The Hamiltonicity of Balanced Bipartite Graphs
Involving Balanced Independent Set

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Abstract

Let $G$ be a balanced bipartite graph of order $2n$ and minimum degree $δ(G) ≥ 5$. If for every balanced independent set $S$ of six vertices $|N(S)| > n + 1$, then $G$ is Hamiltonian. It is an extension of the result of Daniel Brito and Gladys Lárez [3].

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1 Introduction

We use [2] for terminology and notation not defined here. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of a simple, finite and undirected graph $G$. Let $P = a_1b_1a_2b_2, ..., a_lb_l$ be a path of $G$. If all the vertices of $G$ are contained in $P$, then we call $P$ a hamiltonian path. For two vertices $a_j, b_i ∈ P$, the subpath joining $a_j$ and $b_i$ in $P$ is denoted by $P[a_j, b_i]$ and we denote the paths $P[a_j, b_i] − a_j, P[a_j, b_i] − b_i$ and $P[a_j, b_i] − {a_j, b_i}$ by $P(a_j, b_i), P[a_j, b_i]$ and $P(a_j, b_i)$, respectively. The successor and predecessor of a vertex $z$ of $P$ are denoted by $z^+$ and $z^-$, respectively. Let $G$ be a balanced bipartite simple graph of order $2n$, i.e. a graph with a bipartition into two independent vertex
sets of the same cardinality. The set of vertices which are adjacent to \( x \) in \( G \) is denoted by \( N(x) \). \( G \) is connected if between every pair of vertices there is a path that connects them. \( G \) has a perfect matching if there is a set of independent edges that covers all the vertices of \( G \). \( N(S) \) is the neighborhood union of a balanced independent set \( S \) of six vertices. i.e. an independent set containing three vertices from each side of the bipartition. \( G \) is Hamiltonian if it has a cycle that cover all the vertices of \( G \).

The investigation of certain extremal problems involving neighborhood union conditions for balanced independent sets of cardinality four was initiated by Amar et al. \([1]\).

2 Previous lemmas

Lemma 2.1 Let \( G \) be a non-hamiltonian balanced bipartite graph of order \( 2n \) and minimum degree \( \delta(G) \geq 5 \). If \( P \) is a hamiltonian path of \( G \), then there exists a balanced independent set \( S \) of \( P \), such that \( |S| = 6 \)

Proof. Let \( \{A, B\} \) be a balanced bipartition of \( V(G) \) and \( P \) a hamiltonian path of \( G \). Without loss of generality let \( P = a_1b_1a_2b_2, ..., a_nb_n \) be such a path of \( G \). Since \( G \) has minimum degree 5, then \( a_1 \) and \( b_1 \) have both at least 5 neighbors on \( P \). If there are two vertices \( b_i \in N(a_1) \) and \( a_j \in N(b_n) \) such that \( i > j \), then \( P \) is called a crossed path. Otherwise \( P \) is called a non-crossed path.

Caso 1. \( P \) is a non-crossed path.

As \( P \) is a non-crossed path, \( a_1b_n \not\in E(G) \). Without loss of generality, let \( R_1 = \{b_i \in V(P) : b_i \in N(a_1), 2 \leq i \leq p\} \) and \( R_2 = \{a_j \in V(P) : a_j \in N(b_n), p < j \leq n \} \), be the sets of neighbors of \( a_1 \) and \( b_n \) respectively. Therefore \( R = \{b_i \in V(P) : b_i \in R_1\} \cup \{a_j \in V(P) : a_j \in R_2\} \cup \{a_1, b_n\} \) is a balanced independent set of \( G \) with \( |R| \geq 10 \), in consequence there exists a balanced independent set \( S \subseteq R \) such that \( |S| = 6 \).

Caso 2. \( P \) is a crossed path.

