Ulam-Hyers Stability of a 2-Variable *AC*-Mixed Type Functional Equation in Felbin's Type Spaces: Fixed Point Method

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Abstract. In this paper, the authors obtain the generalized Ulam - Hyers stability of a 2 - variable AC - mixed type functional equation

$$f(2x + y, 2z + w) - f(2x - y, 2z - w) = 4[f(x + y, z + w) - f(x - y, z - w)] - 6f(y, w)$$

in Felbin's type spaces using fixed point method.

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1. INTRODUCTION

The investigation of stability problems for functional equations is related to the famous Ulam problem [43] (in 1940), concerning the stability of group homomorphisms, which was first solved by D. H. Hyers [13], in 1941. This stability problem was further generalized by several authors [2, 9, 28, 32, 36]. We cite also other pertinent research works [1, 4, 7, 11, 15, 16, 18, 31, 33, 37, 40, 46, 48].

The general solution and Ulam stability of **mixed type additive and cubic functional equation** of the form

$$3f(x+y+z) + f(-x+y+z) + f(x-y+z) + f(x+y-z) \\ + 4[f(x) + f(y) + f(z)] = 4[f(x+y) + f(x+z) + f(y+z)] \quad (1.1)$$

introduced by J.M. Rassias [29]. The stability of generalized mixed type functional equation of the form

$$f(x+ky) + f(x-ky) = k^2[f(x+y) + f(x-y)] + 2(1-k^2)f(x) \quad (1.2)$$

for fixed integers k with $k \neq 0, \pm 1$ in quasi-Banach spaces was investigated by M. Eshaghi Gordji and H. Khodaie [10]. The mixed type functional equation (1.2) is having the property **additive, quadratic and cubic**.

The solution and stability of a n -dimensional additive functional equation

$$f\left(a \sum_{i=1}^{n-1} x_i - 2ax_n\right) + f\left(2a \sum_{i=1}^{n-1} x_i - ax_n\right) = 3a \left(\sum_{i=1}^{n-1} f(x_i) - f(x_n)\right) \text{ for } n \geq 3 \quad (1.3)$$

where a are integers, $a \geq 1$ with fixed point Alternative was investigated by K. Ravi, M. Arunkumar [34]. Also, Y.S. Jung, I.S. Chang [17] discussed the Hyers - Ulam - Rassias stability for the cubic functional equation

$$f(x+y+2z) + f(x+y-2z) + f(2x) + f(2y) \\ = 2[f(x+y) + 2f(x+z) + 2f(y+z) + 2f(x-z) + 2f(y-z)] \quad (1.4)$$

with the fixed point alternative. Infact an n -dimensional cubic functional equation

$$f\left(\sum_{j=1}^{n-1} x_j + 2x_n\right) + f\left(\sum_{j=1}^{n-1} x_j - 2x_n\right) + \sum_{j=1}^{n-1} f(2x_j) \\ = 2f\left(\sum_{j=1}^{n-1} 2x_j\right) + 4 \sum_{j=1}^{n-1} (f(x_j + x_n) + f(x_j - x_n)) \quad (1.5)$$

and its Hyers - Ulam - Rassias stability with the help of alternative fixed point idea was dealt by H.Y. Chu and D.S. Kang [5]. Recently, F. Moradlou et al., [25] proved the stability of Cauchy functional equation

$$f(x+y) = f(x) + f(y) \quad (1.6)$$

in Felbin's type spaces using fixed point approach.

J.H. Bae and W.G. Park [6] proved the general solution of the 2- variable quadratic functional equation

$$f(x+y, z+w) + f(x-y, z-w) = 2f(x, z) + 2f(y, w) \quad (1.7)$$

and investigated the generalized Hyers-Ulam-Rassias stability of (1.7). The above functional equation has solution

$$f(x, y) = ax^2 + bxy + cy^2. \quad (1.8)$$

Very recently, M. Arunkumar et al., [3] first time introduced and investigated the solution and generalized Ulam-Hyers stability of a 2 - variable AC - mixed type functional equation

$$f(2x + y, 2z + w) - f(2x - y, 2z - w) = 4[f(x + y, z + w) - f(x - y, z - w)] - 6f(y, w) \quad (1.9)$$

having solutions

$$f(x, y) = ax + by \quad (1.10)$$

and

$$f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3 \quad (1.11)$$

in Banach space via direct and fixed point approach.

The solution of the AC functional equation (1.9) is given in the following lemmas.

Lemma 1.1. [3] *If $f : U^2 \rightarrow V$ be a mapping satisfying (1.9) and let $g : U^2 \rightarrow V$ be a mapping given by*

$$g(x, x) = f(2x, 2x) - 8f(x, x) \quad (1.12)$$

for all $x \in U$ then

$$g(2x, 2x) = 2g(x, x) \quad (1.13)$$

for all $x \in U$ such that g is additive.

Lemma 1.2. [3] *If $f : U^2 \rightarrow V$ be a mapping satisfying (1.9) and let $h : U^2 \rightarrow V$ be a mapping given by*

$$h(x, x) = f(2x, 2x) - 2f(x, x) \quad (1.14)$$

for all $x \in U$ then

$$h(2x, 2x) = 8h(x, x) \quad (1.15)$$

for all $x \in U$ such that h is cubic.

Remark 1.3. [3] *If $f : U^2 \rightarrow V$ be a mapping satisfying (1.9) and let $g, h : U^2 \rightarrow V$ be a mapping defined in (1.12) and (1.14) then*

$$f(x, x) = \frac{1}{6}(h(x, x) - g(x, x)) \quad (1.16)$$

for all $x \in U$.

In this paper, the authors established the generalized Ulam-Hyers stability using fixed point method in Felbin's type spaces is discussed in Section 3.

2. FUZZY REAL NUMBER

In this section, we give some preliminaries in the theory of fuzzy real numbers. Furthermore, we give some definition which help to investigate the stability in Felbin's type normed linear spaces.

In [12] Grantner takes the fuzzy real number as a decreasing mapping from the real line to the unit interval or lattice in general. Lowen [23] applies the fuzzy real numbers as non-decreasing, left continuous mapping from the real line to the unit interval so that its supremum over $\mathbb{R}$ is 1. Also fuzzy arithmetic operations on L -fuzzy real line were studied by Rodabaugh [39], where he showed that the binary addition is the only extension of addition to $\mathbb{R}((L))$.

