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An M/G/1 Queue with Two-Stage Heterogeneous Service and Single Working Vacation

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Abstract

We study an M/G/1 queue with two stages of heterogeneous service and single working vacation. Using supplementary variable approach we derive the probability generating function for the number of customers and the average number of customers in the system. Some special cases of interest are discussed.

Mathematics Subject Classification: 60K25, 60K30

Keywords: Poisson arrivals, Two-Stage Heterogeneous Service, Working Vacation, Supplementary Variable Technique

1. Introduction

The $M/G/1$ queue with vacation time has been studied by number of authors. To mention a few references we would name Cooper (1981), Levy et.al. (1975), Yukata Baba (1986), Teghem (1986) and Doshi (1990). Recently a class of semi-vacation policies has been introduced by Servi and Finn. Such a vacation is called working vacation(WV). The server works at a lower rate rather than completely stops service during a vacation. Servi and Finn (2002) studied an $M/M/1$ queue with multiple working vacation and obtained the probability generating function for the number of customers in the system and the waiting time distribution. Some other notable works were done by Wu and Takagi (2006), Tian et.al. (2008) and Afthab Begum (2011).

In many queueing situations all arriving customers require the main service and only some of them may require the subsidiary service provided by the server. Madan (2000) investigated an $M/G/1$ queueing system with second optional service. The single server vacation queueing models with second optional service were analyzed by many authors including Kalyanaraman et.al. (2008) and Thangaraj (2010).

In this paper we study an $M/G/1$ queue with Two-stage Heterogeneous service and single working vacation(SWV). The organization of the paper is as follows. In section 2 we described the model. In section 3 we obtained the steady state probability generating function. In section 4 the performance measures are obtained and in section 5 particular cases have been discussed.

2. The Model description

We consider a single server queueing system in which customers arrive according to a Poisson process with mean arrival rate $\lambda(> 0)$. The service discipline is FCFS. Each arriving customer undergoes the first essential stage (FES) of service which has general distribution with the distribution function $S_{b_1}(x)$, the probability density function $s_{b_1}(x)$ and the Laplace-Stieltjes transform[LST] $S_{b_1}^*(\theta)$ where S_{b_1} is the service time of the first stage service. After the completion of FES of service the customer may opt for the second optional stage (SOS) of service with probability p or the customer

may leave the system without taking the SOS of service with probability $q(p + q = 1)$.

The SOS of service also follows the general distribution with the distribution function $S_{b_2}(x)$, the probability density function $s_{b_2}(x)$ and $S_{b_2}^*(\theta)$ be the LST of $S_{b_2}(x)$, where S_{b_2} is the service time of the second stage service.

Whenever the system becomes empty at a service completion instant the server starts working vacation and the duration of the vacation time follows an exponential distribution with rate η . At a vacation completion instant, if there are customers in the system the server will start a new busy period. Otherwise he waits until a customer arrives. This type of vacation policy is called single working vacation. During the working vacation period the server also provides two stages of service. The first essential stage of service time S_{v_1} of a typical customer follows a general distribution with the distribution function $S_{v_1}(x)$ [the probability density function $s_{v_1}(x)$ and $S_{v_1}^*(\theta)$, the LST] and the SOS of service time S_{v_2} also follows a general distribution with the distribution function $S_{v_2}(x)$ [the probability density function $s_{v_2}(x)$ and $S_{v_2}^*(\theta)$, the LST].

Further, it is noted that the service interrupted at the end of a vacation is lost and it is restarted with a different distribution at the beginning of the following service period. Inter arrival times, service times and working vacation times are mutually independent of each other.

3. The System Size Distribution at a Random Epoch

The system size distribution at an arbitrary time will be treated by the supplementary variable technique. That is, from the joint distribution of the queue length and the remaining service time of the customer in service if the server is busy, or the remaining service time of the customer if the server is on WV.

Assuming that the system is in steady state condition. Let us define the following random variables.

$N(t)$ - the system size at time t .

$S_{b_1}^0(t)$ - the remaining service time for the FES of service in not WV period.

$S_{b_2}^0(t)$ - the remaining service time for the SOS of service in not WV period.

$S_{v_1}^0(t)$ - the remaining service time for the FES of service in WV period.

