Right Derivations on Semirings

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Abstract

Motivated by some works on derivations on rings, Chandramoulesswaran and Thiruveni discussed the notion of derivations on semirings. In this paper, we discuss the notion of right derivations on semirings and prove some simple properties.

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1 Introduction

Richard Dedekind used the notion of semiring in his study of ideals without giving the semiring a formal definition. However, H.S. Vandiver,[5] gave a formal definition and introduced the notion of semiring in 1934. A natural
example of a semiring which is not a ring is the set of natural numbers in $\mathbb{N}$ under usual addition and multiplication of numbers.

Even though, the study of derivations in rings was initiated long back, it got its significance only after Posner [4] who in 1955 established two very striking results. The notion of derivation has also been generalized in various directions. In [3], Jonathan Golan mentioned about the derivation on a semiring. However, nothing more has been said on it. This motived Chandramouleeswaran and Thiruveni to introduce and discuss the notion of derivation on a semiring and its properties[2]. In 1990, Bresar and Vukman [1] firstly introduced the notion of a left- derivation in a ring and proved that a left- derivation of a semiprime ring $R$ must map $R$ into its center. Motivated by this, in this paper, we introduce the notion of right-derivation on a semiring $S$ and prove some simple properties.

2 Preliminaries

In this section we recall some basic definitions on semirings and derivations on it, that are needed for our work.

**Definition 2.1** A semiring $(S, +, \cdot)$ is an algebraic system with a nonempty set $S$ together with two binary operations $+$ and $\cdot$ such that

1. $(S, +)$ is a semi group
2. $(S, \cdot)$ is a semi group
3. For all $a, b, c \in S$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ hold.

**Definition 2.2** A semiring $(S, +, \cdot)$ is said to be additively commutative if $(S, +)$ is a commutative semi group. A semiring $(S, +, \cdot)$ is said to be multiplicatively commutative if $(S, \cdot)$ is a commutative semi group. It is said to be commutative if both $(S, +)$ and $(S, \cdot)$ are commutative.

**Definition 2.3** The semiring $(S, +, \cdot)$ is said to be a semiring with zero, if it has an element $0$ in $S$ such that $x + 0 = x = 0 + x$ and $x \cdot 0 = 0 = 0 \cdot x \ \forall x \in S$.

**Definition 2.4** A semiring $(S, +, \cdot)$ is said to be a semiring with an identity element $1$, if there exists an element $1 \neq 0 \in S$ such that $1 \cdot x = x = x \cdot 1 \ \forall x \in S$.

**Definition 2.5** Let $(S, +, \cdot)$ be a semiring. An element $\alpha$ of $S$ is called additively left cancellative if for all $\alpha, \beta, \gamma \in S, \alpha + \beta = \alpha + \gamma \Rightarrow \beta = \gamma$. If every element of a semiring $S$ is additively left cancellative, it is called an additively left cancellative semiring.
Analogously, one can define an additively right cancellative semiring.

**Definition 2.6** A semiring $(S, +, \cdot)$ is said to be additively cancellative if it is both additively left and right cancellative.

**Definition 2.7** Let $S$ be a semiring. A left $S$–semimodule is a commutative monoid $(M, +, 0_M)$ in which scalar multiplication $S \times M \to M$, denoted by $(s, m) \to sm$, satisfies the following conditions

1. $(ss')m = s(s'm)$
2. $s(m + m') = sm + sm'$
3. $(s + s')m = sm + s'm$
4. $1_S m = m$
5. $s0_M = 0_M = 0_S m \quad \forall s, s' \in S \quad \forall m, m' \in M.$

If $V(M) = M$ then $M$ is an $S$–module where $V(M)$ is the set of all elements of $M$ having additive inverse.

