On \( \mu \)-Operators

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Abstract

In this paper we introduce a new class - [\( \mu \)] - of operators acting on a complex Hilbert space \( H \): If \( T \in L(H) \) then \( T \in [\mu] \) if \( T^2 = -T^*T \). We investigate some basic properties of operators in[\( \mu \)]. We study the relation between the class [\( \mu \)] and some other well known classes of operators acting on \( H \).

Mathematics Subject Classification: 47B15

Keywords: Operator, Normal, 2-Normal, 3-Normal, quasinormal, Adjoint, Hilbert Space

1- Introduction

Let \( H \) be a complex Hilbert space and let \( L(H) \) be the algebra of all bounded linear operators acting on \( H \). If \( T \in L(H) \) then \( T^* \) is its adjoint and \( T = A + iB \) is its Cartesian decomposition. Many classes of operator in \( L(H) \) are defined according to the relation between \( T \) and \( T^* \), for example \( T \) is \textbf{normal} if and only if \( TT^* = T^*T \); \textbf{2-normal} – [ 2 ] – if and only if \( T^2T^* = T^*T^2 \); \textbf{skew-normal} – [ 5 ] - if and only if
$T^2 = T^{*2}$ ; quasinormal –[ 1 ]- if and only if $TT^*T = T^{*}T^2$. In this paper we consider operators in $L(H)$ for which $T^2 = -T^{*2}$. The class of all such operators will be denoted by $[ \mu ]$. In section two we study some of the basic properties of operators in $[ \mu ]$. In section three we study the relation between the class $[ \mu ]$ and some other previously studied classes of operators in $L(H)$.

2. Preliminary notes

We start section two by a characterization of operators in $[ \mu ]$.

**Proposition 2.1** If $T = A + iB \in L(H)$ then $T \in [ \mu ]$ if and only if $A^2 = B^2$.

**Proof.** By direct calculations we have

\[
T^2 = (A^2 - B^2) + i(AB + BA)
\]

and

\[
-T^{*2} = -(A^2 - B^2) + i(AB + BA)
\]

Suppose first that $A^2 = B^2$ then clearly $T^2 = -T^{*2}$. Suppose now that $T^2 = -T^{*2}$ then it follows from (i) and (ii) above that $(A^2 - B^2) = -(A^2 - B^2)$ which implies that $A^2 = B^2$.

**Proposition 2.2** If $T \in L(H)$ such that $T^2 = 0$ then $T \in [ \mu ]$.

**Proof.** Obvious.

**Remark 2.1** It follows from proposition 2.2 that for each real number $a$ each of the following operators acting on the two dimensional Hilbert space $R^2$ is in $[ \mu ]$:

\[
\begin{pmatrix}
 a & \bar{a} \\
 \bar{a} & a
\end{pmatrix},
\begin{pmatrix}
 -a & a \\
 a & -a
\end{pmatrix},
\begin{pmatrix}
 a & a \\
 -a & -a
\end{pmatrix},
\begin{pmatrix}
 a & -a \\
 a & -a
\end{pmatrix},
\begin{pmatrix}
 a & -\bar{a} \\
 -a & a
\end{pmatrix},
\begin{pmatrix}
 -a & -a \\
 -a & a
\end{pmatrix}.
\]

**Proposition 2.3** If $S, T \in L(H)$ are unitarily equivalent and if $T \in [ \mu ]$ then so is $S$.

**Proof.** By assumption, there is a unitary operator $U \in L(H)$ such that $S = U^{-1}TU$ which implies that $S^* = U^*T^*U = U^*U^{-1}$. Thus we have

\[
S^2 = U^{-1}TUU^{-1}TU = U^{-1}T^2U \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (i)
\]

and

\[
-S^{*2} = -U^*T^*U^{-1}U^*U^{-1} = -U^*T^{*2}U^{-1} \quad \ldots \quad (ii)
\]
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Since $U$ is unitary, $U^{-1} = U^*$ and using the fact that $T^2 = -T^{*2}$ we conclude that
$U^{-1}T^2U = -U^*T^{*2}(U^*)^{-1}$. Thus $S^2 = -S^{*2}$, which implies that $S \in [\mu]$.

The following example shows that if $T \in [\mu]$ then it is not necessary that $T + kI \in [\mu]$ for all real numbers:

**Example 2.1** Consider the operators $T = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ acting on $R^2$ then $T \in [\mu]$.
Consider the operators $T + I = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = S$ (say) then by direct calculations one can show
that $S^2 = \begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix} \neq \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix} = -S^{*2}$. Thus $S \not\in [\mu]$.

