Subperiodic Rings with Conditions on Extended Commutators

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Abstract

Let R be a ring with Jacobson radical J and with center C . Let P be the set of potent elements x for which $x^k = x$ for some integer $k > 1$. Let N be the set of nilpotents. A ring R is called *subperiodic* if $R \setminus (J \cup C) \subseteq N + P$. We consider the commutativity behavior of a subperiodic ring with some constraint involving extended commutators.

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Throughout R is a ring with nilpotents N , and with Jacobson radical J and center C . The extended commutator $[x, y]_k$ is defined inductively by $[x, y]_1 = [x, y] = xy - yx$ and $[x, y]_k = [[x, y]_{k-1}, y]$ for all positive integers $k > 1$. An element x is called *potent* if $x^k = x$ for some integer $k > 1$. The ring R is called *subperiodic* if $R \setminus (J \cup C) \subseteq N + P$, where P is the set of potent elements.

In preparation for the proofs of the main theorems, we state the following lemmas.

Lemma 1 ([1]). *Suppose R is a ring in which each element is central or potent. Then R is commutative.*

Lemma 2 ([2]). *Let R be a ring such that both J and R/J are commutative. Then N is an ideal and R/N is commutative.*

Lemma 3 ([3]). *Suppose that R is a ring such that for every element x in R , there exists an integer $n > 1$ such that $x - x^n$ is in the center of R . Then R is commutative.*

Lemma 4. *Suppose R is any ring such that $[x, y]$ commutes with x . Then, for all positive integers n , $[x^n, y] = nx^{n-1}[x, y]$.*

This lemma is readily proved by induction.

Lemma 5. *Suppose R is a subperiodic ring, and suppose that $\sigma : R \rightarrow R_i$ is a homomorphism of R onto a ring R_i . Suppose, further, that the set N of nilpotents of R is an ideal. Then, the set N_i of nilpotents of R_i is contained in $\sigma(J) \cup C_i$, where C_i is the center of R_i .*

Proof. By contradiction. Suppose $d_i \in N_i$, $d_i \notin \sigma(J)$, $d_i \notin C_i$. Let $\sigma(d) = d_i$, ($d \in R$). Then, $d \notin (J \cup C)$, and hence $d = a + b$, $a \in N$, b potent with $b^k = b$ for some integer $k > 1$. Hence,

$$d - a = b = b^k = (d - a)^k.$$

Since N is an ideal and $a \in N$, we conclude that $d - d^k \in N$, and hence

$$\text{For all } d \in R \setminus (J \cup C), d - d^k \in N, k > 1. \quad (1)$$

Let $d_i^n = 0$ (since $d_i \in N_i$). Since $d \notin (J \cup C)$, therefore by (1), $d - d^k \in N$, $k > 1$. Since N is an ideal,

$$(d - d^k) + d^{k-1}(d - d^k) + \cdots + (d^{k-1})^{n-1}(d - d^k) \in N,$$

which implies (as is readily verified)

$$d - d^{n+1}d^{n(k-2)} \in N. \quad (2)$$

Therefore, by (2),

$$d_i - d_i^{n+1}d_i^{n(k-2)} \in \sigma(N). \quad (3)$$

Since $d_i^n = 0$, (3) implies that $d_i \in \sigma(N) \subseteq \sigma(J)$ ($N \subseteq J$ follows from the fact that N is an ideal). So $d_i \in \sigma(J)$, contradiction. This proves the lemma. $\square$

Lemma 6. *Suppose R is a subperiodic ring with central idempotents, and suppose N is an ideal. Let R_i be a subdirectly irreducible ring, and let $\sigma : R \rightarrow R_i$ be an onto homomorphism. Then R_i is of one of the following two types:*

Type 1: $R_i = \sigma(J) \cup C_i$, $C_i =$ center of R_i , or

Type 2: $R_i = \sigma(J) \cup C_i \cup U_i$, $U_i =$ the set of units of R_i .

(Type 1 holds when $1 \notin R_i$, while type 2 holds when $1 \in R_i$.)