**Definition 2.8** Let $S$ be a semiring.

1. $S$ is said to be prime if $aSb = 0 \Rightarrow a = 0$ or $b = 0$.
2. $S$ is said to be semiprime if $aSa = 0 \Rightarrow a = 0$.
3. $S$ is said to be 2 torsion free if $2a = 0, a \in S \Rightarrow a = 0$.

**Definition 2.9** Let $(S, +, \cdot)$ be a semiring and $X$ be $S$ module. A derivation on $S$ is a map $d : S \to X$ satisfying the following conditions

1. $d(x + y) = d(x) + d(y) \quad \forall x, y \in S$
2. $d(xy) = d(x)y + xd(y), \quad \forall x, y \in S.$

**Example 2.10** Let $S$ be a semiring. Let $M_2(S) = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} | a, b, c \in S. \right\}$

Define $d : M_2(S) \to M_2(S)$ is given by $d \left[ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Then $d$ is a derivation on $M_2(S)$.  

3 Right-Derivations

In this section, we introduce the notion of right-(left-)derivations on semirings, to illustrate the concept, discuss examples and prove some simple properties.

Definition 3.1 Let \( S \) be a semiring and \( X \) be \( S \)-module. An additive map \( d_L : S \to X \) is said to be a left-derivation if \( d_L(xy) = xd_L(y) + yd_L(x) \ \forall \ x, y \in S \).

Example 3.2 Let \( S \) be commutative semiring with characteristic 4. Let \( M_2(S) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in S \right\} \).

The map \( d_L : M_2(S) \to M_2(S) \) given by

\[
d_L \left[ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \right] = \begin{pmatrix} 0 & 2b \\ 0 & 0 \end{pmatrix}
\]

is a left-derivation but not a derivation on \( M_2(S) \).

Analogously we can define a right derivation on \( S \) as follows.

Definition 3.3 Let \( S \) be a semiring and \( X \) be a right-\( S \)-module. An additive map \( d_R : S \to X \) is said to be a right-derivation if \( d_R(xy) = d_R(x)y + d_R(y)x \ \forall \ x, y \in S \).

Example 3.4 Let \( S \) be commutative semiring with characteristic 4. Let \( M_2(S) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in S \right\} \).

The map \( d_R : M_2(S) \to M_2(S) \) given by

\[
d_R \left[ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 \\ 2b & 0 \end{pmatrix}
\]

is a right-derivation but not a derivation on \( M_2(S) \).

One can easily prove the following.

Lemma 3.5

1. Sum of two left derivations on an additively commutative semiring is again a left derivation.

2. Sum of two right derivations on an additively commutative semiring is again a right derivation.

Lemma 3.6 Let \( S \) be a semiring, \( X \) be a right-\( S \)-module and \( d_R : S \to X \) be a right-derivation. For any element \( a \in S \), \( d_R(a^n) = nd_R(a)a^{n-1} \).
Proof: We prove this result by induction hypothesis to prove this result.

Take $n = 2$. Then
\[ d_R(a^2) = d_R(a \cdot a) = d_R(a)a + d_R(a)a = 2ad_R(a)a. \]
Assume that $d_R(a^n) = nd_R(a)a^{n-1}$. Now,
\[
\begin{align*}
  d_R(a^{n+1}) &= d_R(a^n \cdot a) \\
  &= d_R(a^n)a + d_R(a)a^n \\
  &= [nd_R(a)a^{n-1}]a + d_R(a)a^n \\
  &= nd_R(a)a^n + d_R(a)a^n \\
  &= (n+1)d_R(a)a^{n+1}
\end{align*}
\]
\[ d_R(a^n) = nd_R(a)a^{n-1} \quad \forall \ n > 0. \]

Analogously we can prove the following.

**Lemma 3.7** Let $S$ be a semiring, $X$ be a left-$S$-module and $d_L : S \rightarrow X$ be a Left-derivation. For any element $a \in S$, $d_L(a^n) = na^{n-1}d_L(a)$.

