On Nearly-Kaehlerian Weyl Spaces

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Abstract

In this work, we consider nearly-Kaehlerian Weyl spaces and show that a nearly-Kaehlerian Weyl space is a Kaehlerian if the almost complex structure is integrable. Also, we give a condition so that an almost semi-Kaehlerian-Weyl space to be Kaehlerian.

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1. Introduction

Let $W_n$ be an $n$-dimensional space with a conformal metric tensor $g$. If the torsion-free connection $D$ on $W_n$ satisfies the compatibility condition

$$D_k g_{ij} = 2 w_k g_{ij}$$

(1.1)

then, $W_n$ is called a Weyl space and denoted by $W_n(g, w)$. Here, $w$ is a 1-form, called complementary co-vector field.

In literature, it is shown that for the renormalization transformation of the metric tensor $g$

$$\tilde{g}_{ij} = \lambda^2 g_{ij}$$

(1.2)
the complementary vector field $w$ is transformed into by the rule

$$\tilde{w}_k = w_k + \partial_k \ln \lambda$$  \hspace{1cm} (1.3)

where $\lambda$ is a scalar function defined on $W_n$ [3], [5], and [6].

A quantity $A$ defined on $W_n(g, w)$ is called a satellite of $g$ of weight $\{p\}$, if it admits a transformation of the form

$$\tilde{A} = \lambda^p A$$  \hspace{1cm} (1.4)

under the renormalization (1.2).

We note that the weight of the metric tensor $g_{ij}$ is $\{2\}$ by the transformation (1.2).

The prolonged (extended) covariant derivative of the satellite $A$ of weight $\{p\}$ is defined by [4], and [8]

$$\tilde{\nabla}_k A = D_k A - p w_k A.$$  \hspace{1cm} (1.5)

From (1.1), (1.2) and (1.5), and taking prolonged covariant derivative of the metric tensor $g_{ij}$, we have

$$\tilde{\nabla}_k g_{ij} = 0.$$  \hspace{1cm} (1.6)

Using the Ricci identity for a covariant vector field $v_k$, we write

$$(\tilde{\nabla}_i \tilde{\nabla}_j - \tilde{\nabla}_j \tilde{\nabla}_i) v_k = -W_{kji}^i t v_t$$  \hspace{1cm} (1.7)

then the curvature tensor of $W_n$ is obtained as

$$W_{kji}^l = \partial_k \Gamma_{ji}^l - \partial_j \Gamma_{ki}^l - \Gamma_{ti}^l \Gamma_{jt}^i + \Gamma_{ji}^t \Gamma_{kt}^l.$$  \hspace{1cm} (1.8)

Here, $\Gamma_{ji}^l$ are the coefficients of the Weyl connection

$$\Gamma_{ji}^l = \left\{ \frac{l}{ji} \right\} - (w_j \delta_i^l + w_i \delta_j^l - g_{ij} w^l)$$  \hspace{1cm} (1.9)

which is symmetric with respect to lower indices. In (1.9), $\left\{ \frac{l}{ji} \right\}$ are the coefficients of the metric connection $\nabla$.

The covariant curvature tensor, the Ricci curvature tensor and the scalar curvature of Weyl space are defined by respectively,

$$W_{kji}^h = W_{kji} g_{lh},$$  \hspace{1cm} (1.10)

$$W_{kji} = W_{kji} m g_{ml},$$  \hspace{1cm} (1.11)

$$W_{ji} = g^{kl} W_{kji} = W_{kji}^k,$$  \hspace{1cm} (1.12)

$$W = g^{ji} W_{ji}.$$  \hspace{1cm} (1.13)
It is also observed that the covariant curvature tensor $W_{ijkl}$ of $W_n$ satisfies the following relations \[1\]

1) $W_{ijkl} + W_{jikl} = 0,$ \hspace{1cm} (1.14)
2) $W_{ijkl} + W_{ijlk} = 2g_{kl}(w_{i,j} - w_{j,i}),$ \hspace{1cm} (1.15)
3) $W_{lijk} + W_{ijlk} + W_{jilk} = 0.$ \hspace{1cm} (1.16)

Furthermore, it can be seen that the anti-symmetric part of the Ricci curvature tensor satisfies the relation \[6\]

$$W_{[ij]} = n\nabla_{[i}w_{j]}.$$ \hspace{1cm} (1.17)

The Ricci curvature tensor is not necessarily symmetric on $W_n$ since the Weyl connection is not metric.