Proof. Write R as a subdirect sum of subdirectly irreducible rings R_i . By (1), we see that for all $d \in R$, $d \in J$ or $d \in C$ or $d - d^k \in N$, $k > 1$. Moreover, if $d - d^k \in N$, say $(d - d^k)^q = 0$, then $d^q = d^{q+1}g(d)$ for some $g(x) \in \mathbb{Z}[x]$, and hence

$$d^q = d^{q+1}g(d) = d^q(dg(d)) = d^q(dg(d))^2 = \dots = d^q e,$$

where $e = (dg(d))^q$ is an idempotent, and thus by (1), we see that

$$\text{For all } d \in R, d \in J \text{ or } d \in C \text{ or } d^q = d^q e, e^2 = e \in d\mathbb{Z}[d], e \in C. \quad (4)$$

In view of the homomorphism $\sigma : R \rightarrow R_i$, let $x_i \in R_i$ and let $\sigma(x) = x_i$, ($x \in R$). By hypothesis, any idempotent $e \in R$ is central in R , and hence $e_i = \sigma(e)$ is central in R_i . By (4), we conclude that

$$\text{For all } d_i \in R_i, d_i \in \sigma(J) \text{ or } d_i \in C_i \text{ or } d_i^q = d_i^q e_i, e_i^2 = e_i \in d_i\mathbb{Z}[d_i], e_i \in C_i. \quad (5)$$

Since e_i is a *central* idempotent in the subdirectly irreducible ring R_i , $e_i = 0$ or $e_i = 1$. If R_i does *not* have an identity, then by (5) and Lemma 5, R_i is as described in type 1 (since $e_i = 0$). On the other hand, if $1 \in R_i$, then R_i is as described in type 2, since $e_i \in d_i\mathbb{Z}[d_i]$. This proves the lemma, since in the latter case, $e_i = (d_i g(d_i))^q = 1$ implies that d_i is indeed a unit in R_i . $\square$

We are now in a position to prove the main theorems.

Theorem 1. *Suppose R is a subperiodic ring with Jacobson radical J , and suppose that J is commutative. Suppose, further, that for all x, y in R , there exist positive integers k, l, m depending on x, y such that*

$$x^k y^l [x, y]_m \text{ is in the center } C, (x, y \text{ in } R). \quad (6)$$

Then N is an ideal and R/N is commutative, and thus R has a nil commutator ideal.

Proof. Let $S = R/J$. Then, by definition,

every *noncentral* element of S is a sum of a nilpotent and potent element. (7)

Since $S = R/J$, S is a subdirect sum of primitive rings S_i .

Case 1. S_i is a division ring. Since (7) holds in S_i , every element of S_i is either central or potent, and hence by Lemma 1, S_i is commutative.

Case 2. S_i is a primitive ring which is not a division ring. Since condition (6) holds in S_i , and since this condition is satisfied by all subrings and by all homomorphic images of S_i , it follows, by Jacobson's Density Theorem [[4]; p.33], that for some division ring D and some integer $q > 1$, the complete matrix ring D_q also satisfies (6). This, however, is false, as can be seen by taking $x = E_{11} + E_{12}$, $y = E_{11}$ in D_q . This implies that S is a subdirect sum of commutative division rings, and hence

$$R/J \text{ is commutative.} \quad (8)$$

Thus, both J and R/J are commutative, and hence by Lemma 2, N is an ideal and R/N is commutative, which proves the theorem. $\square$

Theorem 2. *Suppose R is a subperiodic ring with identity and suppose that J is commutative. Suppose, further, that for all x, y in R , there exist positive integers k, l, m depending on x, y such that*

$$x^k y^l [x, y]_m \text{ is in the center } C, (x, y \text{ in } R). \quad (9)$$

Then R is commutative.

