Products Of Quadratic Polynomials With Roots Modulo Any Integer

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Abstract

We classify products of three quadratic polynomials, each irreducible over $\mathbb{Q}$, which are solvable modulo m for every integer $m > 1$ but have no roots over the rational numbers. Polynomials with this property are known as intersective polynomials. We use Hensel's Lemma and a refined version of Hensel's Lemma to complete the proof.

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1 Introduction

Let $f(x)$ be a monic polynomial with integer coefficients. We are interested in intersective polynomials, that is, those polynomials $f(x)$ which have no root

in the rational numbers but do have a root modulo all positive integers. These polynomials provide counterexamples to the local-global principle. Readers interested in more information on the local-global principle should consult [4, pp. 75-83.]. A common example of this phenomenon is the following product of three quadratic polynomials.

$$f(x) = (x^2 - 2)(x^2 - 17)(x^2 - 34). \quad (1)$$

This example or others very similar appear in [1], [2], [3], [4] and [5]. We note that the three quadratic polynomials in (1) define quadratic subfields of a bicyclic quartic field

$$\mathbb{Q}(\sqrt{2}, \sqrt{17}).$$

This type of product is necessary in order to form intersective polynomials. An explanation of this in terms of Galois theory can be found in [6] or [7]. In this paper we shall treat the general case of these products of quadratic polynomials, which we denote by $f(x)$. Special families will be considered as corollaries. We shall confirm that

$$f(x) \equiv 0 \pmod{m}$$

is solvable for every $m > 1$ by showing that for each prime p and positive integer j , the congruence

$$f(x) \equiv 0 \pmod{p^j}$$

is solvable. General solvability then follows from the Chinese Remainder Theorem. For a given prime p , one of the factors of $f(x)$ is proven to be solvable mod p^j for all positive integers j . We state our main theorem next, recalling the Legendre symbol $\left(\frac{n}{p}\right)$ which takes the values $+1$ if n is a quadratic residue modulo p , -1 if n is a quadratic non-residue modulo p , or 0 if p divides n .

2 Main Result

Theorem 2.1 *Let a, b be squarefree integers, not equal to 1 such that ab is not equal to the square of an integer. Set $\ell = \gcd(a, b)$ and define a_1 and b_1 by $a = a_1\ell$ and $b = b_1\ell$. Set*

$$f(x) = (x^2 - a)(x^2 - b)(x^2 - a_1b_1).$$

Then

$$f(x) \equiv 0 \pmod{m}$$

is solvable modulo every integer $m > 1$ if and only if at least one of a, b, a_1b_1 is congruent to 1 modulo 8 and for every odd prime p at least one of the Legendre symbols

$$\left(\frac{a}{p}\right), \left(\frac{b}{p}\right), \left(\frac{a_1b_1}{p}\right),$$

has the value $+1$.

For the proof of our theorem we will require Hensel's Lemma, both a standard version and a refined version.

Theorem 2.2 (Hensel's Lemma) (see [5, p. 87]) Suppose that $f(x)$ is a polynomial with integral coefficients. If $f(a) \equiv 0 \pmod{p^j}$ and $f'(a) \not\equiv 0 \pmod{p}$, then there is a unique $t \pmod{p}$ such that $f(a + tp^j) \equiv 0 \pmod{p^{j+1}}$.

We call the root a nonsingular if $f'(a) \not\equiv 0 \pmod{p}$. A nonsingular root a of $f(x) \equiv 0 \pmod{p}$ may be lifted to a root modulo p^j , $j = 2, 3, \dots$ by repeatedly applying Hensel's Lemma. We employ a refined version of Hensel's Lemma in the case of a singular root, enabling us to lift our solutions modulo arbitrarily high prime powers.

Theorem 2.3 (Refined Hensel's Lemma) (see [5, p. 89]) Let $f(x)$ be a polynomial with integral coefficients. Suppose that $f(a) \equiv 0 \pmod{p^j}$, that $p^\tau \parallel f'(a)$ and that $j \geq 2\tau + 1$. If $b \equiv a \pmod{p^{j-\tau}}$ then $f(b) \equiv f(a) \pmod{p^j}$ and $p^\tau \parallel f'(b)$. Moreover there is a unique $t \pmod{p}$ such that $f(a + tp^{j-\tau}) \equiv 0 \pmod{p^{j+1}}$.

As noted in [5, p. 89], since the hypotheses of the theorem apply with a replaced by $a + tp^{j-\tau}$ and $(\pmod{p^j})$ replaced by $(\pmod{p^{j+1}})$ but with τ unchanged, the lifting may be repeated and continues indefinitely. We require two simple lemmas for the proof of our theorem.

Lemma 2.4 Suppose that p is an odd prime and r is a squarefree integer. Then the congruence

$$x^2 \equiv r \pmod{p^j}$$

is solvable for $j = 1, 2, \dots$ if and only if $\left(\frac{r}{p}\right) = +1$.

Proof Since r is squarefree, solvability of the given congruence requires $p \nmid r$. Specializing to solvability mod p shows that $\left(\frac{r}{p}\right) = +1$

If $\left(\frac{r}{p}\right) = +1$ then the congruence

$$x^2 \equiv r \pmod{p}$$

is solvable with integer solution $x = x_0$. Since

$$x_0^2 \equiv r \pmod{p}$$

and $p \nmid r$ we see that $p \nmid x_0$ so that x_0 is a nonsingular root of this congruence. Thus we may employ Proposition 1 to lift this solution to a solution of the congruence

$$x^2 \equiv r \pmod{p^j}$$

for $j = 1, 2, \dots$, completing the proof. $\square$

Lemma 2.5 *Suppose that r is a squarefree integer. Then the congruence*

$$x^2 \equiv r \pmod{2^j}$$

is solvable for $j = 1, 2, \dots$ if and only if $r \equiv 1 \pmod{8}$.

