The Alternating Direction Difference Method Based on Variable Substitution for Telegraph Equation

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Abstract. In this paper, a alternating direction difference method based on variable substitution is presented for telegraph equation in the bounded domain, the theocratical analysis show that the scheme is convergence, and we get the discrete L^2 -norm errors in both the approximate solution and its first derivatives for all uniform grids.

Keywords: telegraph equation; alternating direction difference method; error estimates

1. INTRODUCTION

Telegraph equations as a classic mathematical physics equation it is not only of great significance but also has wide application in communication engineering. The research of telegraph equations has gained drastic attention and obtained fruitful achievements for its strong practical background. However, what most of the research adopted was the finite element method and little of that was reported on the alternating direction difference of telegraph equations. In this paper, a new approach of alternating direction difference method was employed to construct the difference scheme of telegraph equations. Due to its high convergence and stability the calculation can be simple, and convenient and the error estimation between the approximation solution and exact solution can be obtained as well.

Consider the following initial-boundary value problem of telegraph equation

$$u_{tt} - \Delta u + \alpha u_t + \lambda u = f(x, y, t), \quad (x, y, t) \in \Omega \times (0, T], \quad (1)$$

$$u(x, y, 0) = u_0(x, y), \quad u_t(x, y, 0) = v_0(x, y), \quad (x, y) \in \Omega, \quad (2)$$

$$u(x, y, t) = 0, \quad (x, y) \in \partial\Omega, t \in (0, T]. \quad (3)$$

where $\alpha = \alpha(x, y, t)$, $\lambda = \lambda(x, y, t)$, Ω is a bounded domain in R^2 , $\partial\Omega$ is the corresponding border, T is a positive constant, $u_0(x, y), v_0(x, y)$ are functions have been given, for $\alpha(x, y, t), \lambda(x, y, t)$ the following conditions are established

(i) there exist $\alpha_0, \alpha_1, \lambda_0, \lambda_1$, s.t. $0 < \alpha_0 \leq \alpha(x, y, t) \leq \alpha_1, 0 < \lambda_0 \leq \lambda(x, y, t) \leq \lambda_1$.

(ii) $f(x, y, t), u(x, y, t)$ are properly smooth.

Let $p(x, y, t) = \frac{\partial u(x, y, t)}{\partial t}$,

So that (1) is equivalent to

$$p_t - \Delta u + \alpha p + \lambda u = f(x, y, t), \quad (x, y, t) \in \Omega \times (0, T]. \quad (4)$$

2. PRELIMINARIES

We suppose $\Omega = (0, 1) \times (0, 1)$, decompose it into uniform grid, put $\Delta x = \Delta y = h = \frac{1}{J}$ as space step, $\Delta t = \tau = \frac{T}{N}$ as time step, with J, N are positive

integer, note $x_i = ih, y_j = jh, i, j = 0, 1, \dots, J, t^n = n\tau, n = 0, 1, \dots, N, g_{i,j}^n = g(x_i, y_j, t^n)$. Further, for convenience, let us denote

$$\begin{aligned}\delta_x U_{i,j}^n &= \frac{U_{i+1,j}^n - U_{i,j}^n}{h}, \quad \delta_t U_{i,j}^n = \frac{U_{i,j}^{n+1} - U_{i,j}^n}{\tau}, \\ \delta_x^2 U_{i,j}^n &= \frac{U_{i+1,j}^n - 2U_{i,j}^n + U_{i-1,j}^n}{h^2}, \quad \delta_t^2 U_{i,j}^n = \frac{U_{i,j}^{n+1} - 2U_{i,j}^n + U_{i,j}^{n-1}}{\tau^2} \\ \Delta_h &= \delta_x^2 + \delta_y^2, \quad \nabla_h = \{\delta_x, \delta_y\}.\end{aligned}$$

Define the following Grid function

$$S = \{f_{i,j} | f_{i,j} = f(x_i, y_j), i, j = 0, \dots, J\}.$$

the discrete inner product and norm

$$\begin{aligned}(u, v) &= \sum_{i,j=1}^{J-1} u_{i,j} v_{i,j} h^2, \quad \|u\|^2 = (u, u). \\ (\nabla_h u, \nabla_h v) &= \sum_{i,j=0}^{J-1} \delta_x u_{i,j} \delta_x v_{i,j} h^2 + \sum_{i,j=0}^{J-1} \delta_y u_{i,j} \delta_y v_{i,j} h^2. \\ \|\nabla_h u\|^2 &= (\nabla_h u, \nabla_h u) = \|u\|_{1x}^2 + \|u\|_{1y}^2.\end{aligned}$$