$S_{v_2}^0(t)$ - the remaining service time for the SOS of service in WV period.

$$Y(t) = \begin{cases} 0 & \text{if the server is idle on WV period} \\ 1 & \text{if the server is idle on not WV period} \\ 2 & \text{if the server is busy for giving FES of service in not WV period} \\ 3 & \text{if the server is busy for giving SOS of service in not WV period} \\ 4 & \text{if the server is busy for giving FES of service in WV period} \\ 5 & \text{if the server is busy for giving SOS of service in WV period} \end{cases}$$

so that the supplementary variables $S_{b_1}^0(t), S_{b_2}^0(t), S_{v_1}^0(t)$ and $S_{v_2}^0(t)$ are introduced in order to obtain bivariate Markov Process $\{N(t), \partial(t)\}$, where

$$\partial(t) = \begin{cases} S_{b_1}^0(t) & \text{if } Y(t) = 2 \\ S_{b_2}^0(t) & \text{if } Y(t) = 3 \\ S_{v_1}^0(t) & \text{if } Y(t) = 4 \\ S_{v_2}^0(t) & \text{if } Y(t) = 5 \end{cases}$$

We define the following limiting probabilities:

$$Q_0 = \lim_{t \rightarrow \infty} \Pr\{N(t) = 0, Y(t) = 0\}$$

$$P_0 = \lim_{t \rightarrow \infty} \Pr\{N(t) = 0, Y(t) = 1\}$$

$$Q_{n,i}(x) = \lim_{t \rightarrow \infty} \Pr\{N(t) = n, Y(t) = \{4, 5\}, x < S_{v_i}^0(t) \leq x + dx\}, i = 1, 2; n \geq 1.$$

$$P_{n,i}(x) = \lim_{t \rightarrow \infty} \Pr\{N(t) = n, Y(t) = \{2, 3\}, x < S_{b_i}^0(t) \leq x + dx\}, i = 1, 2; n \geq 1.$$

By considering the steady state, we have the following system of the differential difference equations.

$$\begin{aligned} (\lambda + \eta)Q_0 &= qP_{1,1}(0) + P_{1,2}(0) + qQ_{1,1}(0) + Q_{1,2}(0) & (1) \\ -\frac{d}{dx}Q_{1,1}(x) &= -(\lambda + \eta)Q_{1,1}(x) + qQ_{2,1}(0)s_{v_1}(x) + Q_{2,2}(0)s_{v_1}(x) + \lambda Q_0 s_{v_1}(x) & (2) \\ -\frac{d}{dx}Q_{n,1}(x) &= -(\lambda + \eta)Q_{n,1}(x) + qQ_{n+1,1}(0)s_{v_1}(x) + Q_{n+1,2}(0)s_{v_1}(x) + \lambda Q_{n-1,1}(x); \end{aligned}$$

$$n > 1 \quad (3)$$

$$-\frac{d}{dx}Q_{1,2}(x) = -(\lambda + \eta)Q_{1,2}(x) + pQ_{1,1}(0)s_{v_2}(x) \quad (4)$$

$$-\frac{d}{dx}Q_{n,2}(x) = -(\lambda + \eta)Q_{n,2}(x) + pQ_{n,1}(0)s_{v_2}(x) + \lambda Q_{n-1,2}(x); n > 1 \quad (5)$$

$$\lambda P_0 = \eta Q_0 \quad (6)$$

$$\begin{aligned} -\frac{d}{dx}P_{1,1}(x) &= -\lambda P_{1,1}(x) + qP_{2,1}(0)s_{b_1}(x) + P_{2,2}(0)s_{b_1}(x) \\ &\quad + \eta s_{b_1}(x) \int_0^\infty Q_{1,1}(y)dy + \lambda P_0 s_{b_1}(x) \end{aligned} \quad (7)$$

$$\begin{aligned} -\frac{d}{dx}P_{n,1}(x) &= -\lambda P_{n,1}(x) + qP_{n+1,1}(0)s_{b_1}(x) + P_{n+1,2}(0)s_{b_1}(x) \\ &\quad + \eta s_{b_1}(x) \int_0^\infty Q_{n,1}(y)dy + \lambda P_{n-1,1}(x); n > 1 \end{aligned} \quad (8)$$

$$-\frac{d}{dx}P_{1,2}(x) = -\lambda P_{1,2}(x) + pP_{1,1}(0)s_{b_2}(x) + \eta s_{b_2}(x) \int_0^\infty Q_{1,2}(y)dy \quad (9)$$

$$\begin{aligned} -\frac{d}{dx}P_{n,2}(x) &= -\lambda P_{n,2}(x) + pP_{n,1}(0)s_{b_2}(x) \\ &\quad + \eta s_{b_2}(x) \int_0^\infty Q_{n,2}(y)dy + \lambda P_{n-1,2}(x); n > 1 \end{aligned} \quad (10)$$

