Generalized Functions for the Fractional Calculus and Dirichlet Averages

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Abstract

The objective of this article is to investigate the Dirichlet averages of the more generalized function for the fractional calculus. (in particular R-function introduced by Hartley and Lorenzo) Representation of such relations are obtained in terms of fractional integral operators in particular Reimann-liouville integrals. Some interesting special cases of the established results associated with generalized Mittag-Leffler functions due to Saxena, et al , Srivastava and Tomovski, are deduced.

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INTRODUCTION

Extension of concept has been a good motivating factor for research and development in mathematical sciences. Examples are the extension of real number system to complex numbers, factorial of integers to gamma functions etc. Extension of integer order derivatives and integrals to arbitrary order. In literature, number of papers have used two important functions for the solution of fractional order differential equations, the well-known Mittag-leffler function $E_q^\gamma(at^q)$ called “Queen function” of fractional calculus and the F-function $F_q(a, t)$ of Hartley and Lorenzo(1998). As these functions provided direct solutions and important understanding for the fundamental linear fractional order differential equations and for the related initial value problem. But in this paper we are taken the generalized function for the fractional calculus called R-function, it is of significant usefulness to develop a generalized function when fractionally differ integrated (by any order) returns itself. Such a function would greatly ease the analysis of fractional order differential equation. To overcome this, the following function was proposed by Hartley and Lorenzo, called R-function defined as follows [5&6]

$$R_{q,v}[a, c, t] = \sum_{n=0}^{\infty} \frac{a^n(t-c)^{(n+1)q-1-v}}{\Gamma((n+1)q-v)} \quad (1)$$

The more compact notation

$$R_{q,v}[a, t-c] = \sum_{n=0}^{\infty} \frac{a^n(t-c)^{(n+1)q-1-v}}{\Gamma((n+1)q-v)}$$

When $c=0$, we get

$$R_{q,v}[a, t-c] = \sum_{n=0}^{\infty} \frac{a^n(t-c)^{(n+1)q-1-v}}{\Gamma((n+1)q-v)}$$

Put $v=q-1$, we get Mittag-Leffler function

$$R_{q,q-1}[a, t] = \sum_{n=0}^{\infty} \frac{a^n(t)^{nq}}{\Gamma(nq+1)} = E(at^q)$$

Taking $a=1$, $v=q-\beta$

$$R_{q,v}[1, t] = t^{\beta-1} E_{q,\beta}(t^q)$$

Definitions and preliminaries used in the paper

A typical combination of gamma functions is given a name, because it has simple and useful integral representation

$$B(x, y) = \int_0^1 u^{x-1} (1-u)^{y-1} d(u), \quad \operatorname{Re}(x) > 0, \operatorname{Re}(y) > 0. \quad (2)$$

If x and y are real and +ve, then normalized integrand of (2) occurs in statistics as the type-I beta density function. For complex values of x, y the integrand defines complex measure on the unit interval, as

$$d_\mu(u) = \frac{u^{x-1} (1-u)^{y-1} d(u)}{B(x, y)}$$

$$\int_0^1 d_\mu(u) = 1, \quad x, y \in \mathbb{C}$$

Where $\mathbb{C} = \{z \in \mathbb{C}; z > 0\}$

And $\mathbb{R} = \{x \in \mathbb{R}; x > 0\}$.

$$\text{In the same manner, } d_\mu(u, v) = \frac{u^{x-1} v^{y-1} (1-u-v)^{z-1} d(v) d(u)}{B(x, y, z)}$$

$$\int_0^1 d_\mu(u, v) = 1, \quad x, y, z \in \mathbb{C}$$

Where the triangular region of integration is $\{(u, v) : u \geq 0, v \geq 0; u+v \leq 1\}$ known as the standard simplex in $\mathbb{R}^2$.