Proof. First, we show that

$$\text{All idempotents of } R \text{ are central.} \quad (10)$$

To prove this, let $e^2 = e$, $x \in R$, and let $f = e + ex - exe$. Then $f^2 = f$, $ef = f$, $fe = e$. By setting $x = f$, $y = e$ in (9), we obtain $ex = exe$ (as is readily verified). Repeating this argument with e now replaced by $1 - e$, we see that $xe = exe$, and hence e is central, proving (10). Moreover,

$$N \text{ is an ideal (by Theorem 1).} \quad (11)$$

As is well known,

$$R \cong \text{a subdirect sum of subdirectly irreducible rings } R_i. \quad (12)$$

Let $\sigma : R \rightarrow R_i$ be the natural homomorphism of R onto R_i . In view of Lemma 6 and (10), (11), (12), we see that

$$R_i = \sigma(J) \cup C_i \cup U_i \quad (C_i = \text{center of } R_i, U_i = \text{set of units in } R_i). \quad (13)$$

Our next goal is to prove that

$$\text{The set } N_i \text{ of nilpotents of } R_i \text{ is an ideal.} \quad (14)$$

To prove this, let $a_i \in N_i$, $x_i \in R$. Then, by (13), $a_i x_i \in \sigma(J) \cup C_i \cup U_i$. If $a_i x_i \in C_i$, then $a_i x_i \in N_i$. Also, if $a_i x_i \in \sigma(J)$, then $(a_i x_i)^2 \in (\sigma(J))^2 \subseteq C_i$ (since $\sigma(J)$ is a commutative ideal), which implies that, for some $y_i \in R_i$, $((a_i x_i)^2)^k = (a_i x_i)^2 (a_i x_i)^2 \dots (a_i x_i)^2 = a_i^k y_i$ for all positive integers k . Since $a_i \in N_i$, let $a_i^k = 0$ in the above equation to obtain $(a_i x_i)^{2k} = a_i^k y_i = 0$, which implies that $a_i x_i \in N_i$. Finally, suppose $a_i x_i = u_i$, where u_i is a unit in R_i , and let $a_i^\delta = 0$, δ minimal. If $\delta = 1$, then $a_i x_i = 0 \in N_i$. Suppose then that $\delta > 1$. Note that

$$0 = a_i^\delta x_i = a_i^{\delta-1} (a_i x_i) = a_i^{\delta-1} u_i, \quad (u_i \text{ is a unit}),$$

and hence $a_i^{\delta-1} = 0$, contradicting the minimality of δ . This contradiction shows that $a_i x_i \in N_i$. Similarly $x_i a_i \in N_i$. Moreover, by Lemma 5, $N_i \subseteq \sigma(J) \cup C_i$ (see (11)), and hence N_i is commutative (since J is commutative) and (14) is proved. We claim that

$$\text{For all } a_i \in \sigma(J), b_i \in R_i, [a_i, b_i]_q = 0 \text{ for some } q \geq 1. \quad (15)$$

To prove this, note that (15) is trivially satisfied if $b_i \in C_i$ or if $b_i \in \sigma(J)$, and hence by (13),

$$b_i \text{ is a unit in } R_i. \quad (16)$$

Note that R_i inherits (9) in R , and hence for some positive integers k, l, m , we have $(1 + a_i)^k b_i^l [1 + a_i, b_i]_m \in C_i$ (the center of R_i), which implies that

$$(1 + a_i)^k b_i^l [a_i, b_i]_m \text{ commutes with } b_i. \quad (17)$$

Moreover, $(1 + a_i)^k = 1 + a_i g(a_i)$ for some $g(x) \in \mathbb{Z}[x]$, and hence by (17),

$$(1 + a_i g(a_i)) b_i^l [a_i, b_i]_m \text{ commutes with } b_i. \quad (18)$$

Since $a_i \in \sigma(J)$, $(a_i g(a_i)) b_i^l [a_i, b_i]_m \in (\sigma(J))^2 \subseteq C_i$, and hence by (18),

$$b_i^l [a_i, b_i]_m \text{ commutes with } b_i. \quad (19)$$

Thus,

$$(b_i^l [a_i, b_i]_m) b_i = b_i (b_i^l [a_i, b_i]_m). \quad (20)$$

In view of (16), we see that (20) implies that $[[a_i, b_i]_m, b_i] = 0$, and thus, $[a_i, b_i]_{m+1} = 0$, which proves (15).