3. ALTERNATING DIRECTION DIFFERENCE SCHEME AND ERROR ESTIMATES

The alternating direction difference scheme of problem (1) – (3) is find $(U^n, P^n) \in S \times S$ s.t.

$$\delta_t P_{i,j}^n - \Delta_h U_{i,j}^{n+1} + \alpha_{i,j}^n P_{i,j}^n + \lambda_{i,j}^n U_{i,j}^n + \tau^3 \delta_x^2 \delta_y^2 (P_{i,j}^{n+1} - P_{i,j}^n) = f_{i,j}^n, \quad (5)$$

$$\delta_t U_{i,j}^n = P_{i,j}^{n+1}, \quad (6)$$

$$U_{i,j}^0 = u_0(x_i, y_j), \quad P_{i,j}^0 = (u_t)_{i,j}^0 = v_0(x_i, y_j), \quad (7)$$

$$U_{0,j}^n = U_{1,j}^n = U_{j,0}^n = U_{j,1}^n = 0. \quad (8)$$

where $\tau^3 \delta_x^2 \delta_y^2 (P_{i,j}^{n+1} - P_{i,j}^n)$ is an additional dissipation term.

Substituting (6) into (5), we have

$$(I - \tau^2 \delta_x^2)(I - \tau^2 \delta_y^2) P_{i,j}^{n+1} = (I - \tau \alpha_{i,j}^n) P_{i,j}^n - \tau \lambda_{i,j}^n U_{i,j}^n + \tau(\delta_x^2 + \delta_y^2) U_{i,j}^n + \tau^4 \delta_x^2 \delta_y^2 P_{i,j}^n + \tau f_{i,j}^n$$

introduce a new variable $P_{i,j}^{n*}$ and hence

$$(I - \tau^2 \delta_x^2) P_{i,j}^{n*} = (I - \tau \alpha_{i,j}^n) P_{i,j}^n - \tau \lambda_{i,j}^n U_{i,j}^n + \tau(\delta_x^2 + \delta_y^2) U_{i,j}^n + \tau^4 \delta_x^2 \delta_y^2 P_{i,j}^n + \tau f_{i,j}^n, \quad (9)$$

$$(I - \tau^2 \delta_y^2) P_{i,j}^{n+1} = P_{i,j}^{n*}, \quad (10)$$

$$\delta_t U_{i,j}^n = P_{i,j}^{n+1}. \quad (11)$$

where I is unit operator.

The alternating direction schemes can turn multi-dimensional problem into one-dimensional problem and iterative matrix of x, y are triple diagonal matrices which have same structure, so they are easy to program for computing.

At the point (x_i, y_j, t^n) , (4) and p can be written as

$$\delta_t p_{i,j}^n - \Delta_h u_{i,j}^{n+1} + \alpha_{i,j}^n p_{i,j}^n + \lambda_{i,j}^n u_{i,j}^n + \tau^3 \delta_x^2 \delta_y^2 (p_{i,j}^{n+1} - p_{i,j}^n) = f_{i,j}^n + R_1, \quad (12)$$

$$\delta_t u_{i,j}^n = p_{i,j}^{n+1} + R_2. \quad (13)$$

where

$$\begin{aligned} R_1 &= \delta_t p_{i,j}^n - (p_t)_{i,j}^n + (\Delta u)_{i,j}^n - \Delta_h u_{i,j}^n + \Delta_h u_{i,j}^n - \Delta_h u_{i,j}^{n+1} + \tau^3 \delta_x^2 \delta_y^2 (p_{i,j}^{n+1} - p_{i,j}^n) \\ &= R_{11} + R_{12} + R_{13} + R_{14}. \end{aligned}$$

$$\begin{aligned} R_2 &= \delta_t u_{i,j}^n - (u_t)_{i,j}^n + p_{i,j}^n - p_{i,j}^{n+1} \\ &= R_{21} + R_{22}. \end{aligned}$$