The standard simplex in $\mathbb{R}^3$ is $\{(u, v, w) : u \geq 0, v \geq 0; w \geq 0; u+v+w \leq 1\}$ a solid tetrahedron with vertices $(0,0,0), (1,0,0), (0,1,0), (0,0,1)$.

Dirichlet measure: In general, let $b \in \mathbb{C}^k$, $k \geq 2$, and $E = E_{k-1}$ be the standard simplex in $\mathbb{R}^{k-1}$. The complex measure μ_b , defined on E , by

$$d_{\mu_b}(u) = \frac{1}{B(b)} \int_0^1 u_1^{b_1-1} u_2^{b_2-1} \dots u_{k-1}^{b_{k-1}-1} (1-u_1-u_2-\dots-u_{k-1})^{b_k-1} d(u_1) d(u_2) \dots d(u_{k-1}),$$

$$\mu_b(E) = \int_E d_{\mu_b}(u) = 1$$

Dirichlet averages : A Dirichlet average of a function denotes a certain kind of integral average with respect to a Dirichlet measure. The concept of Dirichlet average was introduced by Carlson in 1977 and others [1] as follows

$$F(b, z) = \int f(uoz) d_{\mu_b}(u).$$

where μ_b is a Dirichlet measure on the standard simplex E in $\mathbb{R}^{k-1}$ and

$$uoz = \sum_{i=1}^k u_i z_i, \quad u_k = 1 - u_1 - u_2 - \dots - u_{k-1}.$$

The quantity uoz is both convex combination of $z_1, z_2, z_3, \dots, z_k$ and linear function of independent variables $u_1, u_2, \dots, u_{k-1}$.

For, $n=1$, $F(b,z) = f(z)$, for $n=2$, we have

$$d_{\mu_{\beta,\beta'}}(u) = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')} u^{\beta-1} (1-u)^{\beta'-1} d(u).$$

Carlson [1] investigate the average for *the power function*

$$f(z) = z^k, k \in \mathbb{R}.$$

$$R_k(b;z) = \int_{E_{n-1}} (uoz)^k d_{\mu_{\beta}}(u),$$

For $n=2$, he proved

$$R_k(\beta, \beta'; x, y) = \frac{1}{B(\beta, \beta')} \int_0^1 (ux + (1-u)y)^k u^{\beta-1} (1-u)^{\beta'-1} d(u).$$

Where $\beta, \beta' \in \mathbb{C}$, $\min [R(\beta), R(\beta')] > 0$, $x, y \in \mathbb{R}$.

Our paper is devoted to the study of the Dirichlet averages of the generalized function for the fractional calculus (R-Function) (1) in the form

$$M_{q,v}[(a, c); (\beta, \beta'; x, y)] = \int_{E_1} R_{q,v}[a, c; uoz] d_{\mu_{\beta,\beta'}}(u) \quad (2)$$

Reimann-Liouville fractional integral of order $\alpha \in \mathbb{C}$, $R(\alpha) > 0$.

$$(I_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, (x > a, \alpha \in \mathbb{R}). \quad (3)$$

Representation of R_k and $M_{q,v}$ in terms of Reimann-Liouville Fractional Integrals

In this section we deduced representations for the Dirichlet averages $R_k(\beta, \beta'; x, y)$ and $M_{q,v}(\beta, \beta'; x, y)$ with fractional integral operators.

Theorem: Let $(t-c), q, v, \beta, \beta' \in \mathbb{C}$, $R(\beta) > 0$, $R(\beta') > 0$, finally, let x, y be such that $x > y > 0$. Then Dirichlet averages of R-function (1) is given by

$$M_{q,v}[(a, c); (\beta, \beta'; x, y)] = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)(x-y)^{\beta+\beta'-1}} [(I_{0+}^{\beta} t^{\beta-1} R_{q,v}(a; y+t))(x-y), \\ t > c \geq 0, q \geq 0.$$

