\beta_1, \eta_1), \dots, (\beta_m, \eta_m); z] = \sum_{n=0}^{\infty} \frac{(\rho)_n z^n}{\prod_{i=1}^m \Gamma(\eta_i n + \beta_i) (n!)^m},$$

$${}_1 K_{1,1}^{(\beta,\eta)^m} [\rho; 1, (\beta_1, \eta_1), \dots, (\beta_m, \eta_m); z] = E_{\rho}[(\eta, \beta)_m; z].$$

Which is the Generalized Mittag-Leffler function studied by [6].

(d) If we set $k = 1, p = q = 1, a_1 = b_1 = 1$ and $\eta_i = \frac{1}{\alpha_i}$ in (11), then we obtain

$${}_1 K_{1,1}^{(\beta,\eta)^m} [1; 1, (\beta_1, \frac{1}{\alpha_1}), \dots, (\beta_m, \frac{1}{\alpha_m}); z] = \sum_{n=0}^{\infty} \frac{z^n}{\prod_{i=1}^m \Gamma(n \frac{1}{\alpha_i} + \beta_i)},$$

$${}_1 K_{1,1}^{(\beta,\eta)^m} [1; 1, (\beta_1, \frac{1}{\alpha_1}), \dots, (\beta_m, \frac{1}{\alpha_m}); z] = E_{(\frac{1}{\alpha_i}), (\beta_i)}^{(m)} [z].$$

Which is the Multi-Index Mittag-Leffler function studied by [15].

(e) If we set $k = 1, m = 1$ in (11), we have

$${}_p K_{q,1}^{(\beta,\eta)^1} [a_1, \dots, a_p; b_1, \dots, b_q, (\beta, \eta); z] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_n z^n}{\prod_{r=1}^q (b_r)_n \Gamma(\eta n + \beta)}.$$

$${}_p K_{q,1}^{(\beta,\eta)^1} [a_1, \dots, a_p; b_1, \dots, b_q, (\beta, \eta); z] = {}_p M_q^{\eta, \beta} [a_1, \dots, a_p; b_1, \dots, b_q; z].$$

Which is Generalized M-Series defined by [13].

(f) If we set $k = 1, m = 1$ and $\beta = 1$ in (11), we have

$${}_p K_{q,1}^{(\beta,\eta)^1} [a_1, \dots, a_p; b_1, \dots, b_q, (1, \eta); z] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_n z^n}{\prod_{r=1}^q (b_r)_n \Gamma(\eta n + 1)}.$$

$${}_pK_{q,1}^{(\beta,\eta)_1}[a_1, \dots, a_p; b_1, \dots, b_q, (1, \eta); z] = {}_pM_q^\eta[a_1, \dots, a_p; b_1, \dots, b_q; z].$$

Which is M-Series defined by [14].

(g) If we set $p = q = m = 1, a_1 = \delta, b_1 = k$ in (11), then we obtain

$${}_1K_{1,1}^{(\beta,\eta)_1}[\delta; k, (\beta, \eta); z] = \sum_{n=0}^{\infty} \frac{(\delta)_{n,k}}{\Gamma_k(\eta n + \beta)(n!)} \left(\frac{z}{k}\right)^n,$$

$${}_1K_{1,1}^{(\beta,\eta)_1}[\delta; k, (\beta, \eta); z] = E_{k,\eta,\beta}^\delta\left[\frac{z}{k}\right].$$

Which is the *K*-Mittag-Leffler function studied by [1].

(h) If we set $k = 1, p = q = m = 1, a_1 = \delta, b_1 = 1$ in (11), then we obtain

$${}_1K_{1,1}^{(\beta,\eta)_1}[\delta; 1, (\beta, \eta); z] = \sum_{n=0}^{\infty} \frac{(\delta)_n z^n}{\Gamma(\eta n + \beta)(n!)},$$

$${}_1K_{1,1}^{(\beta,\eta)_1}[\delta; 1, (\beta, \eta); z] = E_{\eta,\beta}^\delta[z].$$

Which is the Generalized Mittag-Leffler function studied by [8].