We define Laplace Stieltjes transforms and probability generating functions as follows,

$$S_{b_i}^*(\theta) = \int_0^\infty e^{-\theta x} s_{b_i}(x) dx; \quad i = 1, 2$$

$$S_{v_i}^*(\theta) = \int_0^\infty e^{-\theta x} s_{v_i}(x) dx; \quad i = 1, 2$$

$$Q_{n,i}^*(\theta) = \int_0^\infty e^{-\theta x} Q_{n,i}(x) dx; \quad i = 1, 2$$

$$P_{n,i}^*(\theta) = \int_0^\infty e^{-\theta x} P_{n,i}(x) dx; \quad i = 1, 2$$

$$Q_i^*(z, \theta) = \sum_{n=1}^\infty z^n Q_{n,i}^*(\theta); \quad i = 1, 2$$

$$Q_i(z, 0) = \sum_{n=1}^\infty z^n Q_{n,i}(0); \quad i = 1, 2$$

$$P_i^*(z, \theta) = \sum_{n=1}^\infty z^n P_{n,i}^*(\theta); \quad i = 1, 2$$

$$P_i(z, 0) = \sum_{n=1}^\infty z^n P_{n,i}(0); \quad i = 1, 2$$

Taking the LST of (2) to (5) and (7) to (10) we have

$$\begin{aligned} \theta Q_{1,1}^*(\theta) - Q_{1,1}(0) &= (\lambda + \eta)Q_{1,1}^*(\theta) - qQ_{2,1}(0)S_{v_1}^*(\theta) - Q_{2,2}(0)S_{v_1}^*(\theta) - \lambda Q_0 S_{v_1}^*(\theta) \\ (11) \end{aligned}$$

$$\theta Q_{n,1}^*(\theta) - Q_{n,1}(0) = (\lambda + \eta)Q_{n,1}^*(\theta) - qQ_{n+1,1}(0)S_{v_1}^*(\theta) - Q_{n+1,2}(0)S_{v_1}^*(\theta) - \lambda Q_{n-1,1}^*(\theta);$$

$$n > 1 \quad (12)$$

$$\theta Q_{1,2}^*(\theta) - Q_{1,2}(0) = (\lambda + \eta)Q_{1,2}^*(\theta) - pQ_{1,1}(0)S_{v_2}^*(\theta) \quad (13)$$

$$\theta Q_{n,2}^*(\theta) - Q_{n,2}(0) = (\lambda + \eta)Q_{n,2}^*(\theta) - pQ_{n,1}(0)S_{v_2}^*(\theta) - \lambda Q_{n-1,2}^*(\theta); \quad n > 1 \quad (14)$$

$$\begin{aligned} \theta P_{1,1}^*(\theta) - P_{1,1}(0) &= \lambda P_{1,1}^*(\theta) - qP_{2,1}(0)S_{b_1}^*(\theta) \\ &\quad - P_{2,2}(0)S_{b_1}^*(\theta) - \lambda P_{0,b_1}^*(\theta) - \eta S_{b_1}^*(\theta) \int_0^\infty Q_{1,1}(y)dy \end{aligned} \quad (15)$$

$$\begin{aligned} \theta P_{n,1}^*(\theta) - P_{n,1}(0) &= \lambda P_{n,1}^*(\theta) - qP_{n+1,1}(0)S_{b_1}^*(\theta) - P_{n+1,2}(0)S_{b_1}^*(\theta) \\ &\quad - \eta S_{b_1}^*(\theta) \int_0^\infty Q_{n,1}(y)dy - \lambda P_{n-1,1}^*(\theta); \quad n > 1 \end{aligned} \quad (16)$$

$$\theta P_{1,2}^*(\theta) - P_{1,2}(0) = \lambda P_{1,2}^*(\theta) - pP_{1,1}(0)S_{b_2}^*(\theta) - \eta S_{b_2}^*(\theta) \int_0^\infty Q_{1,2}(y)dy \quad (17)$$

$$\begin{aligned} \theta P_{n,2}^*(\theta) - P_{n,2}(0) &= \lambda P_{n,2}^*(\theta) - pP_{n,1}(0)S_{b_2}^*(\theta) \\ &\quad - \eta S_{b_2}^*(\theta) \int_0^\infty Q_{n,2}(y)dy - \lambda P_{n-1,2}^*(\theta); \quad n > 1 \end{aligned} \quad (18)$$

z^n times (12) summing over n from 2 to ∞ is added up with z times (11) we get

$$\begin{aligned} [\theta - (\lambda - \lambda z + \eta)]Q_1^*(z, \theta) &= \\ Q_1(z, 0)[1 - \frac{q}{z}S_{v_1}^*(\theta)] + S_{v_1}^*(\theta) &\left[qQ_{1,1}(0) - \frac{Q_2(z, 0)}{z} + Q_{1,2}(0) - \lambda Q_0z \right] \end{aligned} \quad (19)$$

z^n times (14) summing over n from 2 to ∞ is added up with z times (13) we get

$$[\theta - (\lambda - \lambda z + \eta)]Q_2^*(z, \theta) = Q_2(z, 0) - pS_{v_2}^*(\theta)Q_1(z, 0) \quad (20)$$