**Theorem 3.8** Let $S$ be a semiring and $X$ be a 2-torsion free right-$S$-module. If $d_R : S \rightarrow X$ is a right-derivation then $a, b, c \in R$, then

1. $d_R(ab + ba) = 2d_R(b)a + 2d_R(a)b.$
2. $d_R(aba) = d_R(b)a + 3d_R(a)ba - d_R(a)ab.$
3. $d_R(abc + cba) = d_R(a)(cb + bc) + 3d_R(b)ca + 3d_R(c)ba - d_R(c)ab - d_R(b)ac.$
4. $d_R(a)(ab - ba) = d_R(a)(ab - ba)a.$
5. $(ba - ab)(d_R(ba) - d_R(a)b - d_R(b)a) = 0.$

Proof:
\[
\begin{align*}
  d_R(ab) &= d_R(a)b + d_R(b)a \\
  d_R(ab + ba) &= d_R(ab) + d_R(ba) \\
  &= d_R(a)b + d_R(b)a + d_R(b)a + d_R(a)b \\
  &= 2d_R(a)b + 2d_R(b)a \quad \cdots (1)
\end{align*}
\]
Replace $b$ by $ab + ba$ in (1),
\[
\begin{align*}
  d_R(a(ab + ba)) &= 2d_R(a)(ab + ba) + 2d_R(ab + ba)a \\
  &= 2d_R(a)ab + 6d_R(a)ba + 4d_R(b)a^2 \quad \cdots (2)
\end{align*}
\]
\[
\begin{align*}
  d_R(a^2b + aba + aba + ba) &= d_R(a^2b) + 2d_R(aba) + d_R(ba^2) \\
  &= d_R(a^2)b + d_R(b)a^2 + 2d_R(aba) + d_R(b)a^2 + d_R(a^2)b \\
  &= 2d_R(b)a^2 + 4d_R(a)ab + 2d_R(aba) \quad \cdots (3)
\end{align*}
\]
From (2) and (3)

\[
2d_R(b)a^2 + 4d_R(a)ab + 2d_R(aba) = 2d_R(a)ab + 6d_R(a)ba + 4d_R(b)a^2
\]

\[
2d_R(aba) = 2d_R(b)a^2 - 2d_R(a)ab + 6d_R(a)ba
\]

\[
d_R(aba) = d_R(b)a^2 + 3d_R(a)ba - d_R(a)ab.
\]

Replace \( b \) by \( cb + bc \) in (1)

\[
d_R(a(cb + bc) + (cb + bc)a) = 2d_R(a)(cb + bc) + 2d_R(cb + bc)a
\]

\[
= 2d_R(a)cb + 2d_R(a)bc + 4d_R(c)ba + 4d_R(b)ca \cdots \cdots (4)
\]

\[
d_R(acb + abc + cba + bca) = d_R(abc) + d_R(abc + cba) + d_R(bca)
\]

\[
= d_R(ac)b + d_R(b)ac + d_R(abc + cba) + d_R(bc)a + d_R(a)bc
\]

\[
= d_R(a)cb + d_R(c)ab + d_R(b)ac + d_R(abc + cba) +
\]

\[
d_R(b)ca + d_R(c)ba + d_R(a)bc \cdots \cdots (5)
\]

From (4) and (5)

\[
2d_R(a)cb + 2d_R(a)bc + 4d_R(c)ba + 4d_R(b)ca
\]

\[
= d_R(a)cb + d_R(c)ab + d_R(b)ac +
\]

\[
d_R(abc + cba) = d_R(a)(cb + bc) + 3d_R(b)ca + 3d_R(c)ba - d_R(c)ab - d_R(b)ac \cdots \cdots (6)
\]