(i) If we set $k = 1, p = q = m = 1, a_1 = b_1 = 1$ in (11), then we obtain

$${}_1K_{1,1}^{(\beta,\eta)_1}[1; 1, (\beta, \eta); z] = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\eta n + \beta)},$$

$${}_1K_{1,1}^{(\beta,\eta)_1}[1; 1, (\beta, \eta); z] = E_{\eta,\beta}[z].$$

Which is the Mittag-Leffler function studied by [16].

(j) If we set $k = 1, p = q = m = 1, a_1 = b_1 = 1$ and $\beta = 1$ in (11), then we obtain

$${}_1K_{1,1}^{(\beta,\eta)_1}[1; 1, (1, \eta); z] = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\eta n + 1)},$$

$${}_1K_{1,1}^{(\beta,\eta)_1}[1; 1, (1, \eta); z] = E_\eta[z].$$

Which is the Mittag-Leffler function studied by [7].

2.2 Fractional Claculus of Multiparameter K-Mittag-Leffler Function

We will prove, the fractional integration and differentiation of ${}_pK_{q,k}^{(\beta,\eta)_m}[z]$ function with power multipliers into the function of the same form.

Theorem 1. Let $\alpha > 0, a \in R, k \in R_+ = (0, \infty); a_j, b_r, \beta_i \in C; \eta_i \in R, Re(\beta_i) > 0; (j = 1, 2, \dots, p; r = 1, 2, \dots, q; i = 1, 2, \dots, m)$. Then there holds the relation,

$$(I_{0+}^\alpha [t^{\frac{\beta_1}{k}-1} {}_pK_{q,k}^{(\beta,\eta)_m}[at^{\frac{\eta_1}{k}}]])(x) = k^\alpha x^{(\frac{\beta_1}{k}+\alpha-1)} {}_pK_{q,k}^{(\beta,\eta)_m}[ax^{\frac{\eta_1}{k}}]. \quad (12)$$

Proff. By virtue of (5) and (11), we have

$$A \equiv (I_{0+}^\alpha [t^{\frac{\beta_1}{k}-1} {}_pK_{q,k}^{(\beta,\eta)_m}[at^{\frac{\eta_1}{k}}]])(x),$$

$$A \equiv (I_{0+}^\alpha [t^{\frac{\beta_1}{k}-1} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k} (at^{\frac{\eta_1}{k}})^n}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n + \beta_i)}])(x),$$

on interchanging the order of the integration and summation, and using (9), we obtain

$$A \equiv \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k}}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n + \beta_i)} \frac{\Gamma(\frac{n\eta_1+\beta_1}{k}) a^n x^{(\frac{n\eta_1+\beta_1}{k}+\alpha-1)}}{\Gamma(\frac{n\eta_1+\beta_1}{k} + \alpha)},$$

using (4) and rearranging the terms, we have

$$A \equiv k^\alpha x^{(\frac{\beta_1}{k}+\alpha-1)} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k}}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=2}^m \Gamma_k(\eta_i n + \beta_i)} \frac{(ax^{\frac{\eta_1}{k}})^n}{\Gamma_k(n\eta_1 + \beta_1 + \alpha k)},$$

$$A \equiv k^\alpha x^{(\frac{\beta_1}{k}+\alpha-1)} {}_pK_{q,k}^{(\beta,\eta)_m}[a_1, \dots, a_p; b_1, \dots, b_q, (\beta_1 + \alpha k, \eta_1), (\beta_2, \eta_2), \dots, (\beta_m, \eta_m); ax^{\frac{\eta_1}{k}}],$$

which proves the theorem.