Inserting $\theta = (\lambda - \lambda z + \eta)$ in (19) and (20) we get

$$Q_1(z, 0) = \left[\frac{zS_{v_1}^*(\lambda - \lambda z + \eta)[\lambda Q_0z - qQ_{1,1}(0) - Q_{1,2}(0)]}{z - (1 - p)S_{v_1}^*(\lambda - \lambda z + \eta) - pS_{v_1}^*(\lambda - \lambda z + \eta)S_{v_2}^*(\lambda - \lambda z + \eta)} \right] \quad (21)$$

The denominator of the above equation has a unique root z_1 in $(0, 1)$. Therefore

$$qQ_{1,1}(0) + Q_{1,2}(0) = \lambda Q_0z_1$$

Substituting this in (21) we have

$$Q_1(z, 0) = \left[\frac{\lambda z Q_0 S_{v_1}^*(\lambda - \lambda z + \eta)(z - z_1)}{z - (1 - p)S_{v_1}^*(\lambda - \lambda z + \eta) - pS_{v_1}^*(\lambda - \lambda z + \eta)S_{v_2}^*(\lambda - \lambda z + \eta)} \right] \quad (22)$$

and

$$Q_2(z, 0) = \left[\frac{pS_{v_2}^*(\lambda - \lambda z + \eta)\lambda z Q_0(z - z_1)S_{v_1}^*(\lambda - \lambda z + \eta)}{z - (1-p)S_{v_1}^*(\lambda - \lambda z + \eta) - pS_{v_1}^*(\lambda - \lambda z + \eta)S_{v_2}^*(\lambda - \lambda z + \eta)} \right]$$

(23) Substituting (22) and (23) in (19) and (20) and putting $\theta = 0$, we have

$$Q_1^*(z, 0) = \left[\frac{\lambda z Q_0(z - z_1)[1 - S_{v_1}^*(\lambda - \lambda z + \eta)]}{(\lambda - \lambda z + \eta)[z - (1-p)S_{v_1}^*(\lambda - \lambda z + \eta) - pS_{v_1}^*(\lambda - \lambda z + \eta)S_{v_2}^*(\lambda - \lambda z + \eta)]} \right]$$

$$Q_2^*(z, 0) = \left[\frac{p\lambda Q_0 z(z - z_1)S_{v_1}^*(\lambda - \lambda z + \eta)[1 - S_{v_2}^*(\lambda - \lambda z + \eta)]}{(\lambda - \lambda z + \eta)[z - (1-p)S_{v_1}^*(\lambda - \lambda z + \eta) - pS_{v_1}^*(\lambda - \lambda z + \eta)S_{v_2}^*(\lambda - \lambda z + \eta)]} \right]$$

z^n times (16) summing over n from 2 to ∞ is added up with z times (15) we get

$$\begin{aligned} [\theta - (\lambda - \lambda z)]P_1^*(z, \theta) &= P_1(z, 0) \left[\frac{z - qS_{b_1}^*(\theta)}{z} \right] \\ &\quad - S_{b_1}^*(\theta) \left[\eta Q_1^*(z, 0) + \lambda z P_0 - qP_{1,1}(0) + \frac{P_2(z, 0)}{z} - P_{1,2}(0) \right] \end{aligned}$$

z^n times (18) summing over n from 2 to ∞ is added up with z times (17) we get

$$[\theta - (\lambda - \lambda z)]P_2^*(z, \theta) = P_2(z, 0) - pS_{b_2}^*(\theta)P_1(z, 0) - \eta S_{b_2}^*(\theta)Q_2^*(z, 0) \quad (27)$$

Inserting $\theta = (\lambda - \lambda z)$ in (26) and (27) and also substituting $qP_{1,1}(0) + P_{1,2}(0) = Q_0[\lambda(1 - z_1) + \eta]$ in (26) we get

$$P_1(z, 0) = \left[\frac{S_{b_1}^*(\lambda - \lambda z)[\eta z Q_1^*(z, 0) + \eta S_{b_2}^*(\lambda - \lambda z)Q_2^*(z, 0) - \lambda z Q_0(1 - z_1) - \eta z(1 - z)Q_0]}{z - (1-p)S_{b_1}^*(\lambda - \lambda z) - pS_{b_1}^*(\lambda - \lambda z)S_{b_2}^*(\lambda - \lambda z)} \right]$$