As \( P \) is a crossed path, there are vertices \( b_i \in N(a_1) \) and \( a_j \in N(b_n) \) such that \( i > j \). Without loss of generality, suppose \( i = \max\{l : b_l \in N(a_1)\} \), \( j = \min\{l : a_l \in N(b_n)\} \), \( r = \max\{l : b_l \in N(a_1), 1 \leq l \leq j-1\} \), \( p = \min\{l : a_l \in N(b_n), i + 1 \leq l \leq n\} \); then \( b_i, b_r \in N(a_1) \) and \( a_j, a_p \in N(b_n) \).

Let \( I_1 = P(a_2, b_{r-1}) \), \( I_2 = P[a_{j+1}, b_{i-1}] \) and \( I_3 = P[a_{p+1}, b_{n-1}] \).

Since \( G \) has minimum degree \( \delta(G) \geq 5 \), then \( a_1 \) and \( b_n \) have both at least two more neighbors on \( P \), let these form the nonempty sets \( R_1 = \{b_l \in V(P) : b_l \in N(a_1)\} \) and \( R_2 = \{a_l \in V(P) : a_l \in N(b_n)\} \).

Subcase 2.1. \( R_1 \cap (I_1 \cup I_2) \neq \emptyset \) and \( R_2 \cap I_2 = \emptyset \)

Then \( R = \{b_l^- \in V(P) : b_l \in R_1 \cap (I_1 \cup I_2)\} \cup \{a_l^+ \in V(P) : a_l \in R_2 \cap I_3\} \cup \{a_1, b_r^-, a_p^+, b_n\} \) is a balanced independent set of \( G \) with \( |R| \geq 8 \),...
and consequently there exists a balanced independent set $S \subseteq R$ such that $|S| = 6$.

**Subcase 2.2.** $R_1 \cap I_1 = \emptyset$ and $R_2 \cap I_3 = \emptyset$

Since $G$ has minimum degree $\delta(G) \geq 5$, then $|R_1 \cap I_2| \geq 2$ and $|R_2 \cap I_2| \geq 2$

Without loss of generality, suppose $b_q, b_s \in R_1 \cap I_2$ and $a_u, a_v \in R_2 \cap I_2$.

We divide into five subcases.

**Subcase 2.2.1.** $q < s < u < v$.

Then $R = \{b_l^- \in V(P) : b_l \in R_1 \cap I_2, l \leq s\} \cup \{a_l^+ \in V(P) : a_l \in R_2 \cap I_2, u \leq l\} \cup \{a_1, b_r^-, a_s^+, b_a\}$ is a balanced independent set of $G$ with $|R| \geq 8$, and consequently there exists a balanced independent set $S \subseteq R$ such that $|S| = 6$.

**Subcase 2.2.2.** $q < u < s < v$.

Then $R = \{b_l^- \in V(P) : b_l \in R_1 \cap I_2, q \leq l < u\} \cup \{a_l^+ \in V(P) : a_l \in R_2 \cap I_2, s < l \leq v\} \cup \{a_1, b_r^-, a_s^+, b_a\}$ is a balanced independent set of $G$ with $|R| \geq 6$, and consequently there exists a balanced independent set $S \subseteq R$ such that $|S| = 6$.

**Subcase 2.2.3.** $u < q < s < v$.

Then $R = \{b_l^- \in V(P) : b_l \in R_1 \cap I_2, u \leq l < s\} \cup \{a_l^+ \in V(P) : a_l \in R_2 \cap I_2, s < l \leq v\} \cup \{a_1, b_r^-, a_s^+, b_a\}$ is a balanced independent set of $G$ with $|R| \geq 6$, and consequently there exists a balanced independent set $S \subseteq R$ such that $|S| = 6$.

**Subcase 2.2.4.** $u < q < v < s$.

Then $R = \{b_l^- \in V(P) : b_l \in R_1 \cap I_2, u \leq l < v\} \cup \{a_l^+ \in V(P) : a_l \in R_2 \cap I_2, q < l < s\} \cup \{a_1, b_r^-, a_s^+, b_a\}$ is a balanced independent set of $G$ with $|R| \geq 6$, and consequently there exists a balanced independent set $S \subseteq R$ such that $|S| = 6$.