Theorem 2. Let $\alpha > 0, a \in R, k \in R_+ = (0, \infty); a_j, b_r, \beta_i \in C; \eta_i \in R, Re(\beta_i) > 0; (j = 1, 2, \dots, p; r = 1, 2, \dots, q; i = 1, 2, \dots, m)$. Then there holds the relation,

$$(I_-^\alpha [t^{-\alpha-\frac{\beta_1}{k}} {}_pK_{q,k}^{(\beta,\eta)_m}[at^{-\frac{\eta_1}{k}}]])(x) = k^\alpha x^{(-\frac{\beta_1}{k})} {}_pK_{q,k}^{(\beta,\eta)_m}[ax^{-\frac{\eta_1}{k}}]. \quad (13)$$

Proff. By virtue of (6) and (11), we have

$$A \equiv (I_-^\alpha [t^{-\alpha-\frac{\beta_1}{k}} {}_pK_{q,k}^{(\beta,\eta)_m}[at^{-\frac{\eta_1}{k}}]])(x),$$

$$A \equiv (I_-^\alpha [t^{-\alpha - \frac{\beta_1}{k}} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k} (at^{-\frac{\eta_1}{k}})^n}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n + \beta_i)}])(x),$$

on interchanging the order of the integration and summation, and using (10), we obtain

$$A \equiv \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k}}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n + \beta_i)} \frac{\Gamma(\frac{n\eta_1 + \beta_1}{k}) a^n x^{-(\frac{n\eta_1 + \beta_1}{k})}}{\Gamma(\frac{n\eta_1 + \beta_1}{k} + \alpha)},$$

using (4) and rearranging the terms, we have

$$A \equiv k^\alpha x^{(-\frac{\beta_1}{k})} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k}}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=2}^m \Gamma_k(\eta_i n + \beta_i)} \frac{(ax^{-\frac{\eta_1}{k}})^n}{\Gamma_k(n\eta_1 + \beta_1 + \alpha k)},$$

$$A \equiv k^\alpha x^{(-\frac{\beta_1}{k})} {}_p K_{q,k}^{(\beta, \eta)_m} [a_1, \dots, a_p; b_1, \dots, b_q, (\beta_1 + \alpha k, \eta_1), (\beta_2, \eta_2) \dots, (\beta_m, \eta_m); ax^{-\frac{\eta_1}{k}}],$$

which proves the theorem.

Theorem 3. Let $\alpha > 0, a \in R, k \in R_+ = (0, \infty); a_j, b_r, \beta_i \in C; \eta_i \in R, \operatorname{Re}(\beta_i) > 0; (j = 1, 2, \dots, p; r = 1, 2, \dots, q; i = 1, 2, \dots, m)$. Then there holds the relation,

$$(D_{0+}^\alpha [t^{\frac{\beta_1}{k}-1} {}_p K_{q,k}^{(\beta, \eta)_m} [at^{\frac{\eta_1}{k}}]])(x) = k^{-\alpha} x^{(\frac{\beta_1}{k}-\alpha-1)} {}_p K_{q,k}^{(\beta, \eta)_m} [ax^{\frac{\eta_1}{k}}]. \quad (14)$$

Proff. By virtue of (7) and (11), we have

$$A \equiv (D_{0+}^\alpha [t^{\frac{\beta_1}{k}-1} {}_p K_{q,k}^{(\beta, \eta)_m} [at^{\frac{\eta_1}{k}}]])(x),$$

$$A \equiv \frac{1}{\Gamma(s-\alpha)} \left(\frac{d}{dx}\right)^s \int_0^x \frac{t^{\frac{\beta_1}{k}-1}}{(x-t)^{1+\alpha-s}} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k} (at^{\frac{\eta_1}{k}})^n}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n + \beta_i)} dt,$$

where $s = [\alpha] + 1$

$$A \equiv \frac{1}{\Gamma(s-\alpha)} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k} a^n}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n + \beta_i)} \left(\frac{d}{dx}\right)^s \int_0^x \frac{t^{\frac{\beta_1 + n\eta_1}{k}-1}}{(x-t)^{1+\alpha-s}} dt,$$

put $t = ux$ the above expression transforms in to the form

$$A \equiv \frac{1}{\Gamma(s-\alpha)} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k} a^n}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n + \beta_i)} \left(\frac{d}{dx}\right)^s x^{\frac{\beta_1 + n\eta_1}{k} + s - \alpha - 1}$$