Set $e = u - U, \theta = p - P$, Subtracting (5)(6) from (9)(10), we obtain the error equations

$$\delta_t \theta_{i,j}^n - \Delta_h e_{i,j}^{n+1} + \alpha \theta_{i,j}^n + \lambda e_{i,j}^n + \tau^3 \delta_x^2 \delta_y^2 (\theta_{i,j}^{n+1} - \theta_{i,j}^n) = R_1, \quad (14)$$

$$\delta_t e_{i,j}^n = \theta_{i,j}^{n+1} + R_2. \quad (15)$$

Make inner product with θ^{n+1} on both sides of (14), and hence

$$\begin{aligned} &(\delta_t \theta^n, \theta^{n+1}) - (\Delta_h e^{n+1}, \delta_t e^n) + \tau^3 (\delta_x^2 \delta_y^2 (\theta^{n+1} - \theta^n), \theta^{n+1}) + (\lambda^n e^n, \theta^{n+1}) \\ &= (R_1, \theta^{n+1}) - (\alpha^n \theta^n, \theta^{n+1}) - (\Delta_h e^{n+1}, R_2). \end{aligned} \quad (16)$$

Now we will estimate one by one.

$$(\delta_t \theta^n, \theta^{n+1}) \geq \frac{1}{2\tau} (\|\theta^{n+1}\|^2 - \|\theta^n\|^2). \quad (17)$$

$$\begin{aligned} -(\Delta_h e^{n+1}, \delta_t e^n) &= \frac{1}{\tau} \sum_{i=0}^{J-1} \sum_{j=1}^{J-1} \delta_x e_{i,j}^{n+1} \delta_x (e_{i,j}^{n+1} - e_{i,j}^n) h^2 + \frac{1}{\tau} \sum_{i=0}^{J-1} \sum_{j=1}^{J-1} \delta_y e_{i,j}^{n+1} \delta_y (e_{i,j}^{n+1} - e_{i,j}^n) h^2 \\ &\geq \frac{1}{2\tau} \sum_{i=0}^{J-1} \sum_{j=0}^{J-1} (\delta_x e_{i,j}^{n+1})^2 h^2 - \frac{1}{2\tau} \sum_{i=0}^{J-1} \sum_{j=0}^{J-1} (\delta_x e_{i,j}^n)^2 h^2 \\ &\quad + \frac{1}{2\tau} \sum_{i=0}^{J-1} \sum_{j=0}^{J-1} (\delta_y e_{i,j}^{n+1})^2 h^2 - \frac{1}{2\tau} \sum_{i=0}^{J-1} \sum_{j=0}^{J-1} (\delta_y e_{i,j}^n)^2 h^2 \\ &= \frac{1}{2\tau} (\|\nabla_h e^{n+1}\|^2 - \|\nabla_h e^n\|^2). \end{aligned} \quad (18)$$

$$\begin{aligned} \tau^3 (\delta_x^2 \delta_y^2 (\theta^{n+1} - \theta^n), \theta^{n+1}) &= \tau^3 (\delta_x \delta_y \theta^{n+1}, \delta_x \delta_y \theta^{n+1}) - \tau^3 (\delta_x \delta_y \theta^n, \delta_x \delta_y \theta^{n+1}) \\ &\geq \tau^3 \|\delta_x \delta_y \theta^{n+1}\|^2 - \frac{\tau^3}{2} (\|\delta_x \delta_y \theta^{n+1}\|^2 + \|\delta_x \delta_y \theta^n\|^2) \\ &= \frac{\tau^3}{2} (\|\delta_x \delta_y \theta^{n+1}\|^2 - \|\delta_x \delta_y \theta^n\|^2). \end{aligned} \quad (19)$$

By ε - inequality and λ is bounded, we have

$$(\lambda^n e^n, \theta^{n+1}) \leq \frac{\lambda^2}{4\varepsilon} \|e^n\|^2 + \varepsilon \|\theta^{n+1}\|^2. \quad (20)$$