Recall that $\sigma : R \rightarrow R_i$ is the natural homomorphism of the ground ring R onto the subdirectly irreducible ring R_i . Our next objective is to prove that

$$\sigma(J) \text{ is contained in the center of } R_i. \quad (21)$$

Suppose not. Then,

$$[a_i, b_i] \neq 0 \text{ for some } a_i \in \sigma(J), b_i \in R_i. \quad (22)$$

Since $[a_i, b_i] \neq 0$, not both $2b_i$ and $3b_i$ commute with a_i . Assume, without loss of generality, that

$$[a_i, 2b_i] \neq 0, (a_i \in \sigma(J), b_i \in R_i). \quad (23)$$

In view of (13), we see that (22) and (23) imply that

$$b_i \text{ and } 2b_i \text{ are both units in } R_i, \text{ since } \sigma(J) \text{ is commutative.} \quad (24)$$

Since N is an ideal (see (11)), all the hypotheses of Lemma 5 hold in R , and hence by (1) of Lemma 5,

$$\text{For all } d \in R, d \in J \text{ or } d \in C \text{ or } d - d^k \in N, k > 1. \quad (25)$$

Note that, since R_i inherits (25) from R , we have:

$$\text{For all } d_i \in R_i, d_i \in \sigma(J) \text{ or } d_i \in C_i \text{ or } d_i - d_i^k \in N_i, k > 1. \quad (26)$$

Moreover, since $\sigma(J)$ is commutative, (22) and (23) imply

$$b_i \notin \sigma(J) \cup C_i \text{ and } 2b_i \notin \sigma(J) \cup C_i. \quad (27)$$

By (26) and (27), we conclude that for some positive integers k_i, l_i

$$b_i - b_i^{k_i} \in N_i \text{ and } (2b_i) - (2b_i)^{l_i} \in N_i, k_i > 1, l_i > 1. \quad (28)$$

By (14), N_i is an ideal of R_i . Let $\bar{x}_i = x_i + N_i$, for any $x_i \in R_i$. Then, by (28),

$$(\bar{b}_i)^{k_i} = \bar{b}_i \text{ and } (\bar{2b}_i)^{l_i} = \bar{2b}_i, k_i > 1, l_i > 1. \quad (29)$$

Note that by (29),

$$(\bar{b}_i)^{(k_i-1)(l_i-1)+1} = \bar{b}_i \text{ and } (2\bar{b}_i)^{(k_i-1)(l_i-1)+1} = 2\bar{b}_i.$$

Thus,

$$2^{(k_i-1)(l_i-1)+1}\bar{b}_i = 2^{(k_i-1)(l_i-1)+1}(\bar{b}_i)^{(k_i-1)(l_i-1)+1} = 2\bar{b}_i.$$

Hence,

$$(2^{(k_i-1)(l_i-1)+1} - 2)\bar{b}_i = \bar{0}.$$

Since b_i is a unit (see (24)), $\bar{b}_i$ a unit also. Thus,

$$(2^{(k_i-1)(l_i-1)+1} - 2) \cdot \bar{1} = \bar{0} \quad (k_i > 1, l_i > 1),$$

which implies that

$$(2^{(k_i-1)(l_i-1)+1} - 2) \cdot 1 \text{ is nilpotent,}$$

and hence R_i is not of zero characteristic. Since R_i is subdirectly irreducible, we conclude that

$$\text{Characteristic of } R_i = p^k, \quad p \text{ prime.} \quad (30)$$

In view of (24), (29), and (30), it follows that the subring $\langle \bar{b}_i \rangle$ generated by the unit $\bar{b}_i$ is a finite commutative ring with identity which has no nonzero nilpotents, and hence

$$\langle \bar{b}_i \rangle = \bigoplus_{j=1}^t GF(p^{k_j}), \quad t \text{ finite.} \quad (31)$$