We here quote some definitions of \[2\], and \[7\].

Let $W_n$ be a Weyl space of dimension $n = 2m (m \geq 1)$. A tensor $F^i_j$ of type $(1,1)$ with weight \{0\} is called an almost complex structure on $W_n$ if the tensor $F^i_j$ satisfies the condition

$$F^i_j F^k_j = -\delta^k_i,$$ \hspace{1cm} (1.18)

and $W_n$ is called an almost complex Weyl space. If $W_n$ admits an almost complex structure $F^i_j$ satisfying the condition

$$g_{ij} F^i_h F^j_k = g_{hk},$$ \hspace{1cm} (1.19)

then $F^i_j$ is called an almost Hermitian structure with a Hermitian metric $g$ on $W_n$. It is said that an almost Hermitian structure $F^i_j$ is a Kaehlerian structure (respectively, space) if $F^i_j$ satisfies the condition

$$\nabla_k F^i_j = 0 \hspace{1cm} \text{for all } i, j, k,$$ \hspace{1cm} (1.20)

then $W_n$ becomes a Kaehlerian space which we denote by $KW_n$.

An almost Hermitian structure $F^i_j$ is called an almost Kaehlerian structure on $W_n$ (respectively, almost Kaehlerian Weyl space) if the tensor $F^i_j$ satisfies

$$F_{hij} = \nabla_h F_{ij} + \nabla_i F_{jh} + \nabla_j F_{hi} = 0.$$ \hspace{1cm} (1.21)

An almost Hermitian structure $F^i_j$ is called an almost semi-Kaehlerian structure on $W_n$ (respectively, almost semi-Kaehlerian Weyl space denoted by $SKW_n$) if the tensor $F^i_j$ satisfies

$$F_i = \nabla_j F^j_i = 0.$$ \hspace{1cm} (1.22)
The contravariant and covariant tensors $F_{ij}$ and $F^{ij}$ are of weight $\{2\}$ and $\{-2\}$, respectively, and defined by means of metric tensor as

\begin{align*}
F_{ij} &= g_{jk} F^k_i, \\
F^{ij} &= g^{ih} F^j_h.
\end{align*}

(1.23) (1.24)

So it can be seen easily that,

\begin{align*}
F_{ij} &= -F_{ji}, \\
F^{ij} &= -F^{ji}.
\end{align*}

(1.25) (1.26)

The Nijenhuis torsion tensor of the Kaehlerian structure $F^h_i$ on $W_n$ with the weight $\{0\}$ is defined by

\begin{align*}
N_{ij}^k &= F^h_i(\hat{\nabla}_h F^k_j - \hat{\nabla}_j F^k_h) - F^h_j(\hat{\nabla}_h F^k_i - \hat{\nabla}_i F^k_h),
\end{align*}

(1.27)

and it is said that an almost complex structure is integrable on $W_n$ if it has no torsion [2].

If we take the complementary vector field $w_k = 0$ in the prolonged covariant derivative in (1.5) then the above definitions reduce to those of Riemannian spaces [4, 8].

Now, we give some definitions of nearly-Kaehlerian structures and almost $L$-structures on $W_n$ and some theorems concerning these structures.

2. Almost $L$-structures and nearly-Kaehlerian structures on Weyl spaces

An almost Hermitian structure $F^j_i$ of weight $\{0\}$ with a Hermitian metric $g_{ij}$ on $W_n$ satisfying

\begin{align*}
G_{ij}^k &\equiv \hat{\nabla}_i F^k_j + \hat{\nabla}_j F^k_i = 0,
\end{align*}

(2.1)

is called a nearly-Kaehlerian structure or an almost Tachibana structure (respectively, nearly-Kaehlerian Weyl space denoted by $\text{NK}W_n$).

An almost Hermitian structure $F^j_i$ of weight $\{0\}$ on $W_n$ satisfying

\begin{align*}
[\hat{\nabla}_j, \hat{\nabla}_k] F^h_i &\equiv (\hat{\nabla}_j \hat{\nabla}_k - \hat{\nabla}_k \hat{\nabla}_j) F^h_i = 0,
\end{align*}

(2.2)

is called an almost $L$-structure and a Weyl space admitting an almost $L$-structure is called almost $L$-Weyl space denoted by $\text{LW}_n$. 