Hoehle [14] especially emphasized the role of fuzzy real numbers as modeling a fuzzy threshold softening the notion of Dedekind cut. In this paper a fuzzy real number is taken as a fuzzy normal and convex mapping from the real line to the unit interval. The concept of the fuzzy metric space has been studied by Kaleva [19, 20] by using fuzzy number as a fuzzy set on the real axis. Kaleva also has recently showed that a fuzzy metric space can be embedded in a complete fuzzy metric space [21].

In [8], Felbin introduced the concept of fuzzy normed linear space (FNLS); Xiao and Zhu [44] studied its linear topological structures and some basic properties of a fuzzy normed linear space. It is known that theories of classical normed space and Menger probabilistic normed spaces are special cases of fuzzy normed linear spaces.

Let η be a fuzzy subset on $\mathbb{R}$, i.e., a mapping $\eta : \mathbb{R} \rightarrow [0, 1]$ associating with each real number t its grade of membership ηt .

Definition 2.1. [8] A fuzzy subset η on R is called a fuzzy real number, whose α -level set is denoted by $[\eta]_\alpha$

$$i.e., [\eta]_\alpha = \{t : \eta(t) \geq \alpha\},$$

if it satisfies two axioms:

(N1) There exists $t_0 \in \mathbb{R}$ such that $\eta(t_0) = 1$.

(N2) For each $\alpha \in (0, 1]$, $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$ where $-\infty < \eta_\alpha^- \leq \eta_\alpha^+ < +\infty$.

The set of all fuzzy real numbers denoted by $F(\mathbb{R})$. If $\eta \in F(\mathbb{R})$ and $\eta(t) = 0$ whenever $t < 0$, then η is called a non-negative fuzzy real number and $F^*(\mathbb{R})$ denotes the set of all non-negative fuzzy real numbers.

The number $\bar{0}$ stands for the fuzzy real number as:

$$\bar{0} = \begin{cases} t, & t = 0, \\ 0, & t \neq 0. \end{cases}$$

Clearly, $\bar{0} \in F^*(\mathbb{R})$. Also the set of all real numbers can be embedded in $F(\mathbb{R})$ because if $r \in (-\infty, \infty)$, then $\bar{r} \in F(\mathbb{R})$ satisfies $\bar{r}(t) = \bar{0}(t - r)$.

Definition 2.2. [8] Fuzzy arithmetic operations $\oplus, \ominus, \otimes, \oslash$ on $F(\mathbb{R}) \times F(\mathbb{R})$ can be defined as:

- (1) $(\eta \oplus \delta)(t) = \sup_{s \in \mathbb{R}} \{\eta(s) \wedge \delta(t - s)\}, t \in \mathbb{R},$
- (2) $(\eta \ominus \delta)(t) = \sup_{s \in \mathbb{R}} \{\eta(s) \wedge \delta(s - t)\}, t \in \mathbb{R},$
- (3) $(\eta \otimes \delta)(t) = \sup_{s \in \mathbb{R}} \{\eta(s) \wedge \delta(t/s)\}, t \in \mathbb{R},$
- (4) $(\eta \oslash \delta)(t) = \sup_{s \in \mathbb{R}} \{\eta(st) \wedge \delta(s)\}, t \in \mathbb{R}.$

The additive and multiplicative identities in $F(\mathbb{R})$ are $\bar{0}$ and $\bar{1}$, respectively. Let $\ominus\eta$ be defined as $\bar{0} \ominus \eta$. It is clear that $\eta \ominus \delta = \eta \oplus (\ominus\delta)$.

Definition 2.3. [8] For $k \in \mathbb{R} \setminus 0$, fuzzy scalar multiplication $k \odot \eta$ is defined as $(k \odot \eta)(t) = \eta(t/k)$ and $0 \odot \eta$ is defined to be 0.

Lemma 2.4. Let η, δ be fuzzy real numbers. Then

$$\forall t \in \mathbb{R}, \eta(t) = \delta(t) \Leftrightarrow \forall \alpha \in (0, 1], [\eta]_\alpha = [\delta]_\alpha.$$

Lemma 2.5. Let $\eta, \delta \in F(\mathbb{R})$ and $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$, $[\delta]_\alpha = [\delta_\alpha^-, \delta_\alpha^+]$. Then

- (i) $[\eta \oplus \delta]_\alpha = [\eta_\alpha^- + \delta_\alpha^-, \eta_\alpha^+ + \delta_\alpha^+]$,
- (ii) $[\eta \ominus \delta]_\alpha = [\eta_\alpha^- - \delta_\alpha^-, \eta_\alpha^+ - \delta_\alpha^+]$,
- (iii) $[\eta \otimes \delta]_\alpha = [\eta_\alpha^- \delta_\alpha^-, \eta_\alpha^+ \delta_\alpha^+]$, $\eta, \delta \in F^*(\mathbb{R})$,
- (iv) $[\bar{1} \oslash \delta]_\alpha = [1/\delta_\alpha^+, 1/\delta_\alpha^-]$, $\delta_\alpha^- > 0$.

Definition 2.6. [8] Define a partial ordering $\preceq$ in $F(\mathbb{R})$ by $\eta \preceq \delta$ if and only if $\eta_\alpha^- \leq \delta_\alpha^-$ and $\eta_\alpha^+ \leq \delta_\alpha^+$ for all $\alpha \in (0, 1]$. The strict inequality in $F(\mathbb{R})$ is defined by $\eta \prec \delta$ if and only if $\eta_\alpha^- < \delta_\alpha^-$ and $\eta_\alpha^+ < \delta_\alpha^+$ for all $\alpha \in (0, 1]$.

Definition 2.7. [44] Let X be a real linear space, L and $\mathbb{R}$ (respectively, left norm and right norm) be symmetric and non-decreasing mappings from $[0, 1] \times [0, 1] \rightarrow [0, 1]$ satisfying $L(0, 0) = 0$, $\mathbb{R}(1, 1) = 1$. Then $\|\cdot\|$ is called a fuzzy norm and $(X, \|\cdot\|, L, \mathbb{R})$ is a fuzzy normed linear space (abbreviated to FNLS) if the mapping $\|\cdot\| : X \rightarrow F^*(\mathbb{R})$ satisfies the following axioms, where $[\|x\|]_\alpha = [\|x\|_\alpha^-, \|x\|_\alpha^+]$ for $x \in X$ and $\alpha \in (0, 1]$:

- (A1) $\|x\| = 0$ if and only if $x = 0$,
- (A2) $\|rx\| = |r| \odot \|x\|$ for all $x \in X$ and $r \in (-\infty, \infty)$,
- (A3) For all $x, y \in X$,
 - (A3L) if $s \leq \|x\|_1^-, t \leq \|y\|_1^-$ and $s + t \leq \|x\|_1^- + \|y\|_1^-$, then $\|x + y\|(s + t) \geq L(\|x\|(s), \|y\|(t))$,
 - (A3R) if $s \geq \|x\|_1^+, t \geq \|y\|_1^+$ and $s + t \geq \|x\|_1^+ + \|y\|_1^+$, then $\|x + y\|(s + t) \leq \mathbb{R}(\|x\|(s), \|y\|(t))$,

Lemma 2.8. [45] Let $(X, \|\cdot\|, L, \mathbb{R})$ be an FNLS, and suppose that

- (R1) $\mathbb{R}(a, b) \leq \max(a, b)$,
- (R2) $\forall \alpha \in (0, 1], \exists \beta \in (0, \alpha]$ such that $\mathbb{R}(\beta, y) \leq \alpha$ for all $y \in (0, \alpha)$,
- (R3) $\lim_{a \rightarrow 0^+} \mathbb{R}(a, a) = 0$.