The following example shows that $[\mu]$ is not closed under addition or multiplication.

**Example 2.2** Consider the two operators $T = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$, $F = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ acting on $R^2$ then $T, F \in [\mu]$. Consider $T + F = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = S$, (say) then $S^2 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \neq \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} = -S^{*2}$. Thus $T + F \not\in [\mu]$.

Now consider $TF = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = 2I$, then $(TF)^2 = 4I \neq -4I = -(TF)^{*2}$. Thus $TF \not\in [\mu]$.
Notice that $TF = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \neq \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} = -FT$.

**Proposition 2.4** If $T, F \in [\mu]$ such that $TF = -FT$ then $T + F \in [\mu]$.

**Proof.** Since $TF = -FT$ then $TF + FT = 0$ which implies that

$T^*F + F^*T = 0$. Now

$(T + F)^2 = T^2 + TF + FT + F^2 = T^2 + F^2$
$-(T + F)^{*2} = -(T^{*2} + T^*F^* + F^*T^* + F^{*2}) = -(T^{*2} + F^{*2})$

Since $-(T^{*2} + F^{*2}) = T^2 + F^2$, we have $(T + F)^2 = -(T + F)^{*2}$ which implies that $T + F \in [\mu]$.

**Proposition 2.5** The direct sum and the tensor product of two operators in $[\mu]$ are in $[\mu]$.

**Proof.** Let $x = x_1 \oplus x_2$ be an element of $H \oplus H$ and let $T, S \in [\mu]$. then

$(T \oplus S)^2x = (T \oplus S)^2(x_1 \oplus x_2)$
Thus \((T \oplus S)^2 = -(T \oplus S)^{-2}\). Hence \(T \oplus S \in [\mu]\).

Also

\[
(T \otimes S)^2 x = (T \otimes S)^2 (x_1 \otimes x_2) = (T^2 \otimes S^2) (x_1 \otimes x_2) = T^2 x_1 \otimes S^2 x_2 = -(T^* \otimes S^*)(x_1 \otimes x_2)
\]

Thus \((T \otimes S)^2 = -(T \otimes S)^{-2}\). Hence \(T \otimes S \in [\mu]\).

The following example shows that \([\mu]\) is not convex:

**Example 2.3** Consider the two operators \(T = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}, F = \begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix}\) acting on \(R^2\) then \(T, F \in [\mu]\). Consider \(\frac{1}{2}T + \frac{1}{2}F = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = S\), (say) then \(S^2 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \neq \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} = -S^2\). Thus \(S \not\in [\mu]\).

**Proposition 2.6** The class \([\mu]\) is closed in the strong operator topology.

**Proof.** Let \(\{\mu_n\}\) be a sequence of operators in \([\mu]\) that converges strongly to an operator \(S\) in \(L(H)\) i.e. \(\mu_n \overset{\text{s}}{\rightharpoonup} S\) then

\[
\|\mu_n x - S x\| \to 0 \text{ as } n \to \infty \text{ for each } x \in H. \text{ Thus } \|\mu_n x - S x\| = \| (\mu_n - S)^* x\| \leq \| (\mu_n - S)^* x\| = \|
\mu_n - S\| \|x\| \to 0 \text{ as } n \to \infty. \text{ Thus } \mu_n \overset{s}{\rightharpoonup} S^*\).

Since the product of operators is sequentially continuous in the strong operators topology \(-[1]-, \mu_n^2 \overset{s}{\rightharpoonup} S^2\), which implies that \(-\mu_n^{*2} \overset{s}{\rightharpoonup} -S^{*2}\), and \(\mu_n^2 \overset{s}{\rightharpoonup} S^2\). Since \(\{\mu_n\}\) is a sequence of operators in \([\mu]\) then \(-\mu_n^{*2} = \mu_{n}^{2}\) which implies that \(\mu_n^2 \overset{s}{\rightharpoonup} -S^{*2}\). Since the limit is unique, \(S^2 = -S^{*2}\). Thus \(S \in [\mu]\) which implies that \([\mu]\) is closed in the strong operator topology.
3. Main results

In this section we study the relation between the class $[\mu]$ and some other classes of operators in $L(H)$. We start by showing that the class $[\mu]$ and some other classes of operators in $L(H)$ are independent.

**Proposition 3.1** If $T \in L(H)$ is hermitian such that $T \in [\mu]$ then $T = 0$.

Proof. Since $T \in [\mu], T^2 = -T^*T$. Since $T$ is hermitian, the last equation implies that $T^2 = 0$ which implies (since $T$ is hermitian) that $T = 0$.