Since an almost Hermitian structure $F^j_i$ on $KW_n$ satisfies $\nabla_k F^j_i = 0$, for all $i, j, k$, then the Kaehlerian structure (respectively, the space $KW_n$) is an almost $L$-structure (respectively, the space $LW_n$).

**Theorem 2.1.** A nearly-Kaehlerian structure (respectively, the space $NKW_n$) is an almost semi-Kaehlerian Weyl structure (respectively, the space $NKW_n$).

**Proof.** In (2.1), contracting with respect to $i$ and $k$ gives

$$\nabla^i_i F^j_i = -\nabla^i_j F^j_i. \quad (2.3)$$

On the other hand, from (1.21), we get $F^m_i = F^m_{ij} g^{jm}$ and contracting with respect to $m$ and $i$, we obtain $F^i_i = F^j_i g^{ji} = 0$. Hence, it follows that $\nabla^i_i F^j_i = 0$, which shows that the structure $F^j_i$ is an almost semi-Kaehlerian Weyl structure.

**Theorem 2.2.** A nearly-Kaehlerian Weyl space is a Kaehlerian-Weyl space if the structure is integrable.

**Proof.** Using the Nijhenuis torsion tensor of the nearly-Kaehlerian structure $F^h_i$ we obtain on $W_n$ [1]

$$N^k_{ij} = F^h_i (\nabla^h_j F^k_j - \nabla^h_j F^i_k) - F^h_j (\nabla^h_i F^k_i - \nabla^h_i F^j_k), \quad (2.4)$$

and also using the defining condition (2.1) we obtain

$$\nabla^h_i F^j_k = -\nabla^h_j F^i_k \quad (2.5)$$

and

$$\nabla^h_i F^j_k = -\nabla^h_j F^i_k. \quad (2.6)$$

Replacing (2.6) in (2.4) we obtain

$$N^k_{ij} = -2 F^h_i \nabla^j_k F^i_k + 2 F^h_j \nabla^i_k F^j_k. \quad (2.7)$$

Since

$$F^h_j F^i_k = -\delta^k_j, \quad (2.8)$$

taking the prolonged covariant derivative of (2.8) gives

$$(\nabla^h_i F^k_j) F^h_j = -(\nabla^h_i F^j_h) F^k_h. \quad (2.9)$$

Similarly, taking the prolonged covariant derivative of

$$F^h_i F^k_j = -\delta^k_i, \quad (2.10)$$
yields
\[(\hat{\nabla}_j F^k_h) F^i_h = - (\hat{\nabla}_j F^i_h) F^k_h \] (2.11)
and by means of (2.1) we obtain
\[(\hat{\nabla}_j F^k_h) F^i_h = - (\hat{\nabla}_j F^i_h) F^k_h = (\hat{\nabla}_i F^j_h) F^k_h. \] (2.12)
On the other hand, substituting (2.9) and (2.12) in (2.4) we conclude that
\[N_{ij}^k = - 4(\hat{\nabla}_i F^j_h) F^k_h. \] (2.13)
By assumption, since the nearly-Kaehlerian structure is integrable, the Nijenhuis torsion tensor \(N_{ij}^k\) is zero. Hence, we obtain
\[(\hat{\nabla}_i F^j_h) F^k_h = 0. \] (2.14)
If we multiply (2.14) by \(F^m_k\), and use (1.18) we thus obtain
\[\hat{\nabla}_i F^m_j = 0 \text{ for all } i, j, m, \] (2.15)
which gives that the structure \(F_i^j\) is Kaehlerian.

We can express the defining condition (2.2) of an almost \(L\)-structure on \(LW_n\) in terms of the Weyl and Ricci curvature tensors.

**Theorem 2.3.** An almost Hermitian structure \(F_i^j\) is an almost \(L\)-structure on \(LW_n\) if and only if
\[F_i^a W_{kja}^h = F_a^h W_{kji}^a. \] (2.16)