Proof Since r is squarefree, solvability of the given congruence modulo all powers of 2 requires r to be odd. Since odd squares are congruent to 1 modulo 8, we have $r \equiv 1 \pmod{8}$. If $r \equiv 1 \pmod{8}$ then the congruence

$$x^2 \equiv r \pmod{2^3}$$

is solvable with solution $x = 1$. Although $x = 1$ is a singular root of this congruence, we may employ Proposition 2, with $j = 3$ and $\tau = 1$ and lift this root to a solution of

$$x^2 \equiv r \pmod{2^j}$$

modulo all powers of 2 as required. $\square$

We now give the proof of our theorem.

Proof Assume that $f(x) = (x^2 - a)(x^2 - b)(x^2 - a_1b_1) \equiv 0 \pmod{m}$ is solvable for every integer $m > 1$. Then in particular this polynomial congruence is solvable modulo powers of 2 so that at least one of the factors of $f(x)$ is solvable modulo 8. Since a congruence of the form $x^2 \equiv r \pmod{8}$ is solvable only if $r \equiv 1 \pmod{8}$, Lemma 2 establishes the first statement in our theorem.

Next, for any odd prime p , we must have at least one of the factors of $f(x)$, say $(x^2 - r)$, solvable modulo p^j , $j = 1, 2, 3, \dots$. As r is squarefree we have $p \nmid r$ and thus $\left(\frac{r}{p}\right) = +1$ as required. This establishes the second statement in our theorem.

On the other hand, assuming that p is an odd prime and one of

$$\left(\frac{a}{p}\right), \left(\frac{b}{p}\right), \left(\frac{a_1 b_1}{p}\right)$$

has the value $+1$, we may use Lemma 1 to deduce solvability of a factor of $f(x)$ and hence $f(x)$ modulo p^j , $j = 1, 2, 3, \dots$. Similarly, assuming that the constant term of one of the factors of $f(x)$ is of the form $8k + 1$ we may use Lemma 2 to conclude that this factor is solvable modulo 2^j , $j = 1, 2, 3, \dots$ and conclude that $f(x)$ is solvable modulo 2^j , $j = 1, 2, 3, \dots$.

Solvability modulo m for any positive integer m follows from the Chinese Remainder Theorem. This concludes the proof of our theorem. $\square$

3 Corollaries

We use our theorem to construct three simple families of polynomials, solvable modulo every positive integer m .

Corollary 3.1 *Let n be an odd squarefree integer with $n \neq 1$. Then*

$$f(x) = (x^2 - 2)(x^2 - n)(x^2 - 2n)$$

is solvable modulo m for every positive integer m if and only if the prime factors of n have the form $8k \pm 1$ and $n \equiv 1 \pmod{8}$.

Proof If $f(x)$ has a root modulo every positive integer m , then our theorem requires that one factor of $f(x)$ have constant term with the form $8k + 1$. Clearly this term must be n . Suppose that p is an odd prime. Our theorem implies that one of the Legendre symbols

$$\left(\frac{2}{p}\right), \left(\frac{n}{p}\right), \left(\frac{2n}{p}\right)$$

has the value $+1$. This is automatic if $p \nmid n$ by multiplicativity of the Legendre symbol. If $p \mid n$ then from our theorem we deduce that

$$\left(\frac{2}{p}\right) = 1$$

implying that

$$p \equiv \pm 1 \pmod{8}.$$

Reversing the argument clearly shows that $f(x)$ is solvable modulo p^j , for all primes p and all positive integers j . General solvability modulo m follows from the Chinese Remainder Theorem. $\square$

The proofs of the next two corollaries are similar to the proof of Corollary 3.1.

Corollary 3.2 *Let n be a squarefree integer with $n \neq \pm 1$. Then*

$$f(x) = (x^2 + 1)(x^2 - n)(x^2 + n)$$

is solvable modulo m for every positive integer if and only if the prime factors of n have the form $4k + 1$ and $n \equiv \pm 1 \pmod{8}$.

Corollary 3.3 *Let n be an odd squarefree integer with $n \neq 1$. Then*

$$f(x) = (x^2 + 2)(x^2 - n)(x^2 + 2n)$$

is solvable modulo m for every positive integer if and only if the prime factors of n have the form $8k + 1$ or $8k + 3$ and $n \equiv 1 \pmod{8}$.

We finish by giving some examples of polynomials obtained from our corollaries.

n	Corollary	Polynomial
17	3.1	$(x^2 - 2)(x^2 - 17)(x^2 - 34)$
-7	3.1	$(x^2 - 2)(x^2 + 7)(x^2 + 14)$
17	3.2	$(x^2 + 1)(x^2 - 17)(x^2 + 17)$
-41	3.2	$(x^2 + 1)(x^2 + 41)(x^2 - 41)$
17	3.3	$(x^2 + 2)(x^2 - 17)(x^2 + 34)$
33	3.3	$(x^2 + 2)(x^2 - 33)(x^2 + 66)$

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