Let $\alpha = k_1 k_2 \dots k_t$. Then, by (31) and (30), $\bar{b}_i^{p^{k\alpha}} = \bar{b}_i$, and thus by Lemma 5 and (11),

$$b_i^{p^{k\alpha}} - b_i \in N_i \subseteq \sigma(J) \cup C_i. \quad (32)$$

Returning to (15), let β be the least positive integer such that

$$[a_i, b_i]_\beta = 0 \quad (\beta \text{ minimal}), \quad a_i \in \sigma(J). \quad (33)$$

We claim that

$$\beta \leq 2. \quad (34)$$

Suppose not. Then $\beta > 2$, and hence by (32),

$$[b_i^{p^{k\alpha}} - b_i, [a_i, b_i]_{\beta-2}] = 0, \quad a_i \in \sigma(J), \quad b_i \in R_i, \quad (35)$$

which implies

$$[[a_i, b_i]_{\beta-2}, b_i^{p^{k\alpha}}] = [[a_i, b_i]_{\beta-2}, b_i] \quad (36)$$

By (33), we see that $[[a_i, b_i]_{\beta-2}, b_i]$ commutes with b_i , and hence by (36) and Lemma 4,

$$p^{k\alpha} b_i^{p^{k\alpha}-1} [[a_i, b_i]_{\beta-2}, b_i] = [[a_i, b_i]_{\beta-2}, b_i],$$

which implies that $0 = [[a_i, b_i]_{\beta-2}, b_i]$, by (30). Thus, $[a_i, b_i]_{\beta-1} = 0$, which contradicts the *minimality* of β (see (33)). This contradiction proves that $\beta \leq 2$, and thus

$$[a_i, b_i] \text{ commutes with } b_i \quad (37)$$

By (22), $a_i \in \sigma(J)$, and hence by (32) and the fact that $\sigma(J)$ is commutative, we conclude that

$$[a_i, b_i^{p^{k\alpha}} - b_i] = 0,$$

which implies that

$$[a_i, b_i^{p^{k\alpha}}] = [a_i, b_i]. \quad (38)$$

Combining (38), (37), (30), and Lemma 4, we obtain $[a_i, b_i] = 0$, which contradicts (22). This contradiction proves (21).

To complete the proof, note that by (26), we have:

$$\text{For all } d_i \in R_i, \quad d_i \in \sigma(J) \text{ or } d_i \in C_i \text{ or } d_i - d_i^k \in N_i, \quad k > 1. \quad (39)$$

Moreover, since $\sigma : R \rightarrow R_i$ is an onto homomorphism (see (12)) and since N is an ideal (see (11)), it follows by Lemma 5 that $N_i \subseteq \sigma(J) \cup C_i \subseteq C_i$, by (21). Combining this with (21), (39), and Lemma 3, we see that R_i is commutative, and the ground ring R is commutative. This proves the theorem. $\square$

The next theorem removes the hypothesis that $1 \in R$ in Theorem 2.

Theorem 3. *Suppose R is a subperiodic ring and suppose J is commutative. Suppose, further, that for all x, y in R , there exist positive integers k, l, m depending on x and y such that*

$$\text{Both } x^k y^l [x, y]_m \text{ and } [x, y]_m x^k y^l \text{ are central, } (x, y \text{ in } R). \quad (40)$$

Then R is commutative.

Proof. First, we show that

$$\text{All idempotents of } R \text{ are central.} \quad (41)$$

As we saw at the beginning of the proof of Theorem 2,

$$ex = exe \text{ for all idempotents } e \text{ and all } x \text{ in } R. \quad (42)$$

(That proof did *not* assume that R has an identity.) Next, we prove that

$$xe = exe \text{ for all idempotents } e \text{ and all } x \text{ in } R. \quad (43)$$

Let $f = e + xe - exe$, $e^2 = e$. Then $f^2 = f$, $ef = e$, $fe = f$. By setting $x = e$, $y = f$ in the hypothesis “ $[x, y]_m x^k y^l$ is central” in (40), we obtain (as is readily verified), $xe = exe$, which proves (41). Also, by Theorem 1,

$$N \text{ is an ideal.} \quad (44)$$

Write R is a subdirect sum of subdirectly irreducible rings R_i . Let $\sigma : R \rightarrow R_i$ be the natural homomorphism of R onto R_i . In view of (41), (44), and Lemma 6, R_i must be of the following two types:

Type 1. $R_i = \sigma(J) \cup C_i$, (C_i = center of R_i), or

Type 2. $R_i = \sigma(J) \cup C_i \cup U_i$, (U_i = set of units in R_i).