and

$$\begin{aligned} P_2(z, 0) &= [S_{b_2}^*(\lambda - \lambda z) \left\{ pzS_{b_1}^*(\lambda - \lambda z)[\eta Q_1^*(z, 0) - \eta(1 - z)Q_0 - \lambda Q_0(1 - z_1)] \right. \\ &\quad \left. + \eta Q_2^*(z, 0)[z - (1-p)S_{b_1}^*(\lambda - \lambda z)] \right\}] \\ &\quad \overline{z - (1-p)S_{b_1}^*(\lambda - \lambda z) - pS_{b_1}^*(\lambda - \lambda z)S_{b_2}^*(\lambda - \lambda z)} \end{aligned} \quad (29)$$

Substituting (28) and (29) in (26) and (27) and inserting $\theta = 0$, we get

$$P_1^*(z, 0) = \frac{Q_0}{D_1(z)D_2(z)} \left\{ z[1 - S_{b_1}^*(\lambda - \lambda z)] \{ \lambda \eta z(z - z_1)[1 - S_{v_1}^*(\lambda - \lambda z + \eta)] \right. \\ \left. + \lambda p \eta S_{b_2}^*(\lambda - \lambda z)(z - z_1) S_{v_1}^*(\lambda - \lambda z + \eta)(1 - S_{v_2}^*(\lambda - \lambda z + \eta)) \right. \\ \left. - [\eta(1 - z) + \lambda(1 - z_1)](\lambda - \lambda z + \eta) \right. \\ \left. \times [z - (1 - p)S_{v_1}^*(\lambda - \lambda z + \eta) - pS_{v_1}^*(\lambda - \lambda z + \eta) S_{v_2}^*(\lambda - \lambda z + \eta)] \right\} \quad (30)$$

$$P_2^*(z, 0) = \frac{Q_0}{D_1(z)D_2(z)} \left\{ pz[1 - S_{b_2}^*(\lambda - \lambda z)] \{ \lambda \eta(z - z_1) S_{v_1}^*(\lambda - \lambda z + \eta) \right. \\ \times (1 - S_{v_2}^*(\lambda - \lambda z + \eta))(z - (1 - p)S_{b_1}^*(\lambda - \lambda z)) \\ + \lambda \eta z(z - z_1) S_{b_1}^*(\lambda - \lambda z)(1 - S_{v_1}^*(\lambda - \lambda z + \eta)) \\ - [\eta(1 - z) + \lambda(1 - z_1)](\lambda - \lambda z + \eta) S_{b_1}^*(\lambda - \lambda z) \\ \times [z - (1 - p)S_{v_1}^*(\lambda - \lambda z + \eta) \\ \left. - pS_{v_1}^*(\lambda - \lambda z + \eta) S_{v_2}^*(\lambda - \lambda z + \eta)] \right\} \quad (31)$$

where

$$D_1(z) = (\lambda - \lambda z)[z - (1 - p)S_{v_1}^*(\lambda - \lambda z + \eta) - pS_{v_1}^*(\lambda - \lambda z + \eta) S_{v_2}^*(\lambda - \lambda z + \eta)] \quad (32)$$

$$D_2(z) = (\lambda - \lambda z + \eta)[z - (1 - p)S_{b_1}^*(\lambda - \lambda z) - pS_{b_1}^*(\lambda - \lambda z) S_{b_2}^*(\lambda - \lambda z)] \quad (33)$$

We define

$$P_B(z) = P_1^*(z, 0) + P_2^*(z, 0) + P_0 \quad (34)$$

as the probability generating function for the number of customers in the system when the server is on not WV period and

$$P_V(z) = Q_1^*(z, 0) + Q_2^*(z, 0) + Q_0 \quad (35)$$

as the probability generating function for the number of customers in the system when the server is on WV period then

$$P(z) = P_B(z) + P_V(z) \quad (36)$$

as the probability generating function for the number of customers in the system. We shall now use the normalizing condition $P(1) = 1$ to determine the only unknown Q_0 , which appears in (36). Substituting $z = 1$ in (36) and using L'hospital's rule we obtain on simplification

$$Q_0 = \frac{(1 - \rho_b)}{\left[\frac{\eta}{\lambda} + \frac{(\lambda - \lambda z_1 + \eta)}{\eta} - \frac{\lambda(1 - z_1)S_{v_1}^*(\eta)[E(S_{b_1}) + pS_{v_2}^*(\eta)E(S_{b_2})]}{1 - (1 - p)S_{v_1}^*(\eta) - pS_{v_1}^*(\eta)S_{v_2}^*(\eta)} \right]} \quad (37)$$