Replace \( c \) by \( ba \) in (6)

\[
d_R(abba + baba) = d_R(a)(bab + bba) + 3d_R(b)baa +
\]

\[
3d_R(ba)ba - d_R(b)aba - d_R(b)aba
\]

\[
= d_R(a)(ba + b^2a) + 3d_R(b)ba^2 +
\]

\[
3d_R(ba)ba - d_R(b)aba - d_R(b)aba \cdots \cdots (7)
\]

\[
d_R(ab^2a + (ba)^2) = d_R(ab^2a) + d_R(ba)^2
\]

\[
= d_R(ab^2a) + d_R(a)ab^2 + 2d_R(ba)ba
\]

\[
= d_R(a)b^2a + d_R(b)ba^2 + d_R(a)ab^2 + 2d_R(ba)ba \cdots \cdots (8)
\]

From (7) and (8)

\[
d_R(a)bab + d_R(a)b^2a + 3d_R(b)ba^2 + 3d_R(ba)ba - d_R(aba)b - d_R(b)ab
\]

\[
= d_R(a)b^2a + 2d_R(b)ba^2 + d_R(a)ab^2 + 2d_R(ba)ba
\]

\[
d_R(ba)bab - d_R(ba)ab = d_R(b)aba - d_R(a)bab - d_R(a)b^2a - 3d_R(b)ba^2 +
\]

\[
d_R(a)b^2a + 2d_R(b)ba^2 + d_R(a)ab^2
\]

\[
d_R(ba)(ba - ab) = d_R(b)(ab - ba)a + d_R(a)(ab - ba)b \cdots \cdots (9)
\]
Proof: Let $S$ be an additively commutative and cancellative semiring.

**Theorem 3.9** Let $S$ be a semiring and $X$ be a 2-torsion free left $S$-module. If \( d_L : S \to X \) is a left-derivation then \( a, b, c \in R \), then

1. \( d_L(ab + ba) = 2ad_L(b) + 2bd_L(a) \).
2. \( d_L(aba) = a2d_L(b) + 3abd_L(a) - bad_L(a) \).
3. \( d_L(abc + cba) = 3abd_L(c) + 3acd_L(b) + (bc + cb)d_L(a) - bad_L(c) - cad_L(b) \).
4. \( (ab - ba)ad_L(a) = a(ba - ab)d_L(a) \).
5. \( (ab - ba)(d_L(ab) - ad_L(b) - bd_L(a)) = 0 \).

**Theorem 3.10** Let $S$ be an additively commutative and cancellative semiring. Let $X$ be a right $S$-module. Let \( d_R : S \to X \) be a non zero right-derivation. Suppose that $xSa = 0$ with $a \in S, x \in X$ \( \Rightarrow a = 0 \) or $x = 0$. Then $S$ is commutative.
Let \(d_R : S \rightarrow X\) be a non zero right-derivation on \(S\).

\[
d_R(aba) = d_R(a)ba + d_R(ba)a
= d_R(a)ba + d_R(b)a^2 + d_R(a)ba
= 2d_R(a)ba + d_R(b)a^2 \quad \cdots \cdots (1)
\]

\[
d_R(aba) = d_R(ab)a + d_R(a)ab
= d_R(a)ba + d_R(b)a^2 + d_R(a)ab
= d_R(a)(ba + ab) + d_R(b)a^2 \quad \cdots \cdots (2)
\]

From (1) and (2) \(2d_R(a)ba + d_R(b)a^2 = d_R(a)(ab + ba) + d_R(b)a^2 \quad \cdots \cdots (3)\)

Replacing \(b\) by \(cb\), and using (3) we get

\[
d_R(a)[a, b] = 0 \quad \forall \ a, b \in S \quad \cdots \cdots (3)
\]

Since \(d_R \neq 0\), we get \([a, b] = 0 \quad \forall \ a, b \in S\), thus proving that \(S\) is commutative.

Analogously we can prove the following.

**Theorem 3.11** Let \(S\) be an additively commutative and cancellative semiring. Let \(X\) be a left \(S\)-module. Let \(d_L : S \rightarrow X\) be a non zero left derivation. Then \(S\) is commutative.

**References**


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