Then $(R1) \Rightarrow (R2) \Rightarrow (R3)$ but not conversely.

Lemma 2.9. [45] Let $(X, \|\cdot\|, L, \mathbb{R})$ be an FNLS. Then we have the following:

- (A) If $\mathbb{R}(a, b) \leq \max(a, b)$, then $\forall \alpha \in (0, 1], \|x + y\|_\alpha^+ \leq \|x\|_\alpha^+ + \|y\|_\alpha^+$ for all $x, y \in X$.
- (B) If (R2) then for each $\alpha \in (0, 1]$ there is $\beta \in (0, \alpha]$ such that $\|x + y\|_\alpha^+ \leq \|x\|_\beta^+ + \|y\|_\alpha^+$ for all $x, y \in X$.
- (C) If $\lim_{a \rightarrow 0^+} \mathbb{R}(a, a) = 0$, then for each $\alpha \in (0, 1]$ there is $\beta \in (0, \alpha]$ such that $\|x + y\|_\alpha^+ \leq \|x\|_\beta^+ + \|y\|_\beta^+$ for all $x, y \in X$.

Lemma 2.10. [45] Let $(X, \|\cdot\|, L, \mathbb{R})$ be an FNLS, and suppose that

- (L1) $L(a, b) \geq \min(a, b)$,
- (L2) $\forall \alpha \in (0, 1], \exists \beta \in [\alpha, 1]$ such that $L(\beta, \gamma) \geq \alpha$ for all $\gamma \in [\alpha, 1)$,
- (L3) $\lim_{a \rightarrow 1^-} L(a, a) = 1$.

Then $(L1) \Rightarrow (L2) \Rightarrow (L3)$.

Lemma 2.11. [45] Let $(X, \|\cdot\|, L, R)$ be an FNLS. Then we have the following:

- (A) If $L(a, b) \geq \min(a, b)$, then $\forall \alpha \in (0, 1]$, $\|x + y\|_{\alpha}^{-} \leq \|x\|_{\alpha}^{-} + \|y\|_{\alpha}^{-}$ for all $x, y \in X$.
- (B) If (L2) then for each $\alpha \in (0, 1]$ there is $\beta \in [\alpha, 1]$ such that $\|x + y\|_{\alpha}^{-} \leq \|x\|_{\beta}^{-} + \|y\|_{\alpha}^{-}$ for all $x, y \in X$.
- (C) If $\lim_{a \rightarrow 1^{-}} R(a, a) = 1$, then for each $\alpha \in (0, 1]$ there is $\beta \in [\alpha, 1]$ such that $\|x + y\|_{\alpha}^{-} \leq \|x\|_{\beta}^{-} + \|y\|_{\beta}^{-}$ for all $x, y \in X$.

Lemma 2.12. [45] Let $(X, \|\cdot\|, L, R)$ be an FNLS. Then:

- (a) If $R(a, b) \geq \max(a, b)$ and $\forall \alpha \in (0, 1]$, $\|x + y\|_{\alpha}^{+} \leq \|x\|_{\alpha}^{+} + \|y\|_{\alpha}^{+}$ for all $x, y \in X$ then (A3R).
- (b) If $L(a, b) \leq \min(a, b)$ and $\forall \alpha \in (0, 1]$ $\|x + y\|_{\alpha}^{-} \leq \|x\|_{\alpha}^{-} + \|y\|_{\alpha}^{-}$ for all $x, y \in X$ then (A3L).

Theorem 2.13. [41] Let $(X, \|\cdot\|, L, R)$ be an FNLS and $\lim_{a \rightarrow 0^{+}} R(a, a) = 0$. Then $(X, \|\cdot\|, L, R)$ is a Hausdorff topological vector space, whose neighborhood base of origin is $\{N(\epsilon, \alpha) : \epsilon > 0, \alpha \in (0, 1]\}$, where $N(\epsilon, \alpha) = \{x : \|x\|_{\alpha}^{+} \leq \epsilon\}$.

Definition 2.14. Let $(X, \|\cdot\|, L, R)$ be an FNLS. A sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ converges to $x \in X$, if $\lim_{n \rightarrow \infty} \|x_n - x\|_{\alpha}^{+} = 0$, for every $\alpha \in (0, 1]$ denoted by $\lim_{n \rightarrow \infty} x_n = x$.

Definition 2.15. Let $(X, \|\cdot\|, L, R)$ be an FNLS. A sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ is called a Cauchy sequence if $\lim_{m, n \rightarrow \infty} \|x_m - x_n\|_{\alpha}^{+} = 0$ for every $\alpha \in (0, 1]$.

Definition 2.16. Let $(X, \|\cdot\|, L, R)$ be an FNLS. A subset $A \subseteq X$ is said to be complete if every Cauchy sequence in A , converges in A . The fuzzy normed space $(X, \|\cdot\|, L, R)$ is said to be a fuzzy Banach space if it is complete.

3. STABILITY RESULTS: FIXED POINT METHOD

In this section, we apply a fixed point method for achieving stability of the 2-variable AC functional equation (1.9).

Now, we present the following theorem due to B. Margolis and J.B. Diaz [24] for fixed point Theory.

Theorem 3.1. [24] Suppose that for a complete generalized metric space (Ω, δ) and a strictly contractive mapping $T : \Omega \rightarrow \Omega$ with Lipschitz constant L . Then, for each given $x \in \Omega$, either

$$d(T^n x, T^{n+1} x) = \infty \quad \forall n \geq 0,$$

or there exists a natural number n_0 such that

- (A1) $d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$;
- (A2) The sequence $(T^n x)$ is convergent to a fixed point y^* of T
- (A3) y^* is the unique fixed point of T in the set $\Delta = \{y \in \Omega : d(T^{n_0} x, y) < \infty\}$;
- (A4) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.