Then by *Holder* inequality and ε - inequality, we obtain

$$\begin{aligned} (R_{11}, \theta^{n+1}) &= \sum_{i=1}^{J-1} \sum_{j=1}^{J-1} \frac{1}{\tau} \int_{t^n}^{t^{n+1}} (t^{n+1} - s) \frac{\partial^2 p_{i,j}}{\partial s^2} ds \cdot \theta_{i,j}^{n+1} \cdot h^2 \\ &\leq \sum_{i=0}^{J-1} \sum_{j=0}^{J-1} \frac{1}{\tau} \left(\int_{t^n}^{t^{n+1}} \left(\frac{\partial^2 p_{i,j}}{\partial s^2} \right)^2 ds \right)^{\frac{1}{2}} \cdot \left[\int_{t^n}^{t^{n+1}} (t^{n+1} - s)^2 ds \right]^{\frac{1}{2}} \cdot \theta_{i,j}^{n+1} \cdot h^2 \\ &\leq C \|\theta^{n+1}\|^2 + C\tau \|p_{tt}\|_{L^2(t^n, t^{n+1}; \|\cdot\|)}^2. \\ (R_{12}, \theta^{n+1}) &= \sum_{i=0}^{J-1} \sum_{j=0}^{J-1} \left[\left(\frac{\partial^2 u}{\partial x^2} \right)_{i,j}^n + \left(\frac{\partial^2 u}{\partial y^2} \right)_{i,j}^n - (\delta_x^2 + \delta_y^2) u_{i,j}^n \right] \cdot \theta_{i,j}^{n+1} \cdot h^2 \\ &\leq \sum_{i=0}^{J-1} \sum_{j=0}^{J-1} C \cdot \frac{1}{h^2} \left[\int_{x_i}^{x_{i+1}} (x_{i+1} - s)^2 \frac{\partial^3 u_{i,j}}{\partial s^3} ds + \int_{y_j}^{y_{j+1}} (y_{j+1} - s)^2 \frac{\partial^3 u_{i,j}}{\partial s^3} ds \right] \cdot \theta_{i,j}^{n+1} \cdot h^2 \\ &\leq Ch^4 \left(\left\| \frac{\partial^3 u}{\partial x^3} \right\|^2 + \left\| \frac{\partial^3 u}{\partial y^3} \right\|^2 \right) + C \|\theta^{n+1}\|^2. \end{aligned}$$

and similarly

$$(R_{13}, \theta^{n+1}) \leq Ch^4 \left(\left\| \frac{\partial^3 u}{\partial x^2 \partial t} \right\|_{L^2(t^n, t^{n+1}; \|\cdot\|)}^2 + \left\| \frac{\partial^3 u}{\partial y^2 \partial t} \right\|_{L^2(t^n, t^{n+1}; \|\cdot\|)}^2 \right) + C \|\theta^{n+1}\|^2.$$

$$\begin{aligned} (R_{14}, \theta^{n+1}) &= \tau^3 \sum_{i=1}^{J-1} \sum_{j=1}^{J-1} \delta_x^2 \delta_y^2 (p_{i,j}^{n+1} - p_{i,j}^n) \cdot \theta_{i,j}^{n+1} \cdot h^2 \\ &= \tau^3 \sum_{i=0}^{J-1} \sum_{j=0}^{J-1} \int_{t^n}^{t^{n+1}} \delta_x \delta_y \left(\frac{\partial p_{i,j}}{\partial s} \right) ds \cdot \delta_x \delta_y \theta_{i,j}^{n+1} \cdot h^2 \\ &\leq C\tau^3 \|\delta_x \delta_y \theta^{n+1}\|^2 + C\tau^3 \left\| \frac{\partial^3 p}{\partial x \partial y \partial t} \right\|_{L^2(t^n, t^{n+1}; \|\cdot\|)}^2. \end{aligned}$$

By ε - inequality and α is bounded, we have

$$|-(\alpha^n \theta^n, \theta^{n+1})| \leq \varepsilon \|\theta^{n+1}\|^2 + \frac{\alpha_1^2}{4\varepsilon} \|\theta^n\|^2. \quad (21)$$