**Subcase 2.2.5.** $u < v < q < s$.

Then $R = \{b_l^- \in V(P) : b_l \in R_1 \cap I_2, v < l \leq s\} \cup \{a_l^+ \in V(P) : a_l \in R_2 \cap I_2, l < q\} \cup \{a_1, b_i^+\}$ is a balanced independent set of $G$ with $|R| \geq 6$, and consequently there exists a balanced independent set $S \subseteq R$ such that $|S| = 6$.

In all cases, there exist balanced independent sets, $S$, of cardinality 6, and the lemma is proved.

**Lemma 2.2** Let $G$ be a balanced bipartite graph of order $2n$ and minimum degree $\delta(G) \geq 5$. If every balanced independent set $S$ such that $|S| = 6$, we have $|N(S)| \geq n$, then $G$ is connected.

**Proof** Let $\{A, B\}$ be a balanced bipartition of $V(G)$ and suppose $G$ is not connected, then there exist distinct components $W$ and $T$ of $G$. As $G$ has minimum degree 5, each component has at least 5 vertices in $A$ and 5 vertices in $B$. We may assume that $|V(W) \cap A| \geq |V(T) \cap B|$. Then every set $R$ containing 5 vertices of $V(W) \cap A$ and 5 vertices of $V(T) \cap B$ is a
balanced independent set, and consequently there exists a balanced independent set \( S \subseteq R \) such that \(| S | = 6\) with \(| N(S) | < n\) contradicting the hypothesis.

**Lemma 2.3** Let \( G \) be a balanced bipartite graph of order \( 2n \) and minimum degree \( \delta(G) \geq 2 \). If for every balanced independent set \( S \) such that \(| S | = 6\), we have \(| N(S) | \geq n\), then \( G \) has a perfect matching.

**Proof.** Let \( \{A, B\} \) be a balanced bipartition of \( V(G) \). By Koning-Hall Theorem (see Swamy and Thulasiraman [4]) it is sufficient to show that for every subset \( W \subseteq A \) we have \(| N(W) | \geq | W | \). Indeed, if \(| N(W) | < | W |\), then \( 3 \leq W \leq n - 1 \). So every set \( R \) containing 3 vertices of \( W \) and 3 vertices of \( B - N(W) \) is a balanced independent set, and consequently there exists a balanced independent set \( S \subseteq R \) such that \(| S | = 6\) with \(| N(S) | < n\), contradicting the hypothesis.

### 3 Main Results

**Theorem 3.1** Let \( G \) be a balanced bipartite graph of order \( 2n \) and minimum degree \( \delta(G) \geq 5 \). If for every balanced independent set \( S \) such that \(| S | = 6\), we have \(| N(S) | > n\), then \( G \) is Hamiltonian or \( G \) contains a spanning subgraph consisting of a cycle and a isolated edge.

**Proof.** Let \( \{A, B\} \) be a balanced bipartition of \( V(G) \). From Lemma 2.2, \( G \) is connected. Suppose \( G \) is not Hamiltonian. Let \( G = (A, B, E) \) be a graph with maximum size among all the graphs that are not hamiltonian of order \( 2n \) satisfying the condition of the theorem. By maximality of \(| E(G) |\), for every two nonadjacent vertices \( a \in A \) and \( b \in B \), the graph \( G + ab \) is Hamiltonian, and, furthermore, every Hamiltonian cycle of \( G + ab \) contains the edge \( ab \). Thus there is an \( a - b \) path \( P : a = a_1b_1a_2b_2, ..., a_nb_n = b \) in \( G \) that contains every vertex of \( G \), with \( a_1, a_2, ..., a_n \in A \) and \( b_1, b_2, b_3, ..., b_n \in B \). By Lemma 2.1 there exists a balanced independent set \( S \) of \( P \), such that \(| S | = 6\). Without loss of generality let be \( \{b_j, b_s, b_i\} \subseteq (N(a_i) \cap P) \) and \( \{a_j, a_u, a_v\} \subseteq (N(b_n) \cap P) \) such that \( i = \max\{l : b_l \in N(a_j), j = \min\{l : a_l \in N(b_n)\}\} \) and \( j < u < v < q < s < i \) with \( S = \{a^+_j, a^+_u, a^+_v, b^+_i, b^+_s, b^+_j\} \) the balanced independent set. and since \(| N(S) | > n\), then \( a^+_w, w \in \{j, u, v\}, \) is adjacent to \( S \) or \( b^+_z, z \in \{q, s, i\}, \) is adjacent to \( S \). Then we have the cycle \( C = a^+_w b^+_z P b^+_n a^+_w P^{-1} a^+_b P^{-1} a^+_w \) and the edge is \( b^+_z b^+_i \) or the cycle \( C = b^+_z a^+_u P a^+_z P b^+_n a^+_w P b^+_z \) and the edge is \( a^+_w a^+_z \) for some \( w \in \{j, u, v\} \) and \( z \in \{q, s, i\} \).