Using the above theorem, we now obtain the generalized Ulam - Hyers stability of (1.9).

Through out this section let U be a normed space and V be a Banach space. Define a mapping $F : U^2 \rightarrow V$ by

$$\begin{aligned} F(x, y, z, w) = & f(2x + y, 2z + w) - f(2x - y, 2z - w) - 4f(x + y, z + w) \\ & + 4f(x - y, z - w) + 6f(y, w) \end{aligned}$$

for all $x, y, z, w \in U$.

Theorem 3.2. Let $F : U^2 \rightarrow V$ be a mapping for which there exist a function $\varphi : U^4 \rightarrow F^*(R)$ with the condition

$$\lim_{n \rightarrow \infty} \frac{1}{\mu_i^n} \varphi(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n w)_\alpha^+ = 0 \quad (3.1)$$

where $\mu_i = 2$ if $i = 0$ and $\mu_1 = \frac{1}{2}$ if $i = 1$ such that the functional inequality

$$\|F(x, y, z, w)\|_\alpha^+ \preceq \varphi(x, y, z, w)_\alpha^+ \quad (3.2)$$

for all $x, y, z, w \in U$ and $\alpha \in (0, 1]$. If there exists $L = L(i) < 1$ such that the function

$$x \rightarrow \gamma(x)_\alpha^+ = \frac{1}{2} \odot \psi(x)_\alpha^+,$$

has the property

$$\gamma(x)_\alpha^+ \preceq L\mu_i \odot \gamma(\mu_i x)_\alpha^+. \quad (3.3)$$

Then there exists a unique 2-variable additive mapping $A : U^2 \rightarrow V$ satisfying the functional equation (1.9) and

$$\|f(2x, 2x) - 8f(x, x) - A(x, x)\|_\alpha^+ \preceq \frac{L^{1-i}}{1-L} \gamma(x)_\alpha^+ \quad (3.4)$$

for all $x \in U$, where $\psi(x)_\alpha^+$ is defined in (3.9), for all $x \in U$.

Proof. Consider the set

$$\Omega = \{p/p : U^2 \rightarrow V, p(0, 0) = 0\}$$

and introduce the generalized metric on Ω ,

$$d(p, q) = d_\gamma(p, q) = \inf\{K \in (0, \infty) : \|p(x, x) - q(x, x)\|_\alpha^+ \preceq K\gamma(x)_\alpha^+, x \in U, \forall \alpha \in (0, 1]\}.$$

It is easy to see that (Ω, d) is complete.

Define $T : \Omega^2 \rightarrow \Omega$ by

$$Tp(x, x) = \frac{1}{\mu_i} p(\mu_i x, \mu_i x),$$

for all $x \in U$. Now $p, q \in \Omega$,

$$\begin{aligned} d(p, q) \preceq K &\Rightarrow \|p(x, x) - q(x, x)\|_\alpha^+ \preceq K\gamma(x)_\alpha^+, x \in U. \\ &\Rightarrow \left\| \frac{1}{\mu_i} p(\mu_i x, \mu_i x) - \frac{1}{\mu_i} q(\mu_i x, \mu_i x) \right\|_\alpha^+ \preceq \frac{1}{\mu_i} K\gamma(\mu_i x)_\alpha^+, x \in U, \\ &\Rightarrow \left\| \frac{1}{\mu_i} p(\mu_i x, \mu_i x) - \frac{1}{\mu_i} q(\mu_i x, \mu_i x) \right\|_\alpha^+ \preceq LK\gamma(x)_\alpha^+, x \in U, \\ &\Rightarrow \|Tp(x, x) - Tq(x, x)\|_\alpha^+ \preceq LK\gamma(x)_\alpha^+, x \in U, \\ &\Rightarrow d_\gamma(p, q) \preceq LK. \end{aligned}$$

This implies $d(Tp, Tq) \preceq Ld(p, q)$, for all $p, q \in \Omega$. i.e., T is a strictly contractive mapping on Ω with Lipschitz constant L .

Letting (x, y, z, w) by (x, x, x, x) in (3.2), we obtain

$$\|f(3x, 3x) - 4f(2x, 2x) + 5f(x, x)\|_\alpha^+ \preceq \varphi(x, x, x, x)_\alpha^+ \quad (3.5)$$

for all $x \in U$. Replacing (x, y, z, w) by $(x, 2x, x, 2x)$ in (3.2), we get

$$\|f(4x, 4x) - 4f(3x, 3x) + 6f(2x, 2x) - 4f(x, x)\|_\alpha^+ \preceq \varphi(x, 2x, x, 2x)_\alpha^+ \quad (3.6)$$

for all $x \in U$. Now, from (3.5) and (3.6), we have

$$\begin{aligned} \|f(4x, 4x) - 10f(2x, 2x) + 16f(x, x)\|_{\alpha}^{+} &\preceq 4 \odot \|f(3x, 3x) - 4f(2x, 2x) + 5f(x, x)\|_{\alpha}^{+} \\ &\quad + \|f(4x, 4x) - 4f(3x, 3x) + 6f(2x, 2x) - 4f(x, x)\|_{\alpha}^{+} \\ &\preceq 4 \odot \varphi(x, x, x, x)_{\alpha}^{+} + \varphi(x, 2x, x, 2x)_{\alpha}^{+} \end{aligned} \quad (3.7)$$

for all $x \in U$. From (3.7), we arrive

$$\|f(4x, 4x) - 10f(2x, 2x) + 16f(x, x)\|_{\alpha}^{+} \preceq \psi(x)_{\alpha}^{+} \quad (3.8)$$

where

$$\psi(x)_{\alpha}^{+} = 4 \odot \varphi(x, x, x, x)_{\alpha}^{+} + \varphi(x, 2x, x, 2x)_{\alpha}^{+} \quad (3.9)$$

for all $x \in U$. It is easy to see from (3.8) that

$$\|f(4x, 4x) - 8f(2x, 2x) - 2(f(2x, 2x) - 8f(x, x))\|_{\alpha}^{+} \preceq \psi(x)_{\alpha}^{+} \quad (3.10)$$

for all $x \in U$. Using (1.12) in (3.10), we obtain

$$\|g(2x, 2x) - 2g(x, x)\|_{\alpha}^{+} \preceq \psi(x)_{\alpha}^{+} \quad (3.11)$$

for all $x \in U$. From (3.11), we arrive

$$\left\| \frac{g(2x, 2x)}{2} - g(x, x) \right\|_{\alpha}^{+} \preceq \frac{1}{2} \odot \psi(x)_{\alpha}^{+} \quad (3.12)$$

for all $x \in U$. Using (3.3) for the case $i = 0$ it reduces to

$$\left\| \frac{g(2x, 2x)}{2} - g(x, x) \right\|_{\alpha}^{+} \preceq L\gamma(x)_{\alpha}^{+}$$

for all $x \in U$,

$$\text{i.e., } d_{\psi}(g, Tg) \preceq L \Rightarrow d(g, Tg) \preceq L \preceq L^1 < \infty.$$