Since there are nonzero hermitian operators (such as the identity operator $I$) and since there are nonzero operators in $[\mu]$ (for example any operator in remark 2.1) then we have:

**Proposition 3.2** The class of all hermitian operators and the class $[\mu]$ are independent.

**Proposition 3.3** The class of all normal operators and the class $[\mu]$ are independent.

Proof. A nonzero hermitian operator is a normal operator which is not in $[\mu]$.

The operator $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ is a nonnormal operator which is in $[\mu]$.

**Proposition 3.4** If $T \in [\mu]$ such that $T^2$ is unitarily equivalent to $T^*$ then $T$ is normal.

Proof. Since $T^2$ is unitarily equivalent to $T^*$, there is a unitary operator $U$ such that $T^* = UT^2U^*$ which implies that $T = UT^{*2}U^*$. Now it is easy to show that $TT^* = -UT^{*4}U^* = T^*T$. Thus $T$ is normal.

**Proposition 3.5** The class of all skew-adjoint operators ($T = -T^*$) and the class $[\mu]$ are independent.

Proof. The operator $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ is a non-skew-adjoint operator which is in $[\mu]$.

The operator $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is a skew-adjoint operator which is not in $[\mu]$.

**Proposition 3.6** If $T \in L(H)$ is skew-adjoint such that $T \in [\mu]$ then $T = 0$.
Proof. Let $T \in L(H)$ be skew-adjoint and let $T = A + iB$ be its Cartesian decomposition. Since $T$ is skew-adjoint $T = -T^*$ which implies that $A = 0$. Thus $A^2 = 0$. Since $T \in [\mu]$ then, by Proposition 2.1, $B^2 = 0$ which implies (Since $B$ is hermitian) that $B = 0$. Thus $T = 0$.

Proposition 3.7 The class of all isometric operators and the class $[\mu]$ are independent.

Proof. The identity operator $I$ is an isometric operator and $I \notin [\mu]$.

The operator $T = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ is in $[\mu]$ but $T^*T = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \neq I$.

Proposition 3.8 If $T \in [\mu]$ is idempotent then $T = 0$.

Proof. Since $T \in [\mu]$, $T^2 = -T^*T$. Since $T$ is idempotent, $T^2 = T$ which implies that $-T^*T = -T^2$. Thus $T = -T^*$. Thus $T$ is skew-adjoint. The result now follows from proposition 3.6.

Corollary 3.1 If $T \in [\mu]$ is similar to an idempotent then $T = 0$

Proof. Since any operator similar to an idempotent is idempotent, $T$ is idempotent. The result now follows immediately from proposition 3.8.

Proposition 3.9 The class of all idempotent operators and the class $[\mu]$ are independent.

Proof. We prove the result by the following two examples.

Example 3.1. Consider the operator $S = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ acting on $R^2$ then direct calculations shows that $S^2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = S$. Thus $S$ is idempotent. However it can be shown that $S^2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = -S^* = \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}$. Thus $S \notin [\mu]$.

Example 3.2. The operator $S = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$ acting on $R^2$ is in $[\mu]$ but $S^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq S$. Thus $S$ is not idempotent.
In [2] the author introduced the class of 2-normal operators in $L(H) : T = A + iB \in L(H)$ is called 2-normal if $A^2B = BA^2$ and $B^2A = AB^2$. Several characterizations of 2-normal operators were given in [2] such as: $T \in L(H)$ is 2-normal if and only if $T^2T^* = T^*T^2$; if and only if $T^2$ is normal. The class of all 2-normal operators is denoted by $[2N]$.

**Proposition 3.10** If $T \in L(H)$ is a $\mu$-operator then $T \in [2N]$.

**Proof.** Let $= A + iB$. Since $T \in [\mu], A^2 = B^2$. Multiplying the last equation on the left and then on the right by $B$ we get $A^2B = B^3 = BA^2$. Also Multiplying $A^2 = B^2$ on the left and then on the right by $A$ we get $AB^2 = A^3 = B^2A$. Thus $T$ is 2-normal.

**Corollary 3.2** If $T, T + kI \in [\mu]$ for some nonzero complex number $k$ then $T$ is normal.

**Proof.** Since $T, T + kI \in [\mu]$ then – by proposition 3.10 – $T, T + kI$ are 2-normal operators. The result now follows from ([2], proposition 3.3, p. 193).

**Remark 3.1** The converse of proposition 3.10 is not in general true. A nonzero hermitian operator in $L(H)$ is a 2-normal operator which is not in $[\mu]$.