$$\times \int_0^1 (1-u)^{s-\alpha-1} u^{\frac{\beta_1+n\eta_1}{k}-1} dt,$$

using the Beta function formula, we have

$$A \equiv x^{\frac{\beta_1}{k}-\alpha-1} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k} a^n}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n + \beta_i)} \frac{\Gamma(\frac{\beta_1+n\eta_1}{k})}{\Gamma(\frac{\beta_1+n\eta_1}{k} - \alpha)} x^{\frac{n\eta_1}{k}},$$

using (4) and rearranging the terms, we obtain

$$A \equiv k^{-\alpha} x^{\frac{\beta_1}{k}-\alpha-1} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k}}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=2}^m \Gamma_k(\eta_i n + \beta_i) \Gamma_k(n\eta_1 + \beta_1 - k\alpha)} (ax^{\frac{\eta_1}{k}})^n,$$

$$A \equiv k^{-\alpha} x^{\frac{\beta_1}{k}-\alpha-1} {}_pK_{q,k}^{(\beta,\eta)_m}[a_1, \dots, a_p; b_1, \dots, b_q, (\beta_1 - \alpha k, \eta_1), (\beta_2, \eta_2) \dots, (\beta_m, \eta_m); ax^{\frac{\eta_1}{k}}],$$

which proves the theorem.

Theorem 4. Let $\alpha > 0, a \in R, k \in R_+ = (0, \infty); a_j, b_r, \beta_i \in C; \eta_i \in R, \operatorname{Re}(\beta_i) > 0; (j = 1, 2, \dots, p; r = 1, 2, \dots, q; i = 1, 2, \dots, m)$. Then there holds the relation,

$$(D_-^\alpha [t^{\alpha-\frac{\beta_1}{k}} {}_pK_{q,k}^{(\beta,\eta)_m}[at^{-\frac{\eta_1}{k}}]])(x) = k^{-\alpha} x^{(-\frac{\beta_1}{k})} {}_pK_{q,k}^{(\beta,\eta)_m}[ax^{-\frac{\eta_1}{k}}]. \quad (15)$$

Proof. By virtue of (8) and (11), we have

$$A \equiv (D_-^\alpha [t^{\alpha-\frac{\beta_1}{k}} {}_pK_{q,k}^{(\beta,\eta)_m}[at^{-\frac{\eta_1}{k}}]])(x),$$

$$A \equiv \frac{1}{\Gamma(s-\alpha)} \left(-\frac{d}{dx}\right)^s \int_x^\infty \frac{t^{\alpha-\frac{\beta_1}{k}}}{(t-x)^{1+\alpha-s}} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k} (at^{-\frac{\eta_1}{k}})^n}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n + \beta_i)} dt,$$

where $s = [\alpha] + 1$

$$A \equiv \frac{1}{\Gamma(s-\alpha)} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k} a^n}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n + \beta_i)} \left(-\frac{d}{dx}\right)^s \int_x^\infty \frac{t^{\alpha-\frac{\beta_1+n\eta_1}{k}}}{(t-x)^{1+\alpha-s}} dt,$$

put $t = \frac{x}{u}$ and using the Beta function, the above expression transforms in to the form,

$$A \equiv x^{-\frac{\beta_1}{k}} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k} a^n \Gamma(\frac{\beta_1+n\eta_1}{k} - s) (-1)^s \Gamma(1+s - \frac{\beta_1+n\eta_1}{k}) x^{-\frac{n\eta_1}{k}}}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n + \beta_i) \Gamma(\frac{\beta_1+n\eta_1}{k} - \alpha) \Gamma(1 - \frac{\beta_1+n\eta_1}{k})}, \quad (16)$$

the reflection formula for gamma function, see ([10],1.60),

$$\frac{1}{\Gamma(1 - \frac{\beta_1 + n\eta_1}{k})} = \frac{\Gamma(\frac{\beta_1 + n\eta_1}{k})}{\Gamma(\frac{\beta_1 + n\eta_1}{k})\Gamma(1 - \frac{\beta_1 + n\eta_1}{k})} = \frac{\Gamma(\frac{\beta_1 + n\eta_1}{k}) \sin[(\frac{\beta_1 + n\eta_1}{k})\pi]}{\pi}, \quad (17)$$