Again replacing $x = \frac{x}{2}$ in (3.12), we get,

$$\left\| g(x, x) - 2g\left(\frac{x}{2}, \frac{x}{2}\right) \right\|_{\alpha}^{+} \preceq \psi\left(\frac{x}{2}\right)_{\alpha}^{+} \quad (3.13)$$

Using (3.3) for the case $i = 1$ it reduces to

$$\left\| g(x, x) - 2g\left(\frac{x}{2}, \frac{x}{2}\right) \right\|_{\alpha}^{+} \preceq \gamma(x)_{\alpha}^{+}$$

for all $x \in U$,

$$\text{i.e., } d_{\psi}(g, Tg) \preceq 1 \Rightarrow d(g, Tg) \preceq 1 \preceq L^0 < \infty.$$

In both cases, we arrive

$$d(g, Tg) \preceq L^{1-i}.$$

Therefore (A1) holds.

By (A2), it follows that there exists a fixed point A of T in Ω such that

$$A(x, x) = \lim_{n \rightarrow \infty} \frac{1}{\mu_i^n} (f(\mu_i^{(n+1)}x, \mu_i^{(n+1)}x) - 8f(\mu_i^n x, \mu_i^n x)) \quad (3.14)$$

for all $x \in U$.

To prove $A : U^2 \rightarrow V$ is additive. Replacing (x, y, z, w) by $(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n w)$ in (3.2) and dividing by μ_i^n , it follows from (3.1) that

$$\begin{aligned} \|A(x, y, z, w)\|_\alpha^+ &= \lim_{n \rightarrow \infty} \frac{\|F(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n w)\|_\alpha^+}{\mu_i^n} \\ &\preceq \lim_{n \rightarrow \infty} \frac{\varphi(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n w)_\alpha^+}{\mu_i^n} = 0 \end{aligned}$$

for all $x, y, z, w \in U$ i.e., A satisfies the functional equation (1.9).

By (A3), A is the unique fixed point of T in the set $\Delta = \{A \in \Omega : d(f, A) < \infty\}$, A is the unique function such that

$$\|f(2x, 2x) - 8f(x, x) - A(x, x)\|_\alpha^+ \preceq K\gamma(x)_\alpha^+$$

for all $x \in U$ and $K > 0$. Finally by (A4), we obtain

$$d(f, A) \preceq \frac{1}{1-L} d(f, Tf)$$

this implies

$$d(f, A) \preceq \frac{L^{1-i}}{1-L}$$

which yields

$$\|f(2x, 2x) - 8f(x, x) - A(x, x)\|_\alpha^+ \preceq \frac{L^{1-i}}{1-L} \gamma(x)_\alpha^+$$

this completes the proof of the theorem. $\square$

The following Corollary is an immediate consequence of Theorem 3.2 concerning the stability of (1.9).

Corollary 3.3. *Let $F : U^2 \rightarrow V$ be a mapping and there exists real numbers λ and s such that*

$$\|F(x, y, z, w)\|_\alpha^+ \preceq \begin{cases} \lambda \otimes \{\|x\|^s \oplus \|y\|^s \oplus \|z\|^s \oplus \|w\|^s\}, & s < 1 \text{ or } s > 1; \\ \lambda \otimes \|x\|^s \otimes \|y\|^s \otimes \|z\|^s \otimes \|w\|^s, & s < \frac{1}{4} \text{ or } s > \frac{1}{4}; \\ \lambda \otimes \{\|x\|^s \otimes \|y\|^s \otimes \|z\|^s \otimes \|w\|^s \\ \oplus \{|x|^{4s} \oplus |y|^{4s} \oplus |w|^{4s} \oplus |z|^{4s}\}\}, & s < \frac{1}{4} \text{ or } s > \frac{1}{4}; \end{cases} \quad (3.15)$$

for all $x, y, z, w \in U$, then there exists a unique 2- variable additive function $A : U^2 \rightarrow V$ such that

$$\|f(2x, 2x) - 8f(x, x) - A(x, x)\|_\alpha^+ \preceq \begin{cases} \lambda_\alpha^+ \otimes \frac{2^{(s-1)}(18 \oplus 2^{s+1})(\|x\|^s)_\alpha^+}{|2 - 2^s|}, \\ \lambda_\alpha^+ \otimes \frac{2^{(4s-1)}(4 \oplus 2^{2s})(\|x\|^{4s})_\alpha^+}{|2 - 2^{4s}|}, \\ \lambda_\alpha^+ \otimes \frac{2^{(4s-1)}(22 \oplus 2^{2s} + 2 \odot 2^{4s})(\|x\|^{4s})_\alpha^+}{2 - 2^{4s}} \end{cases} \quad (3.16)$$

for all $x \in U$.

Proof. Setting

$$\varphi(x, y, z, w)_\alpha^+ = \begin{cases} \lambda \otimes \{||x||^s \oplus ||y||^s \oplus ||z||^s \oplus ||w||^s\}, \\ \lambda \otimes ||x||^s \otimes ||y||^s \otimes ||z||^s \otimes ||w||^s \\ \lambda \otimes \{||x||^s \otimes ||y||^s \otimes ||z||^s \otimes ||w||^s \oplus \{||x||^{4s} \oplus ||y||^{4s} \oplus ||w||^{4s} \oplus ||z||^{4s}\}\} \end{cases}$$

for all $x, y, z, w \in U$. Now,

$$\begin{aligned} & \frac{\varphi(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n w)}{\mu_i^n} \\ &= \begin{cases} \frac{\lambda}{\mu_i^n} \otimes \{||\mu_i^n x||^s \oplus ||\mu_i^n y||^s \oplus ||\mu_i^n z||^s \oplus ||\mu_i^n w||^s\}, \\ \frac{\lambda}{\mu_i^n} \otimes ||\mu_i^n x||^s \otimes ||\mu_i^n y||^s \otimes ||\mu_i^n z||^s \otimes ||\mu_i^n w||^s \\ \frac{\lambda}{\mu_i^n} \otimes \{||\mu_i^n x||^s \otimes ||\mu_i^n y||^s \otimes ||\mu_i^n z||^s \otimes ||\mu_i^n w||^s \oplus \{||\mu_i^n x||^{4s} \oplus ||\mu_i^n y||^{4s} \oplus ||\mu_i^n z||^{4s} \oplus ||\mu_i^n w||^{4s}\}\} \end{cases} \\ &= \begin{cases} \rightarrow 0 \text{ as } n \rightarrow \infty, \\ \rightarrow 0 \text{ as } n \rightarrow \infty, \\ \rightarrow 0 \text{ as } n \rightarrow \infty. \end{cases} \end{aligned}$$

Thus, (3.1) is holds.