**Definition 3.1** If $T \in L(H)$ then $T$ is called quasinormal if $TT^*T = T^*T^2$.

**Proposition 3.11** If $T \in L(H)$ such that $T$ is 2-normal and quasinormal then $T$ is normal.

**Proof.** ([2], proposition 2.3, p. 193).

Using proposition 3.11 we conclude two facts:

The first is that there are operators in $[\mu]$ which are not quasinormal since otherwise all operators in $[\mu]$ would be normal which is not true.

The second is that there are quasinormal operators which are not in $[\mu]$ since otherwise all quasinormal operators would be normal which is not true.

From the previous discussion we have:
Proposition 3.12. The class \( [ \mu ] \) and the class of all quasinormal operators are independent.

In [3] the author introduced the class of \( \infty \) operators: \( T \in L(H) \) is called an \( \infty \) operator if \( T^3 = T^* \). The class of all \( \infty \) operators is denoted by \( (\infty) \).

In the following we give an example of a an operator which is in \( [ \mu ] \) but not in \( (\infty) \):

**Example 3.3** Consider the operators \( T = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \) acting on \( R^2 \) then \( T \in [ \mu ] \). Now it is easily shown that \( T^3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq T^* = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \). Thus \( T \notin (\infty) \).

In the following we give an example of a an operator which is in \( (\infty) \) but not in \( [ \mu ] \):

**Example 3.4** Consider the operators \( T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) acting on \( R^2 \) then \( T^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = T^* \). Thus \( T \in (\infty) \). However one can easily show that \( T^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \) while \( -T^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). Thus \( T \notin [ \mu ] \).

Using the last two examples we conclude that

**Proposition 3.13** The two classes \( [ \mu ] \) and \( (\infty) \) are independent.

In [4] the author introduced the class of subprojection operators in \( L(H) : T \in L(H) \) is called a subprojection if \( T^2 = T^* \). The class of all subprojections is denoted by \( S(H) \). In the following we give an example of an operator in \( [ \mu ] \) which is not in \( S(H) \):

**Example 3.5** Consider the operators \( T = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \) acting on \( R^2 \) then – by remark 2.1 - \( T \in [ \mu ] \). However \( T^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = T^* \). Thus \( T \notin S(H) \).

In the following we give an example of an operator in \( S(H) \) which is not in \( [ \mu ] \):
Example 3.6 Consider the operators \( T = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \) acting on \( R^2 \) then \( T^2 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \).

Thus \( T \notin \{ \mu \}. \) However one can easily show that \( T^2 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = \*T. \) Thus \( T \in S(H). \)

We conclude from the last two examples:

**Proposition 3.14** The two classes \( \{ \mu \} \) and \( S(H) \) are independent.

**Proposition 3.15** If \( T \in \{ \mu \} \cap S(H) \) then \( T = 0. \)

**Proof.** Since \( T \in S(H) \), \( T^2 = T^* \) which implies that \( -T^* = -T. \)

Since \( T \in \{ \mu \}, T^2 = -T^* \) which implies that \( T = -T^* \). Thus if \( T = A + iB \) then the last the last equation implies that \( T + T^* = 0 \) which implies that \( A = 0. \) Since \( A^2 = B^2, B^2 = 0 \) which implies (since \( B \) is hermitian) that \( B = 0. \) Thus \( T = 0. \)

In [6] Kutkut introduced a new class of operators which he called the class of parahyponormal operators: \( T \in L(H) \) is called parahyponormal if \( \|Tx\|^2 \leq \|TT^*x\| \) for all \( x \) in \( H \) with \( \|x\| = 1 \), or equivalently, ([8], Theorem 1.1, p74), if and only if for every \( \lambda > 0. \) \( \lambda^2 +T^*T(TT^*)^2 - 2\lambda. \) The class of all parahyponormal operators is denoted by \( \text{Phn}(H). \) In the following we give an example of an operator which is in \( \text{Phn}(H) \) but not in \( \{ \mu \}. \)

**Example 3.7** The operator \( T = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \) acting on \( R^2 \) is in \( \text{Phn}(H) \) ([6], p.83) but direct calculations shows that \( T^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} = -T^*. \) Thus \( T \notin \{ \mu \}. \)
In the following we give an example of an operator which is in $[\mu]$ but not in $\text{Phn}(H)$.

**Example 3.8** The operator $T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is not in $\text{Phn}(H)$ ([6], p.81). However and by direct calculations one can show that $T^2 = 0$. Thus $T \not\in [\mu]$.

From the last two examples we conclude that

**Proposition 3.16** The two classes $[\mu]$ and $\text{Phn}(H)$ are independent.

**References**


*Received: April 11, 2013*