Concircular Trasformation of a Quarter Symmetric Metric Connection in an SP-Sasakian Manifold

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Abstract. In 1982, P. Stavre [8] defined semi-symmetric S-concircular and S-conharmonic connections and established some results. Analogous to this we have defined a concircular transformation on quarter symmetric metric connection in an SP-Sasakian manifold. The invariance property of the η -Einstein tensor and conformal curvature tensor has been shown.

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1. PRELIMINARIES

An n -dimensional differentiable manifold M is called an almost para-contact manifold if it admits an almost para-contact structure (ϕ, ξ, η) consisting of a $(1,1)$ tensor field ϕ , a vector field ξ and 1-form η satisfying

$$(1) \quad \phi^2 X = X - \eta(X)\xi; \quad \phi \circ \xi = 0;$$

$$\eta(\xi) = 1; \quad \eta \circ \phi = 0; \quad \text{rank}(\phi) = n - 1$$

for all $X \in TM$

Let g be the Riemannian metric satisfying

$$(2) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

or equivalently

$$(3) \quad g(\phi X, Y) = g(X, \phi Y); \quad g(X, \xi) = \eta(X)$$

for all $X, Y \in TM$. Then M becomes an almost para-contact Riemannian manifold equipped with the almost para-contact Riemannian structure (ϕ, ξ, η, g) [7] .

If we define,

$$(4) \quad {}'\phi(X, Y) = g(\phi X, Y)$$

then in addition to the above equations, we have

$$(5) \quad {}'\phi(X, Y) = {}'\phi(Y, X); \quad {}'\phi(\phi X, \phi Y) = {}'\phi(X, Y)$$

An almost para-contact Riemannian manifold is called a P-Sasakian manifold [1] if it satisfies

$$(6) \quad (\nabla_X \phi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(Y)\eta(X)\xi \quad \forall X, Y \in TM$$

where ∇ is a Levi-civita connection of the Riemannian metric. From the above equation it follows that

$$(7) \quad \nabla_X \xi = \phi X$$

$$(8) \quad (\nabla_X \eta)Y = g(X, \phi Y) = (\nabla_Y \eta)X$$

Especially, a P-Sasakian manifold M is called a special para-Sasakian manifold or briefly an SP-Sasakian manifold [6] if M admits a 1-form η satisfying

$$(9) \quad (\nabla_X \eta)Y = -g(X, Y) + \eta(X)\eta(Y)$$

For SP-Sasakian manifold we have [2]

$$(10) \quad {}'\phi(X, Y) = -g(X, Y) + \eta(X)\eta(Y)$$

In an n -dimensional P-Sasakian manifold M , the curvature tensor R , the Ricci tensor S satisfy (see [1], [2] and [9])

$$(11) \quad \eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X)$$

$$(12) \quad S(X, \xi) = -(n-1)\eta(X)$$

$$(13) \quad R\xi = -(n-1)\xi$$

for any vector fields $X, Y, Z \in TM$.

A linear connection $\tilde{\nabla}$ in a Riemannian manifold is said to be quarter symmetric connection [5] if its torsion tensor T satisfies

$$(14) \quad T(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] = p(Y)tX - p(X)tY$$

where p is 1-form and t is a tensor of type $(1,1)$. A linear connection $\tilde{\nabla}$ is called a metric connection [10] iff

$$(15) \quad \tilde{\nabla}_X g = 0$$

Definition 1.1. A linear connection $\tilde{\nabla}$ satisfying (14) and (15) is called a quarter symmetric metric connection [4].

Let ∇ be the Riemannian connection with respect to g , then for quarter symmetric metric connection, we have

$$(16) \quad \tilde{\nabla}_X Y = \nabla_X Y + p(Y)tX - 'T(X, Y)P$$

where $'T(X, Y) = g(t(X), Y)$ and $p(X) = g(X, P)$

Putting $t(X) = \phi(X)$, we get

$$(17) \quad \tilde{\nabla}_X Y = \nabla_X Y + p(Y)\phi X - '\phi(X, Y)P$$

2. CURVATURE TENSOR

Let R and $\tilde{R}$ be the curvature tensors of the connections ∇ and $\tilde{\nabla}$ respectively, then we have

$$(18) \quad R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

$$(19) \quad \tilde{R}(X, Y, Z) = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X, Y]} Z$$

Using (4), (10), (16), (18) and (19), we immediately get the following proposition.

Proposition 2.1. *For the quarter symmetric connection (17) the curvature tensor $\tilde{R}$ of $\tilde{\nabla}$ is given by*

$$(20) \quad \begin{aligned} \tilde{R}(X, Y, Z) = R(X, Y, Z) + Q(X, Z)\phi Y - Q(Y, Z)\phi X \\ - ' \phi(Y, Z)QX + ' \phi(X, Z)QY \end{aligned}$$

where,

$$Q(X, Y) = g(QX, Y) = (\nabla_X p)Y - p(Y)p(\phi X) + \frac{1}{2}'\phi(X, Y)p(P) - p(Y)\eta(X)$$

Definition 2.1. Let the quarter symmetric metric connection $\tilde{\nabla}$ satisfies

$$(21) \quad Q(X, Y) = fg(X, Y)$$

where f is a scalar function, then $\tilde{\nabla}$ is called quarter symmetric metric connection with concircular transformation.