But we have $\gamma(x)_\alpha^+ = \frac{1}{2} \psi(x)_\alpha^+$ has the property $\gamma(x)_\alpha^+ \preceq L \cdot \mu_i \odot \gamma(\mu_i x)_\alpha^+$ for all $x \in U$. Hence

$$\begin{aligned} \gamma(x)_\alpha^+ &= \frac{1}{2} \odot \psi(x)_\alpha^+ = \frac{1}{2} \odot (4\varphi(x, x, x, x)_\alpha^+ + \varphi(x, 2x, x, 2x)_\alpha^+) \\ &= \begin{cases} \frac{\lambda}{2} \odot (18||x||^s \oplus 2||2x||^s), \\ \frac{\lambda}{2} \odot (4 \oplus 2^{2s}) ||x||^{4s}, \\ \frac{\lambda}{2} \odot (22 \oplus 2^{2s} + 2 \odot 2^{4s}) ||x||^{4s}. \end{cases} \end{aligned}$$

Now,

$$\begin{aligned} \frac{1}{\mu_i} \gamma(\mu_i x)_\alpha^+ &= \begin{cases} \frac{\lambda}{2\mu_i} \odot \{(18||\mu_i x||^s \oplus 2||2\mu_i x||^s)\}, \\ \frac{\lambda}{2\mu_i} \odot \{(4 \oplus 2^{2s})\} ||\mu_i x||^{4s}, \\ \frac{\lambda}{2\mu_i} \odot \{(22 \oplus 2^{2s} \oplus 2 \odot 2^{4s})\} ||\mu_i x||^{4s}. \end{cases} \\ &= \begin{cases} \mu_i^{s-1} \frac{\lambda}{2} \odot (18 \oplus 2^{s+1}) ||x||^s, \\ \mu_i^{4s-1} \frac{\lambda}{2} \odot (4 \oplus 2^{2s}) ||x||^{4s}, \\ \mu_i^{4s-1} \frac{\lambda}{2} \odot (22 \oplus 2^{2s} \oplus 2 \odot 2^{4s}) ||x||^{4s}. \end{cases} \\ &= \begin{cases} \mu_i^{s-1} \psi(x)_\alpha^+, \\ \mu_i^{4s-1} \psi(x)_\alpha^+, \\ \mu_i^{4s-1} \psi(x)_\alpha^+. \end{cases} \end{aligned}$$

Hence the inequality (3.3) holds either, $L = 2^{s-1}$ for $s < 1$ if $i = 0$ and $L = \frac{1}{2^{s-1}}$ for $s > 1$ if $i = 1$. Now from (3.4), we prove the following cases for condition (i).

Case:1 $L = 2^{s-1}$ for $s < 1$ if $i = 0$

$$\begin{aligned} \|f(2x, 2x) - 8f(x, x) - A(x, x)\|_{\alpha}^{+} &\preceq \lambda_{\alpha}^{+} \otimes \frac{(2^{(s-1)})^{1-0}}{1 - 2^{(s-1)}} \left\{ \frac{18 \oplus 2^{(s+1)}}{2} \right\} (\|x\|^s)_{\alpha}^{+} \\ &= \lambda_{\alpha}^{+} \otimes \frac{2^{(s-1)}}{1 - 2^{(s-1)}} \left\{ \frac{18 \oplus 2^{(s+1)}}{2} \right\} (\|x\|^s)_{\alpha}^{+} \\ &= \lambda_{\alpha}^{+} \otimes \frac{2^{(s-1)} \cdot 2}{2 - 2^s} \left\{ \frac{18 \oplus 2^{(s+1)}}{2} \right\} (\|x\|^s)_{\alpha}^{+} \\ &= \lambda_{\alpha}^{+} \otimes \frac{2^{(s-1)} (18 \oplus 2^{(s+1)}) (\|x\|^s)_{\alpha}^{+}}{2 - 2^s} \end{aligned}$$

Case:2 $L = \frac{1}{2^{s-1}}$ for $s > 1$ if $i = 1$

$$\begin{aligned} \|f(2x, 2x) - 8f(x, x) - A(x, x)\|_{\alpha}^{+} &\preceq \lambda_{\alpha}^{+} \otimes \frac{\left(\frac{1}{2^{(s-1)}}\right)^{1-1}}{1 - \frac{1}{2^{(s-1)}}} \left\{ \frac{18 \oplus 2^{(s+1)}}{2} \right\} (\|x\|^s)_{\alpha}^{+} \\ &= \lambda_{\alpha}^{+} \otimes \frac{2^{(s-1)}}{2^{(s-1)} - 1} \left\{ \frac{18 \oplus 2^{(s+1)}}{2} \right\} (\|x\|^s)_{\alpha}^{+} \\ &= \lambda_{\alpha}^{+} \otimes \frac{2^{(s-1)} \cdot 2}{2^s - 2} \left\{ \frac{18 \oplus 2^{(s+1)}}{2} \right\} (\|x\|^s)_{\alpha}^{+} \\ &= \lambda_{\alpha}^{+} \otimes \frac{2^{(s-1)} (18 \oplus 2^{(s+1)}) (\|x\|^s)_{\alpha}^{+}}{2^s - 2} \end{aligned}$$

In similar manner we can prove the following cases $L = 2^{4s-1}$ for $s < \frac{1}{4}$ if $i = 0$ and $L = \frac{1}{2^{4s-1}}$ for $s > \frac{1}{4}$ if $i = 1$ for conditions (ii) and (iii) respectively. Hence the proof is complete. $\square$