where $\rho_b = \lambda[E(S_{b_1}) + pE(S_{b_2})]$, $E(S_{b_1})$ and $E(S_{b_2})$ are the mean service times of stage 1 and stage 2 respectively. From (37) we obtain the system stability condition,

$$\rho_b < 1. \quad (38)$$

4. Performance Measures

Mean System Length

Let L_v and L_b denote the mean system size during the working vacation and not working vacation period respectively. Then

$$\begin{aligned} L_v &= \frac{d}{dz} P_V(z) \text{ at } z = 1 \\ &= \frac{d}{dz} \left\{ \frac{[N_1(z) + N_2(z)]}{D(z)} \lambda Q_0 \right\} \text{ at } z = 1 \text{ where} \\ N_1(z) &= z(z - z_1)[1 - S_{v_1}^*(\lambda - \lambda z + \eta)] \\ N_2(z) &= pz(z - z_1)S_{v_1}^*(\lambda - \lambda z + \eta)[1 - S_{v_2}^*(\lambda - \lambda z + \eta)] \\ D(z) &= (\lambda - \lambda z + \eta)[z - (1 - p)S_{v_1}^*(\lambda - \lambda z + \eta) - pS_{v_1}^*(\lambda - \lambda z + \eta)S_{v_2}^*(\lambda - \lambda z + \eta)] \\ L_v &= \frac{\lambda Q_0 \left\{ D(1)[N'_1(1) + N'_2(1)] - D'(1)[N_1(1) + N_2(1)] \right\}}{(D(1))^2} \text{ where} \\ N_1(1) &= (1 - z_1)(1 - S_{v_1}^*(\eta)) \\ N_2(1) &= p(1 - z_1)S_{v_1}^*(\eta)(1 - S_{v_2}^*(\eta)) \\ N'_1(1) &= (1 - S_{v_1}^*(\eta)) + (1 - z_1)(1 - s_{v_1}^*(\eta) + \lambda S_{v_1}'^*(\eta)) \\ N'_2(1) &= p \left\{ S_{v_1}^*(\eta)(1 - S_{v_2}^*(\eta)) + (1 - z_1) \left\{ S_{v_1}^*(\eta)(1 - S_{v_2}^*(\eta)) - \lambda S_{v_1}'^*(\eta) \right. \right. \\ &\quad \left. \left. + \lambda(S_{v_1}'^*(\eta)S_{v_2}^*(\eta) + S_{v_1}^*(\eta)S_{v_2}'^*(\eta)) \right\} \right\} \\ D(1) &= \eta \left\{ 1 - (1 - p)S_{v_1}^*(\eta) - pS_{v_1}^*(\eta)S_{v_2}^*(\eta) \right\} \\ D'(1) &= -\lambda \left[1 - (1 - p)S_{v_1}^*(\eta) - pS_{v_1}^*(\eta)S_{v_2}^*(\eta) \right] \\ &\quad + \eta \left[1 + \lambda(1 - p)S_{v_1}'^*(\eta) + p\lambda(S_{v_1}'^*(\eta)S_{v_2}^*(\eta) + S_{v_1}^*(\eta)S_{v_2}'^*(\eta)) \right] \end{aligned}$$

$$\begin{aligned}
L_b &= \frac{d}{dz} P_B(z) \text{ at } z = 1 \\
&= \frac{d}{dz} \left\{ \frac{[N_3(z)N_4(z) + N_5(z)N_6(z)]}{D_1(z)D_2(z)} Q_0 \right\} \text{ at } z = 1 \text{ where} \\
N_3(z) &= z[1 - S_{b_1}^*(\lambda - \lambda z)] \\
N_4(z) &= \lambda \eta z(z - z_1)[1 - S_{v_1}^*(\lambda - \lambda z + \eta)] \\
&\quad + \lambda p \eta S_{b_2}^*(\lambda - \lambda z)(z - z_1) S_{v_1}^*(\lambda - \lambda z + \eta)(1 - S_{v_2}^*(\lambda - \lambda z + \eta)) \\
&\quad - [\eta(1 - z) + \lambda(1 - z_1)](\lambda - \lambda z + \eta) \\
&\quad \times [z - (1 - p)S_{v_1}^*(\lambda - \lambda z + \eta) - pS_{v_1}^*(\lambda - \lambda z + \eta) S_{v_2}^*(\lambda - \lambda z + \eta)] \\
N_5(z) &= pz[1 - S_{b_2}^*(\lambda - \lambda z)] \\
N_6(z) &= \{ \lambda \eta(z - z_1) S_{v_1}^*(\lambda - \lambda z + \eta)(1 - S_{v_2}^*(\lambda - \lambda z + \eta))(z - (1 - p)S_{b_1}^*(\lambda - \lambda z)) \\
&\quad + \lambda \eta z(z - z_1) S_{b_1}^*(\lambda - \lambda z)(1 - S_{v_1}^*(\lambda - \lambda z + \eta)) \\
&\quad - [\eta(1 - z) + \lambda(1 - z_1)](\lambda - \lambda z + \eta) S_{b_1}^*(\lambda - \lambda z) \\
&\quad \times [z - (1 - p)S_{v_1}^*(\lambda - \lambda z + \eta) - pS_{v_1}^*(\lambda - \lambda z + \eta) S_{v_2}^*(\lambda - \lambda z + \eta)] \} \\
D_1(z) \text{ and } D_2(z) &\text{ are given respectively in equations (32) and (33)}
\end{aligned}$$