Clearly, if R_i is of type 1, then R_i is commutative, since J is commutative. So we may assume that

$$1 \in R_i \text{ and } R_i = \sigma(J) \cup C_i \cup U_i, \text{ } (U_i = \text{ set of units in } R_i). \quad (45)$$

We claim that

$$U_i \subseteq C_i \text{ (center of } R_i\text{)}. \quad (46)$$

Suppose not. Let $u_i \in U_i$, $x_i \in R_i$ be such that $[u_i, x_i] \neq 0$. Let $d \in R$ be such that $\sigma(d) = u_i$. Since $[u_i, x_i] \neq 0$, $d \notin C$ (center of R). Also, if $d \in J$, then $u_i = \sigma(d) \in \sigma(J)$, and hence $\sigma(J) = R_i$ (since u_i is a unit), which implies that R_i is commutative (since J is commutative), contradicting the fact that $[u_i, x_i] \neq 0$. This contradiction proves that $d \notin J$. The net result is: $d \notin (J \cup C)$, and hence

$$d = a + b, \text{ } a \in N, \text{ } b \text{ potent with } b^k = b, \text{ } k > 1. \quad (47)$$

Thus, $d - a = b = b^k = (d - a)^k$, $k > 1$. Since N is an ideal (see (44)) and $a \in N$ (see (47)), $d - d^k \in N$. Thus,

$$d - d^k \in N, \text{ } k > 1, \text{ } d \notin (J \cup C), \text{ } d \in R. \quad (48)$$

In view of (41), (44), (48), and (4) of Lemma 6, we conclude that, for some positive integer q ,

$$d^q = d^q e, \quad e^2 = e, \quad e \text{ central in } R \text{ (by (41))}. \quad (49)$$

By above, $\sigma(d) = u_i$, and hence by (49), $u_i^q = u_i^q \sigma(e)$, u_i a unit, which implies that

$$\sigma(e) = 1, \quad (e \in R, \quad 1 \text{ is the identity of } R_i). \quad (50)$$

Thus, by (49), (50), we see that

$$e \text{ is a central idempotent, } e \neq 0, \quad e \in R, \quad (51)$$

and hence eR is a ring *with identity* e , which (as is readily verified) satisfies all the hypotheses of Theorem 2, and hence by Theorem 2,

$$eR \text{ is commutative.} \quad (52)$$

Recall that $\sigma : R \rightarrow R_i$ is an onto homomorphism. Let $x_i, y_i \in R_i$ and let $\sigma(x) = x_i, \sigma(y) = y_i, (x, y \in R)$. Then, by (52), $[ex, ey] = 0$, and hence $[\sigma(ex), \sigma(ey)] = 0$, which implies by (50), $[\sigma(x), \sigma(y)] = 0$, and thus $[x_i, y_i] = 0$. Therefore, R_i is commutative, which contradicts the fact that $[u_i, x_i] \neq 0$ (see above). This contradiction proves (46). By (45) and (46), we conclude that $R_i = \sigma(J) \cup C_i$, and hence R_i is commutative, (see type 1), which implies that the ground ring R is commutative. This proves the theorem. $\square$

Remark 1. In (40), we could have replaced the integers k, l, m in “ $[x, y]_m x^k y^l$ ” by the integers k', l', m' , as a glance at the proof of (41) shows.

We conclude with the following:

Corollary 1. *Suppose R is a subperiodic ring and suppose J is commutative. Suppose that for all x, y in R , there exists a positive integer m , depending on x and y , such that $[x, y]_m = 0$. Then R is commutative.*

Related work appears in [5].

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