By Taylor expansion and *Holder* inequality, we get

$$\begin{aligned} -(\Delta_h e^{n+1}, R_2) &= \sum_{i=1}^{J-1} \sum_{j=1}^{J-1} \nabla_h \left(\frac{1}{\tau} \int_{t^n}^{t^{n+1}} (t^{n+1} - s) \frac{\partial^2 u_{i,j}}{\partial s^2} ds \right) \cdot \nabla_h e_{i,j}^{n+1} \cdot h^2 \\ &\quad + \sum_{i=1}^{J-1} \sum_{j=1}^{J-1} \nabla_h \left(\int_{t^n}^{t^{n+1}} \frac{\partial p_{i,j}}{\partial s} ds \right) \cdot \nabla_h e_{i,j}^{n+1} \cdot h^2 \\ &\leq C\tau h^2 \left(\left\| \frac{\partial^3 u}{\partial x \partial t^2} \right\|_{L^2(t^n, t^{n+1}; \|\cdot\|)}^2 + \left\| \frac{\partial^3 u}{\partial y \partial t^2} \right\|_{L^2(t^n, t^{n+1}; \|\cdot\|)}^2 \right. \\ &\quad \left. + \left\| \frac{\partial^2 p}{\partial x \partial t} \right\|_{L^2(t^n, t^{n+1}; \|\cdot\|)}^2 + \left\| \frac{\partial^2 p}{\partial y \partial t} \right\|_{L^2(t^n, t^{n+1}; \|\cdot\|)}^2 \right) + C \|\nabla_h e^{n+1}\|^2. \quad (22) \end{aligned}$$

Substituting (17) – (22) into (16), and sum on n from 1 to $N - 1$, we obtain

$$\begin{aligned} & \| \theta^N \|^2 - \| \theta^0 \|^2 + \| \nabla_h e^N \|^2 - \| \nabla_h e^0 \|^2 + \tau^4 (\| \delta_x \delta_y \theta^N \|^2 - \| \delta_x \delta_y \theta^0 \|^2) \\ & \leq C\tau \sum_{n=0}^{N-1} \| \theta^{n+1} \|^2 + C\tau^2 (\| p_{tt} \|_{L^2(0,T;\|\cdot\|)}^2 + h^4 \| \frac{\partial^3 u}{\partial x \partial y \partial t} \|_{L^2(0,T;\|\cdot\|)}^2) + C\tau^4 (\sum_{n=0}^{N-1} \| \delta_x \delta_y \theta^{n+1} \|^2 \\ & + \| \frac{\partial^3 p}{\partial x \partial y \partial t} \|_{L^2(0,T;\|\cdot\|)}^2) + C\tau^2 h^2 (\| \frac{\partial^3 u}{\partial x \partial t^2} \|_{L^2(0,T;\|\cdot\|)}^2 + \| \frac{\partial^3 u}{\partial y \partial t^2} \|_{L^2(0,T;\|\cdot\|)}^2) \\ & + \| \frac{\partial^2 p}{\partial x \partial t} \|_{L^2(0,T;\|\cdot\|)}^2 + \| \frac{\partial^2 p}{\partial y \partial t} \|_{L^2(0,T;\|\cdot\|)}^2) + C\tau \sum_{n=0}^{N-1} \| \nabla_h e^{n+1} \|^2. \end{aligned}$$

Choose properly τ s.t. $C\tau \leq \frac{1}{2}$, then

$$\begin{aligned} & \| \theta^N \|^2 + \| \nabla_h e^N \|^2 + \tau^4 \| \delta_x \delta_y \theta^N \|^2 \leq \| \theta^0 \|^2 + \| \nabla_h e^0 \|^2 + \tau^4 \| \delta_x \delta_y \theta^0 \|^2 \\ & + C\tau \sum_{n=0}^{N-1} \| \theta^n \|^2 + C\tau \sum_{n=0}^{N-1} \| \nabla_h e^n \|^2 + C\tau^4 \sum_{n=0}^{N-1} \| \delta_x \delta_y \theta^{n+1} \|^2 \cdot \tau + C(\tau^2 + \tau^2 h^2 + h^4). \end{aligned}$$

By discrete Gronwall inequality, we obtain

$$\| \theta^N \|^2 + \| \nabla_h e^N \|^2 + \tau^4 \| \delta_x \delta_y \theta^N \|^2 \leq C(\tau^2 + \tau^2 h^2 + h^4).$$

Theorem 3.1. *Let u and p are the solution of (1) – (3) and (4) respectively, U^n and P^n are the solution of (5) – (6) respectively, assume that the grid in the solving area is equidistant subdivision, then there exists a constant C independent h, τ , s.t. for any $0 \leq n \leq N$ ($N \leq [\frac{T}{\tau}]$) we have*

$$\| p^n - P^n \|^2 + \| \nabla_h (u^n - U^n) \|^2 \leq C(\tau^2 + \tau^2 h^2 + h^4).$$

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