**Proof.** Since the almost Hermitian structure \(F_i^j\) is a tensor of weight \(\{0\}\), we get
\[\hat{\nabla}_k \hat{\nabla}_j F_i^h - \hat{\nabla}_j \hat{\nabla}_k F_i^h = \nabla_k \nabla_j F_i^h - \nabla_j \nabla_k F_i^h, \] (2.17)
and by using the Ricci identity [3] we have
\[(\nabla_k \nabla_j - \nabla_j \nabla_k) F_i^h = F_i^a W_{kja}^h - F_a^h W_{kji}^a. \] (2.18)
If an almost Hermitian structure \(F_i^j\) is \(L\)-structure on \(LW_n\), then making use of (2.2) we obtain
\[(\nabla_k \nabla_j - \nabla_j \nabla_k) F_i^h = F_i^a W_{kja}^h - F_a^h W_{kji}^a = 0, \] (2.19)
from which it follows that
\[F_i^a W_{kja}^h = F_a^h W_{kji}^a. \] (2.20)
Conversely, suppose (2.16) holds. Then we have
\[
F^a_i W^h_{kja} - F^h_a W^a_{kji} = \nabla_k \nabla_j F^h_i - \nabla_j \nabla_k F^h_i = 0,
\]
which shows that the structure $F^j_i$ is an almost $L$-structure on $LW_n$.

**Theorem 2.4.** On an almost Hermitian Weyl space condition (2.16) is equivalent to
\[
W^h_{kijh} = F^l_j F^t_i W^a_{kilt}.
\]

**Proof.** Suppose (2.16) holds. Multiplying (2.16) by $F^m_i$ gives
\[
W^h_{kjm} = -F^h_a F^m_i W^a_{kji}.
\]
First, multiplying (2.23) by $g_{ht}$ and then using (1.23) and (1.25), respectively, we obtain
\[
W^l_{kjmt} = -F^l_i F^m_i W^a_{kji},
\]
which becomes
\[
W^j_{kijh} = F^l_j F^t_i W^a_{kilt}.
\]
Conversely, suppose that (2.22) holds then, multiplying (2.22) by $F^m_i$, we conclude that
\[
F^m_i W^k_{jim} = -F^p_i W^p_{kjup}.
\]
Multiplying $g^{mn}$ and using (1.11) we obtain
\[
F^m_i W^k_{jim} = g^{pn} F^p_m W^p_{kjup},
\]
and
\[
F^m_i W^k_{jim} = F^p_m W^p_{kjup},
\]
which implies that (2.16) holds.

**Remark.** It can be shown that the covariant curvature tensor $W_{kjim}$ of $W_n$ satisfies the following relation:
\[
W_{kjim} - W^s_{kmkj} = g_{mk}(w_{i,j} - w_{j,i}) + g_{mj}(w_{k,i} - w_{i,k}) + g_{ji}(w_{m,k} - w_{k,m}) + g_{ik}(w_{j,m} - w_{m,j}) + g_{im}(w_{k,j} - w_{j,k}) + g_{kj}(w_{i,m} - w_{m,i}).
\]
where $w_{i,j} - w_{j,i}$ stands for $w_{i,j} - w_{j,i} = \nabla_j w_i - \nabla_i w_j = \nabla_j w_i - \nabla_i w_j = \partial_j w_i - \partial_i w_j$. 


We quote the following theorem from [2].

**Theorem.** If an almost semi-Kaehlerian structure $F^i_j$ satisfies

$$
\nabla_k F^i_j \nabla^i F^j_k = a \nabla_k F^i_j \nabla^i F^i_j,
$$

(2.31)

where $a$ is a non-zero constant, then $Q$ defined by

$$
Q \equiv W + \frac{1}{2} F^{ij} F^{kl} W_{ijkl} + 2g^{kj} (\nabla_j w_k - \nabla_k w_j)
$$

(2.32)

is non-negative (non-positive) according as $a$ is positive (negative).

Further, the structure $F^i_j$ is Kaehlerian on $W_n$ if and only if $Q = 0$. In [2], it is shown that

$$
Q = a (\nabla_k F^i_j)(\nabla^k F^i_j),
$$

(2.33)

where $\nabla^k = g^{jk} \nabla_j$.

In particular, $Q = 0$ if and only if the structure is Kaehlerian on $W_n$.

The following theorem gives a relation between a nearly-Kaehlerian structure on $NKW_n$ and a Kaehlerian structure on $KW_n$.

**Theorem 2.5.** For a nearly-Kaehlerian structure $F^i_j$ on $NKW_n$,

$$
Q \geq 0,
$$

(2.34)

where $Q$ is given by (2.32) and the equality holds if and only the structure $F^i_j$ is Kaehlerian.