and

$$\begin{aligned} \Gamma(\frac{\beta_1 + n\eta_1}{k} - s)\Gamma(1 + s - \frac{\beta_1 + n\eta_1}{k}) &= \frac{\pi}{\sin[(\frac{\beta_1 + n\eta_1}{k} - s)\pi]}, \\ &= \frac{\pi}{\sin[(\frac{\beta_1 + n\eta_1}{k})\pi] \cos(s\pi)} = \frac{\pi(-1)^s}{\sin[(\frac{\beta_1 + n\eta_1}{k})\pi]}, \end{aligned} \quad (18)$$

using (17) and (18) in (16), we obtain

$$A \equiv x^{-\frac{\beta_1}{k}} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k} a^n \Gamma(\frac{\beta_1 + n\eta_1}{k}) x^{-\frac{n\eta_1}{k}}}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=1}^m \Gamma_k(\eta_i n + \beta_i) \Gamma(\frac{\beta_1 + n\eta_1}{k} - \alpha)},$$

using (4) and rearranging the terms, we obtain

$$A \equiv k^{-\alpha} x^{-\frac{\beta_1}{k}} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{n,k}}{\prod_{r=1}^q (b_r)_{n,k} \prod_{i=2}^m \Gamma_k(\eta_i n + \beta_i) \Gamma_k(n\eta_1 + \beta_1 - k\alpha)} (ax^{-\frac{\eta_1}{k}})^n,$$

$$A \equiv k^{-\alpha} x^{-\frac{\beta_1}{k}} {}_p K_{q,k}^{(\beta, \eta)_m} [a_1, \dots, a_p; b_1, \dots, b_q, (\beta_1 - \alpha k, \eta_1), (\beta_2, \eta_2) \dots, (\beta_m, \eta_m); ax^{-\frac{\eta_1}{k}}],$$

which proves the theorem.

2.2.1 Particular Cases

(A) By putting $k = 1$,

(i) in equation (12), then it reduces to the result earlier given by ([3], Theorem 3.1, Equation 6).

(ii) in equation (13), then it reduces to the result earlier given by ([3], Theorem 3.2, Equation 7).

(iii) in equation (14), then it reduces to the result earlier given by ([3], Theorem 3.3, Equation 8).

(iv) in equation (15), then it reduces to the result earlier given by ([3], Theorem 3.4, Equation 9).

(B) By putting $k = 1, p = q = m$ and $b_1 = b_2 = \dots = b_m = 1$,

(i) in equation (12), then it reduces to the result earlier given by ([5], Theorem 4.1, Equation 4.4).

(ii) in equation (14), then it reduces to the result earlier given by ([5], Theorem 4.2, Equation 4.5).

(C) By putting $k = p = q = 1$, $a_1 = \rho$ and $b_1 = 1$,

(i) in equation (12), then it reduces to the result earlier given by ([11], Theorem 1, Equation 2.1).

(ii) in equation (13), then it reduces to the result earlier given by ([11], Theorem 3, Equation 2.4).

(iii) in equation (14), then it reduces to the result earlier given by ([11], Theorem 5, Equation 2.6).

(iv) in equation (15), then it reduces to the result earlier given by ([11], Theorem 7, Equation 2.8).

(D) By putting $k = p = q = m = 1$, $a_1 = \delta$ and $b_1 = 1$,

(i) in (12), then it reduces to the result earlier given by ([12], Theorem 1, Equation 14).

(ii) in (13) then it reduces to the result earlier given by ([12], Theorem 3, Equation 23).

(iii) in (14) then it reduces to the result earlier given by ([12], Theorem 5, Equation 29).

(iv) in (15) then it reduces to the result earlier given by ([12], Theorem 7, Equation 35).

(E) By putting $k = m = 1$ and $\beta = 1$,

(i) in (12), then it reduces to the result earlier given by ([2], Theorem 2.1, Equation 6).

(ii) in (13) then it reduces to the result earlier given by ([2], Theorem 2.2, Equation 10).

(iii) in (14) then it reduces to the result earlier given by ([2], Theorem 2.3, Equation 14).

(vi) in (15) then it reduces to the result earlier given by ([2], Theorem 2.4, Equation 18).

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