Therefore

$$L_b = \frac{Q_0}{2[D'_1(1)]^2[D'_2(1)]^2} \left\{ D'_1(1)D'_2(1)[N''_3(1)N'_4(1) + N'_3(1)N''_4(1) \right. \\
\left. + N''_5(1)N'_6(1) + N'_5(1)N''_6(1)] - (N'_3(1)N'_4(1) \right. \\
\left. + N'_5(1)N'_6(1))[D''_1(1)D'_2(1) + D'_1(1)D''_2(1)] \right\}$$

where

$$\begin{aligned}
N'_3(1) &= -\lambda E(S_{b_1}) \\
N''_3(1) &= -[2\lambda E(S_{b_1}) + \lambda^2 E(S_{b_1}^2)] \\
N'_4(1) &= \eta \lambda \{ p S_{v_1}^*(\eta)(1 - S_{v_2}^*(\eta))(1 + \lambda(1 - z_1)E(S_{b_2})) + (1 - S_{v_1}^*(\eta)) - (1 - z_1)S_{v_1}^*(\eta) \} \\
&\quad + (\eta^2 + \lambda^2(1 - z_1))[1 - (1 - p)S_{v_1}^*(\eta) - pS_{v_1}^*(\eta)S_{v_2}^*(\eta)] \\
N''_4(1) &= 2\eta \lambda \left\{ 1 - S_{v_1}^*(\eta) + p\lambda E(S_{b_2})S_{v_1}^*(\eta)(1 - S_{v_2}^*(\eta)) \right. \\
&\quad \left. + (1 - p)\lambda S_{v_1}^{*'}(\eta) + p\lambda(S_{v_1}^{*'}(\eta)S_{v_2}^*(\eta) + S_{v_1}^*(\eta)S_{v_2}^{*'}(\eta)) \right\} \\
&\quad + \eta \lambda (1 - z_1) \left\{ p\lambda^2 E(S_{b_2}^2)S_{v_1}^*(\eta)(1 - S_{v_2}^*(\eta)) - 2p\lambda^2 E(S_{b_2})S_{v_1}^{*'}(\eta) \right. \\
&\quad \left. + 2p\lambda^2 E(S_{b_2})(S_{v_1}^{*'}(\eta)S_{v_2}^*(\eta) + S_{v_1}^*(\eta)S_{v_2}^{*'}(\eta)) + 2\lambda S_{v_1}^{*'}(\eta) \right\} \\
&\quad - 2\eta \lambda [1 - (1 - p)S_{v_1}^*(\eta) - pS_{v_1}^*(\eta)S_{v_2}^*(\eta)] \\
&\quad + 2(\lambda^2(1 - z_1) + \eta^2) \left\{ 1 + \lambda(1 - p)S_{v_1}^{*'}(\eta) + p\lambda(S_{v_1}^{*'}(\eta)S_{v_2}^*(\eta) + \right.
\end{aligned}$$