**Proof.** Since a nearly-Kaehlerian structure $F^i_j$ is an almost semi-Kaehlerian structure, multiplying (2.1) with $g^{im}$ we obtain

$$
\nabla_i F^{mk} + \nabla^m F^i_k = 0
$$

(2.35)

and also multiplying (2.35) with $g^{in}$ gives

$$
\nabla^n F^{mk} = -\nabla^m F^{mk}.
$$

(2.36)

Making use of (2.2), (2.33) and (2.36) we obtain

$$
\nabla_k F^{ij} (\nabla^i F^j_k) = (\nabla_k F^i_j) (-\nabla^i F^{kj}) = (\nabla_k F^i_j) (\nabla^k F^{ij}),
$$

(2.37)

from which it follows that $a = 1$.

Also, we observe that the equality in (2.34) holds if and only if the structure $F^i_j$ is Kaehlerian.
Theorem 2.6. If an almost $L$-structure $F^j_i$ on $LW_n$ is almost semi-Kaehlerian and satisfies either

$$\hat{\nabla}^k G^j_{hi} = 0$$  \hspace{1cm} (2.38)

or

$$\hat{\nabla}^k F_{hij} = 0,$$  \hspace{1cm} (2.39)

where $\hat{\nabla}^k = g^{jk} \nabla_j$, $F_{hij}$, and $G^j_{hi}$ are defined in (1.21), and (2.1), respectively, then the structure $F^j_i$ is Kaehlerian.

Proof. From the defining condition (2.2) of an almost $L$-structure it follows that

$$\hat{\nabla}^i \hat{\nabla}_h F^j_i = \hat{\nabla}_h \hat{\nabla}^i F^j_i.$$  \hspace{1cm} (2.40)

Multiplying (2.40) by $g^{ik}$ gives

$$\hat{\nabla}^i \hat{\nabla}_h F^j_i = \hat{\nabla}_h \hat{\nabla}^i (\hat{\nabla}^k F^j_k + \hat{\nabla}^i F^j_i) = -g^{ij} \hat{\nabla}_h \hat{\nabla}^k F^j_k.$$  \hspace{1cm} (2.41)

Since $F^j_i$ is an almost semi-Kaehlerian structure, it satisfies $\hat{\nabla}^k F^j_k = 0$ then (2.41) reduces to

$$\hat{\nabla}^i \hat{\nabla}_h F^j_i = 0.$$  \hspace{1cm} (2.42)

i) Suppose (2.38) holds. Multiplying (2.38) by $\delta_k^i$ we obtain

$$\delta_k^i \hat{\nabla}^k G^j_{hi} = \delta_k^i \hat{\nabla}^k (\hat{\nabla}^h F^j_i + \hat{\nabla}^i F^j_h) = 0$$  \hspace{1cm} (2.43)

and using (2.42), (2.43) we find

$$\hat{\nabla}^i \hat{\nabla}_h F^j_i = 0.$$  \hspace{1cm} (2.44)

On the other hand, since an almost Hermitian structure $F^j_i$ satisfying

$$F^{ij} \hat{\nabla}^k \hat{\nabla}_k F^{ij} = 0$$  \hspace{1cm} (2.45)

is Kaehlerian [1], thus (2.44) shows that the structure $F^j_i$ is Kaehlerian.

ii) Suppose (2.39) holds. Taking the prolonged covariant derivative of (2.39) gives

$$\hat{\nabla}^k F_{hij} = \hat{\nabla}^k \hat{\nabla}_h F_{ij} + \hat{\nabla}^k \hat{\nabla}_i F_{jh} + \hat{\nabla}^k \hat{\nabla}_j F_{hi} = 0.$$  \hspace{1cm} (2.46)

Multiplying (2.46) by $\delta_k^i$ and using (1.23), respectively, we obtain

$$\hat{\nabla}^i \hat{\nabla}_h F_{ij} + \hat{\nabla}^i \hat{\nabla}_i F_{jh} + \hat{\nabla}^i \hat{\nabla}_j F_{hi} = 0$$  \hspace{1cm} (2.47)

and then

$$g_{jm} \hat{\nabla}^i \hat{\nabla}_h F^m_i + \hat{\nabla}^i \hat{\nabla}_i F_{jh} - g_{hm} \hat{\nabla}^i \hat{\nabla}_j F^m_i = 0.$$  \hspace{1cm} (2.48)
By means of (2.42), (2.48) yields

\[ \dot{\nabla}^i \dot{\nabla}_i F_{jh} = 0. \]  

(2.49)

Using (2.44), from (2.49) it follows that the structure is Kaehlerian.

References


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