Theorem 3.4. Let $F : U^2 \rightarrow V$ be a mapping for which there exist a function $\varphi : U^4 \rightarrow F^*(R)$ with the condition

$$\lim_{n \rightarrow \infty} \frac{1}{\mu_i^{3n}} \varphi(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n w)_{\alpha}^{+} = 0 \quad (3.17)$$

where $\mu_i = 2$ if $i = 0$ and $\mu_i = \frac{1}{2}$ if $i = 1$ such that the functional inequality

$$\|F(x, y, z, w)\|_{\alpha}^{+} \preceq \varphi(x, y, z, w)_{\alpha}^{+} \quad (3.18)$$

for all $x, y, z, w \in U$. If there exists $L = L(i) < 1$ such that the function

$$x \rightarrow \gamma(x)_{\alpha}^{+} = \frac{1}{2} \odot \psi(x)_{\alpha}^{+},$$

has the property

$$\gamma(x)_{\alpha}^{+} \preceq L \mu_i^3 \odot \gamma(\mu_i x)_{\alpha}^{+}. \quad (3.19)$$

Then there exists a unique 2-variable cubic mapping $C : U^2 \rightarrow V$ satisfying the functional equation (1.9) and

$$\|f(2x, 2x) - 2f(x, x) - C(x, x)\|_{\alpha}^{+} \preceq \frac{L^{1-i}}{1-L} \gamma(x)_{\alpha}^{+} \quad (3.20)$$

for all $x \in U$. The mapping $\psi(x)_{\alpha}^{+}$ is defined in (3.9) for all $x \in U$.

Proof. Consider the set

$$\Omega = \{p/p : U^2 \rightarrow V, p(0, 0) = 0\}$$

and introduce the generalized metric on Ω ,

$$d(p, q) = d_\gamma(p, q) = \inf\{K \in (0, \infty) : \|p(x, x) - q(x, x)\|_\alpha^+ \preceq K\gamma(x)_\alpha^+, x \in U\}.$$

It is easy to see that (Ω, d) is complete. Define $T : \Omega^2 \rightarrow \Omega$ by

$$Tp(x, x) = \frac{1}{\mu_i^3}p(\mu_i x, \mu_i x),$$

for all $x \in U$. Now $p, q \in \Omega$,

$$\begin{aligned} d(p, q) \preceq K &\Rightarrow \|p(x, x) - q(x, x)\|_\alpha^+ \preceq K\gamma(x)_\alpha^+, x \in U. \\ &\Rightarrow \left\| \frac{1}{\mu_i^3}p(\mu_i x, \mu_i x) - \frac{1}{\mu_i^3}q(\mu_i x, \mu_i x) \right\|_\alpha^+ \preceq \frac{1}{\mu_i^3}K\gamma(\mu_i x)_\alpha^+, x \in U, \\ &\Rightarrow \left\| \frac{1}{\mu_i^3}p(\mu_i x, \mu_i x) - \frac{1}{\mu_i^3}q(\mu_i x, \mu_i x) \right\|_\alpha^+ \preceq LK\gamma(x)_\alpha^+, x \in U, \\ &\Rightarrow \|Tp(x, x) - Tq(x, x)\|_\alpha^+ \preceq LK\gamma(x)_\alpha^+, x \in U, \\ &\Rightarrow d_\gamma(p, q) \preceq LK. \end{aligned}$$

This implies $d(Tp, Tq) \preceq Ld(p, q)$, for all $p, q \in \Omega$. i.e., T is a strictly contractive mapping on Ω with Lipschitz constant L .

It follows from (3.8) that

$$\|f(4x, 4x) - 2f(2x, 2x) - 8(f(2x, 2x) - 2f(x, x))\|_\alpha^+ \preceq \psi(x)_\alpha^+ \quad (3.21)$$

for all $x \in U$. Using (1.14) in (3.21), we obtain

$$\left\| \frac{h(2x, 2x)}{8} - h(x, x) \right\|_\alpha^+ \preceq \frac{\psi(x)_\alpha^+}{8} \quad (3.22)$$

for all $x \in U$. Using (3.22) for the case $i = 0$ it reduces to

$$\left\| \frac{h(2x, 2x)}{8} - h(x, x) \right\|_\alpha^+ \preceq L\gamma(x)_\alpha^+$$

for all $x \in U$,

$$\text{i.e., } d_\psi(h, Th) \preceq L \Rightarrow d(h, Th) \preceq L \preceq L^1 < \infty.$$

Again replacing $x = \frac{x}{2}$ in (3.22), we get,

$$\left\| h(x, x) - 8h\left(\frac{x}{2}, \frac{x}{2}\right) \right\|_\alpha^+ \preceq \psi\left(\frac{x}{2}\right)_\alpha^+ \quad (3.23)$$

Using (3.23) for the case $i = 1$ it reduces to

$$\left\| h(x, x) - 8h\left(\frac{x}{2}, \frac{x}{2}\right) \right\|_\alpha^+ \preceq \gamma(x)_\alpha^+$$

for all $x \in U$,

$$\text{i.e., } d_\psi(h, Th) \preceq 1 \Rightarrow d(h, Th) \preceq 1 \preceq L^0 < \infty.$$

In both cases, we arrive

$$d(h, Th) \preceq L^{1-i}.$$

Therefore (A1) holds. The rest of the proof is similar to that of Theorem 3.2. $\square$

The following Corollary is an immediate consequence of Theorem 3.4 concerning the stability of (1.9).

Corollary 3.5. *Let $F : U^2 \rightarrow V$ be a mapping and there exists real numbers λ and s such that*

$$\|F(x, y, z, w)\|_{\alpha}^{+} \preceq \begin{cases} \lambda \otimes \{\|x\|^s \oplus \|y\|^s \oplus \|z\|^s \oplus \|w\|^s\}, & s < 3 \text{ or } s > 3; \\ \lambda \otimes \|x\|^s \otimes \|y\|^s \otimes \|z\|^s \otimes \|w\|^s, & s < \frac{3}{4} \text{ or } s > \frac{3}{4}; \\ \lambda \otimes \{\|x\|^s \otimes \|y\|^s \otimes \|z\|^s \otimes \|w\|^s \\ \oplus \{\|x\|^{4s} \oplus \|y\|^{4s} \oplus \|w\|^{4s} \oplus \|z\|^{4s}\}\}, & s < \frac{3}{4} \text{ or } s > \frac{3}{4}; \end{cases} \quad (3.24)$$

for all $x, y, z, w \in U$, then there exists a unique 2-variable cubic function $C : U^2 \rightarrow V$ such that

$$\|f(2x, 2x) - 2f(x, x) - C(x, x)\|_{\alpha}^{+} \preceq \begin{cases} \lambda_{\alpha}^{+} \otimes \frac{2^{(s-1)}(18 \oplus 2^{s+1})(\|x\|_{\alpha}^{+})}{|8 - 2^s|}, \\ \lambda_{\alpha}^{+} \otimes \frac{2^{(4s-1)}(4 \oplus 2^{2s})(\|x\|_{\alpha}^{4s})}{|8 - 2^{4s}|}, \\ \lambda_{\alpha}^{+} \otimes \frac{2^{(4s-1)}(22 \oplus 2^{2s} + 2 \odot 2^{4s})(\|x\|_{\alpha}^{+})}{|8 - 2^{4s}|} \end{cases} \quad (3.25)$$

for all $x \in U$.