Theorem 2.1. *If a quarter symmetric metric connection, in an SP-Sasakian manifold satisfies (21), then the curvature tensors are related by*

$$(22) \quad \tilde{R}(X, Y, Z) - \tilde{L}(X, Y, Z) = R(X, Y, Z) - L(X, Y, Z)$$

where

$$(23) \quad \begin{aligned} L(X, Y, Z) = \frac{r}{2(n-1)^2} [g(X, Z)\phi Y - g(Y, Z)\phi X \\ - ' \phi(Y, Z)X + ' \phi(X, Z)Y] \end{aligned}$$

and $r, \tilde{r}$ be the scalar curvatures of the connections ∇ and $\tilde{\nabla}$ respectively.

Proof. Let us consider a transformation (17) satisfying (21). Then (20) gives

$$(24) \quad \begin{aligned} \tilde{R}(X, Y, Z) = R(X, Y, Z + f[g(X, Z)\phi Y - g(Y, Z)\phi X \\ - ' \phi(Y, Z)X + ' \phi(X, Z)Y]) \end{aligned}$$

Contracting (24), we get

$$(25) \quad \tilde{Ric}(Y, Z) = Ric(Y, Z) + f[g(\phi Y, Z) + (n-1)g(Y, Z) - n'\phi(Y, Z) + '\phi(Y, Z)]$$

From (4) and (25), we get

$$(26) \quad \tilde{Ric}(Y, Z) = Ric(Y, Z) + f[(n-1)g(Y, Z) - (n-2)'\phi(Y, Z)]$$

This gives

$$(27) \quad \tilde{R}Y = RY + f[(n-1)Y - (n-2)\phi Y]$$

Again by the contraction of (27), we get

$$(28) \quad f = \frac{\tilde{r} - r}{2(n-1)^2}$$

From (23), (25) and (28), we get (22). $\square$

Theorem 2.2. *If $\tilde{\nabla}$ is a quarter symmetric metric connection in an SP-Sasakian manifold satisfying (21), then the conformal curvature tensor is invariant of the transformation (17).*

Proof. From (24), (26) and (27), we have

$$(29) \quad \begin{aligned} \tilde{R}(X, Y, Z) = R(X, Y, Z) + g(X, Z) & \left(\frac{\tilde{R}Y - RY - f(n-1)Y}{-(n-2)} \right) \\ & - g(Y, Z) \left(\frac{\tilde{R}X - RX - f(n-1)X}{-(n-2)} \right) \\ & - \left(\frac{\tilde{Ric}(Y, Z) - Ric(Y, Z) - f(n-1)g(Y, Z)}{-(n-2)} \right) X \\ & + \left(\frac{\tilde{Ric}(X, Z) - Ric(X, Z) - f(n-1)g(X, Z)}{-(n-2)} \right) Y \end{aligned}$$

From (28) and (29), we get

$$\begin{aligned}
(30) \quad & \tilde{R}(X, Y, Z) + \frac{1}{(n-2)}[g(X, Z)\tilde{R}Y - g(Y, Z)\tilde{R}X - \tilde{Ric}(Y, Z)X \\
& + \tilde{Ric}(X, Z)Y] - \frac{\tilde{r}}{(n-1)(n-2)}[g(X, Z)Y - g(Y, Z)X] \\
& = R(X, Y, Z) + \frac{1}{(n-2)}[g(X, Z)RY - g(Y, Z)RX - Ric(Y, Z)X \\
& + Ric(X, Z)Y] - \frac{r}{(n-1)(n-2)}[g(X, Z)Y - g(Y, Z)X]
\end{aligned}$$

Which shows the invariance of conformal curvature tensor. $\square$

Definition 2.2. An almost para-contact metric manifold is said to be η -Einstein if the Ricci tensor S satisfies,

$$(31) \quad S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$$

where a and b are smooth functions on the manifold M [6].

Theorem 2.3. If $\tilde{\nabla}$ is a quarter symmetric metric connection in an SP -Sasakian manifold satisfying (21), then the tensor of η -Einstein given by (31) is invariant of the transformation (17).

Proof. From (10), (26) and (28), we get

$$(32) \quad \tilde{Ric}(Y, Z) = Ric(Y, Z) + \left(\frac{\tilde{r} - r}{2(n-1)^2} \right) [(2n-3)g(Y, Z) - (n-2)\eta(Y)\eta(Z)]$$

Which gives

$$\begin{aligned}
(33) \quad & \tilde{Ric}(Y, Z) - \left(\frac{\tilde{r}}{2(n-1)^2} \right) [(2n-3)g(Y, Z) - (n-2)\eta(Y)\eta(Z)] \\
& = Ric(Y, Z) - \left(\frac{r}{2(n-1)^2} \right) [(2n-3)g(Y, Z) - (n-2)\eta(Y)\eta(Z)]
\end{aligned}$$

Hence (33) shows the invariance of the tensor of η -Einstein. $\square$

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