$$\begin{aligned}
& \left. S_{v_1}^*(\eta) S_{v_2}^{*'}(\eta) \right\} N_5'(1) = -p\lambda E(S_{b_2}) \\
& N_5''(1) = -p[2\lambda E(S_{b_2}) + \lambda^2 E(S_{b_2}^2)] \\
& N_6'(1) = [\eta\lambda + \eta^2 + \lambda^2(1 - z_1)][1 - (1 - p)S_{v_1}^*(\eta) - pS_{v_1}^*(\eta)S_{v_2}^*(\eta)] \\
& \quad - \eta\lambda^2(1 - z_1)E(S_{b_1})S_{v_1}^*(\eta)(1 - S_{v_2}^*(\eta)) - \eta\lambda(1 - z_1)S_{v_1}^*(\eta)S_{v_2}^*(\eta) \\
& N_6''(1) = 2\eta\lambda \left\{ [1 - \lambda(1 - p)E(S_{b_1})]S_{v_1}^*(\eta)(1 - S_{v_2}^*(\eta)) - pS_{v_1}^*(\eta)(1 - S_{v_2}^*(\eta)) \right. \\
& \quad \left. + (1 - p)\lambda S_{v_1}^{*'}(\eta) + p\lambda(S_{v_1}^{*'}(\eta)S_{v_2}^*(\eta) + S_{v_1}^*(\eta)S_{v_2}^{*'}(\eta)) + \lambda E(S_{b_1})(1 - S_{v_1}^*(\eta)) \right\} \\
& \quad + \eta\lambda(1 - z_1) \left\{ p\lambda^2 E(S_{b_1}^2) + 2\lambda S_{v_1}^{*'}(\eta)S_{v_2}^*(\eta) + 2\lambda^2(1 - p)S_{v_1}^{*'}(\eta)E(S_{b_1})(1 - S_{v_2}^*(\eta)) \right. \\
& \quad \left. + 2\lambda E(S_{b_1})(1 - S_{v_1}^*(\eta)) - \lambda^2 E(S_{b_1}^2)S_{v_1}^*(\eta) + 2\lambda^2 E(S_{b_1})S_{v_1}^{*'}(\eta) \right\} \\
& \quad + [1 - (1 - p)S_{v_1}^*(\eta) - pS_{v_1}^*(\eta)S_{v_2}^*(\eta)] \left\{ 2\eta^2 \lambda E(S_{b_1}) - \lambda^3(1 - z_1)\eta E(S_{b_1}^2) \right. \\
& \quad \left. + 2\lambda^3(1 - z_1)E(S_{b_1}) \right\} + 2 \left[\eta^2 + \lambda^2(1 - z_1) - \lambda^2\eta(1 - z_1)E(S_{b_1}) \right] \\
& \quad \times \left\{ 1 + \lambda(1 - p)S_{v_1}^*(\eta) + p\lambda(S_{v_1}^{*'}(\eta)S_{v_2}^*(\eta) + S_{v_1}^*(\eta)S_{v_2}^{*'}(\eta)) \right\} \\
& D_1'(1) = -\lambda[1 - (1 - p)S_{v_1}^*(\eta) - pS_{v_1}^*(\eta)S_{v_2}^*(\eta)] \\
& D_1''(1) = -2\lambda[1 + \lambda(1 - p)S_{v_1}^{*'}(\eta) + p\lambda(S_{v_1}^{*'}(\eta)S_{v_2}^*(\eta) + S_{v_1}^*(\eta)S_{v_2}^{*'}(\eta))] \\
& D_2'(1) = \eta[1 - \lambda E(S_{b_1}) - p\lambda E(S_{b_2})] \\
& D_2''(1) = -2\lambda[1 - \lambda E(S_{b_1}) - p\lambda E(S_{b_2})] - \eta\lambda^2[E(S_{b_1}^2) + pE(S_{b_2}^2) + 2pE(S_{b_1})E(S_{b_2})]
\end{aligned}$$

where $E(S_{b_1})$ and $E(S_{b_2})$ are the mean service times of stage 1 and stage 2 respectively and $E(S_{b_1}^2)$ and $E(S_{b_2}^2)$ are the second moments of the service times of stage 1 and stage 2 respectively.

5. Particular Cases

Case i: If no customer receives the additional second stage of service and the server never takes a vacation then on setting $p = 0$ and $\eta \rightarrow \infty$ in (36) we get

$$P(z) = \frac{(1 - \lambda E(S_{b_1}))(1 - z)S_{b_1}^*(\lambda - \lambda z)}{S_{b_1}^*(\lambda - \lambda z) - z} \quad (39)$$

Equation (39) is well known generating function of the steady state queue length distribution of an M/G/1 queue irrespective of the notations. (Medhi[9])

Caseii : If the server never takes a vacation then taking limit as $\eta \rightarrow \infty$ in (36)

we get

$$P(z) = \frac{\left[z[S_{b_1}^*(\lambda - \lambda z) - 1] + pzS_{b_1}^*(\lambda - \lambda z)[S_{b_2}^*(\lambda - \lambda z) - 1] \right]}{z - (1-p)S_{b_1}^*(\lambda - \lambda z) - pS_{b_1}^*(\lambda - \lambda z)S_{b_2}^*(\lambda - \lambda z)} \times P_0 \quad (40)$$

where $P_0 = 1 - \lambda[E(S_{b_1}) + pE(S_{b_2})]$

Equation (40) is well known generating function of the steady state queue length distribution of an $M/G/1$ queue with second optional service studied by Madan (2000) irrespective of the notations.

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