Hence the proof of the theorem is complete.

As an immediate consequence we have

**Corollary 3.2** Let \( G \) be a balanced bipartite graph of order \( 2n \) and minimum degree \( \delta(G) \geq 5 \). If for every balanced independent set \( S \) such that \(| S | = 6\), we have \(| N(S) | > n\), then \( G \) is traceable.
Theorem 3.3 Let $G$ be a balanced bipartite graph of order $2n$ and minimum degree $\delta(G) \geq 5$. If for every balanced independent set $S$ such that $|S| = 6$, we have $|N(S)| > n + 1$, then $G$ is Hamiltonian.

Proof. Let \{A, B\} be a balanced bipartition of $V(G)$. From Lemma 2.2, $G$ is connected. Suppose $G$ is not Hamiltonian. By Theorem 3.1 there exists a cycle $C$ of length $2n - 2$ such that $G - C$ is one edge. Without loss of generality let $C = a_2b_2, ..., a_nb_n, a_1b_1$ and $a_1b_1 \in E(G - C)$. Since $G$ has minimum degree $\delta(G) \geq 5$, then $a_1$ and $b_1$ have both at least four neighbors on $C$. Let $R = \{b_i^+ \in V(C) : b_i \in N_C(a_1)\} \cup \{a_i^+ \in V(C) : a_i \in N_C(b_1)\}$. It is obvious that $R$ is a balanced independent set. Let $S \subseteq R$ be a balanced independent set such that $|S| = 6$. By Lemma 2.3 there is a matching $M$ in $G$ and since $|N(S)| > n + 1$, there is an edge of $M$, both ends of which are adjacent to $S$ in form cross, thus we obtain a contradiction to the supposition. Therefore $|N(S)| \leq n + 1$, which contradicts the hypothesis of the theorem. This proves the theorem.

As an immediate consequence we have

Corollary 3.4 Let $G$ be a balanced bipartite graph of order $2n$ and minimum degree $\delta(G) \geq 5$. If for every six independent vertices the sum of their degrees is at least $n + 2$, then $G$ is hamiltoniano.

Let $H_{26}$ denote the class of graphs obtained from the graph depicted in Fig. 1, where some (or all) of the sixteen possible edges joining the top to the bottom might be present as well.

From Theorem 2 we conjecture:

Conjecture 3.5. Let $G$ be a balanced bipartite graph of order $2n$ and minimum degree $\delta(G) \geq 5$. If for every balanced independent set $S$ such that $|S| = 6$, we have $|N(S)| > n$, then either $G$ is Hamiltonian or $G \in H_{26}$.

If the assertion of the conjecture is true then the result is the best possible in a condition. Take two complete balanced bipartite graph, choose from each of them one side of the bipartition and join the sides completely. The resulting graph satisfies $|N(S)| = n$ for every balanced independent set $S$ but is not hamiltonian. This shows that imposing a stronger minimum degree or even connectivity condition would not help in this case.
References


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