Proof: $M_{q,v}[(a, c); (\beta, \beta'; x, y)] =$

$$\frac{1}{B(\beta, \beta')} \sum_{n=0}^{\infty} \frac{(a)^n}{\Gamma(n+1)q-v} \int_0^1 [y + u(x-y)]^{q(n+1)-1-v} u^{\beta-1} (1-u)^{\beta'-1} d(u).$$

$$\text{Put } u(x-y) = t$$

$$M_{q,v}[(a, c); (\beta, \beta'; x, y)] =$$

$$\frac{\Gamma(\beta + \beta^l)}{\Gamma(\beta^l) \Gamma(\beta)} \sum_{n=0}^{\infty} \frac{(a)^n}{\Gamma(n+1)q - v} \int_0^{x-y} [y + t - c]^n \left\{ \frac{t}{x-y} \right\}^{\beta-1} \left(1 - \frac{t}{x-y} \right)^{\beta^l-1} \frac{dt}{x-y} \quad (1)$$

Thus, we have

$$M_{q,v}[(a, c); (\beta, \beta^l; x, y)] = \frac{(x-y)^{1-\beta-\beta^l}}{B(\beta, \beta^l)} \int_0^{x-y} t^{\beta-1} R_{q,v}(a; y+t) (x-y-t)^{\beta^l-1} dt$$

Hence

$$M_{q,v}[(a, c); (\beta, \beta^l; x, y)] = \frac{\Gamma(\beta+\beta^l)}{\Gamma(\beta)(x-y)^{\beta+\beta^l-1}} [(I_{0+}^{\beta^l} t^{\beta-1} R_{q,v}(a; y+t)](x-y), \quad t > c \geq 0, q \geq 0.$$

This proves the theorem.

Special Cases

In this section, we consider some particular cases of the above theorem by setting $V = q-1, c = 0$, we get Dirichlet averages of ML-function reported [2]

$$M_{\alpha,\delta}^{\gamma}(\beta, \beta^l; x, y) = \frac{\Gamma(\beta+\beta^l)}{\Gamma(\beta)(x-y)^{\beta+\beta^l-1}} \int_0^{x-y} t^{\beta-1} E_{\alpha,\beta}^{\gamma}(y+t) (x-y-t)^{\beta^l-1} dt.$$

$$M_{\alpha,\delta}^{\gamma}(\beta, \beta^l; x, y) = \frac{\Gamma(\beta+\beta^l)}{\Gamma(\beta)(x-y)^{\beta+\beta^l-1}} \{ I_{0+}^{\alpha} t^{\beta-1} E_{\alpha,\beta}^{\gamma}(y+t) \} (x-y).$$

Further, by setting $y = 0$ in above equations, we get well-known result reported in [3&4] like as

$$M_{\alpha,\delta}^{\gamma}(\beta, \beta^l; x, 0) = \frac{\Gamma(\beta+\beta^l)}{\Gamma(\gamma) \Gamma(\beta)} {}_2\psi_2 \left[\begin{matrix} (\gamma, 1), & (\beta, 1) \\ (\beta + \beta^l, 1), & (\delta, 1) \end{matrix} ; x \right].$$

In particular, when $\beta + \beta^l = \gamma$,

$$M_{\alpha,\delta}^{\gamma}(\beta, \gamma - \beta; x, 0) = \frac{1}{\Gamma(\beta)} {}_1\psi_1 \left[\begin{matrix} (\beta, 1) \\ (\delta, \alpha) \end{matrix} ; x \right].$$

CONCLUSION

This paper has presented a new function for the fractional calculus, it is called the R-function. The R-function is unique in that it contains all of the derivatives and

integrals of the F-function. The R-function has the eigen –property, that is it returns itself on q^{th} order differ-integration. Special cases of the R-function also include the exponential function, the sin , cosine , hyperbolic sine and hyperbolic Cosine functions.

The value of the R-function is clearly demonstrated in the dynamic thermocouple problem where it enables the analyst to directly inverse transform the Laplace domain solution , to obtain the time domain solution.

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