Proof. The proof of the corollary is similar tracing as that of Corollary 3.3. $\square$

Now, we are ready to prove the main fixed point stability results.

Theorem 3.6. *Let $F : U^2 \rightarrow V$ be a mapping for which there exist a function $\varphi : U^4 \rightarrow F^*(R)$ with the conditions (3.1) and (3.17) where $\mu_i = 2$ if $i = 0$ and $\mu_1 = \frac{1}{2}$ if $i = 1$ such that the functional inequality*

$$\|F(x, y, z, w)\|_{\alpha}^{+} \preceq \varphi(x, y, z, w)_{\alpha}^{+} \quad (3.26)$$

for all $x, y, z, w \in U$. If there exists $L = L(i) < 1$ such that the function

$$x \rightarrow \gamma(x)_{\alpha}^{+} = \frac{1}{2} \odot \psi(x)_{\alpha}^{+},$$

has the properties (3.3) and (3.19) Then there exists a unique 2-variable additive mapping $A : U^2 \rightarrow V$ and a unique 2-variable cubic mapping $C : U^2 \rightarrow V$ satisfying the functional equation (1.9) and

$$\|f(x, x) - A(x, x) - C(x, x)\|_{\alpha}^{+} \preceq \frac{1}{3} \odot \frac{L^{1-i}}{1-L} \gamma(x)_{\alpha}^{+} \quad (3.27)$$

for all $x \in U$. The mapping $\psi(x)_{\alpha}^{+}$ is defined in (3.9) for all $x \in U$.

Proof. By Theorems 3.2 and 3.4, there exists a unique 2-variable additive function $A_1 : U^2 \rightarrow V$ and a unique 2-variable cubic function $C_1 : U^2 \rightarrow V$ such that

$$\|f(2x, 2x) - 8f(x, x) - A_1(x, x)\|_{\alpha}^{+} \preceq \frac{L^{1-i}}{1-L} \gamma(x)_{\alpha}^{+} \quad (3.28)$$

and

$$\|f(2x, 2x) - 2f(x, x) - C_1(x, x)\|_{\alpha}^{+} \preceq \frac{L^{1-i}}{1-L} \gamma(x)_{\alpha}^{+} \quad (3.29)$$

for all $x \in U$. Now from (3.28) and (3.29), one can see that

$$\begin{aligned} & \left\| f(x, x) + \frac{1}{6}A_1(x, x) - \frac{1}{6}C_1(x, x) \right\|_{\alpha}^+ \\ &= \left\| \left\{ -\frac{f(2x, 2x)}{6} + \frac{8f(x, x)}{6} + \frac{A_1(x, x)}{6} \right\} + \left\{ \frac{f(2x, 2x)}{6} - \frac{2f(x, x)}{6} - \frac{C_1(x, x)}{6} \right\} \right\|_{\alpha}^+ \\ &\leq \frac{1}{6} \odot \{ \|f(2x, 2x) - 8f(x, x) - A_1(x, x)\|_{\alpha}^+ + \|f(2x, 2x) - 2f(x, x) - C_1(x, x)\|_{\alpha}^+ \} \\ &\leq \frac{1}{6} \odot \left\{ \frac{L^{1-i}}{1-L} \gamma(x)_{\alpha}^+ + \frac{L^{1-i}}{1-L} \gamma(x)_{\alpha}^+ \right\} \end{aligned}$$

for all $x \in U$. Thus we obtain (3.27) by defining $A(x, x) = \frac{-1}{6}A_1(x, x)$ and $C(x, x) = \frac{1}{6}C_1(x, x)$, $\psi(x)_{\alpha}^+$ is defined in (3.9) for all $x \in U$. $\square$

The following Corollary is an immediate consequence of Theorem 3.6, using Corollaries 3.3 and 3.5 concerning the stability of (1.9).

Corollary 3.7. *Let $F : U^2 \rightarrow V$ be a mapping and there exists real numbers λ and s such that*

$$\|F(x, y, z, w)\|_{\alpha}^+ \preceq \begin{cases} \lambda \otimes \{ \|x\|^s \oplus \|y\|^s \oplus \|z\|^s \oplus \|w\|^s \}, & s \neq 1, 3; \\ \lambda \otimes \|x\|^s \otimes \|y\|^s \otimes \|z\|^s \otimes \|w\|^s, & s \neq \frac{1}{4}, \frac{3}{4}; \\ \lambda \otimes \{ \|x\|^s \otimes \|y\|^s \otimes \|z\|^s \otimes \|w\|^s \\ \oplus \{ \|x\|^{4s} \oplus \|y\|^{4s} \oplus \|w\|^{4s} \oplus \|z\|^{4s} \} \}, & s \neq \frac{1}{4}, \frac{3}{4}; \end{cases} \quad (3.30)$$

for all $x, y, z, w \in U$, then there exists a unique 2-variable additive mapping $A : U^2 \rightarrow V$ and a unique 2-variable cubic mapping $C : U^2 \rightarrow V$ such that

$$\begin{aligned} & \|f(x, x) - A(x, x) - C(x, x)\| \\ &\leq \begin{cases} \lambda_{\alpha}^+ \otimes \frac{2^{(s-1)}(18 \oplus 2^{s+1})}{3} \left(\frac{1}{|2-2^s|} + \frac{1}{|8-2^s|} \right) (\|x\|^s)_{\alpha}^+, \\ \lambda_{\alpha}^+ \otimes \frac{2^{(4s-1)}(4 \oplus 2^{2s})}{3} \left(\frac{1}{|2-2^s|} + \frac{1}{|8-2^s|} \right) (\|x\|^{4s})_{\alpha}^+, \\ \lambda_{\alpha}^+ \otimes \frac{2^{(4s-1)}(22 \oplus 2^{2s} \oplus 2 \odot 2^{4s})}{3} \left(\frac{1}{|2-2^s|} + \frac{1}{|8-2^s|} \right) (\|x\|^{4s})_{\alpha}^+ \end{cases} \quad (3.31) \end{aligned}$$

for all $x \in U$.

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