Estimating the Algebraic Connectivity of Triangle-Free Graphs

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Abstract

The second smallest eigenvalue of the Laplacian matrix of a graph is called its algebraic connectivity. In this article, we present an estimation for the algebraic connectivity of a triangle-free graph. Then, we give some examples to illustrate our results and make a comparison with some known lower bounds.

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1. Introduction

Let $G=(V, E)$ be a simple connected graph with $n=|V|$ vertices and $m=|E|$ edges. For a vertex $v \in V$, let d_v denote the degree of v . The Laplacian matrix of G is defined as $L=D-A$, where A is the adjacency matrix of G and D is the diagonal matrix of vertex degrees. Clearly, L is symmetric and positive semidefinite and consequently its eigenvalues, which are called the Laplacian eigenvalues of G , are real nonnegative numbers and since each row sum of L is equal to 0, then 0 is the smallest eigenvalue of L . Denote the eigenvalues of L by $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n = 0$. The spectrum of the Laplacian matrix is important in graph theory because it has a close relation to numerous graph invariants, such as diameter, chromatic number, maximum cut, spanning trees, connectivity, etc. (for more details, see [2,3] and the references therein). Thus, good lower and upper bounds and approximations for the eigenvalues of L are needed in many applications. Among them is λ_{n-1} . It is well known that G is connected if and only if $\lambda_{n-1} > 0$. This fact led Fiedler[2,3] to define the algebraic connectivity of G to be equal to λ_{n-1} . Usually, the algebraic connectivity of a graph G is denoted by $a(G)$. There where attempts to find a lower bound for $a(G)$ in terms of simple properties of graphs such as diameter, which we denote here by δ , and the number of vertices and edges of the graph (see [1] for a comprehensive review). Among the known lower bounds for the algebraic connectivity, $a(G)$, are

(1) Mohar's bound [8]

$$a(G) \geq \frac{4}{n\delta} \quad (1)$$

(2) Lu, Zhang and Tian bound [7]

$$a(G) \geq \frac{2n}{2 + (n-1)n\delta - 2m\delta} \quad (2)$$

In this paper, we present an estimation for the algebraic connectivity of triangle-free graphs based on degrees of vertices. Then we apply our results to some examples of triangle-free graphs and make a comparison with the above mentioned bounds and refer to some known results to conclude that our estimations are good in some sense.

2. Main Results

We will employ the following result, known as the Cauchy interlacing property. For a proof, see [5, p. 189].

Lemma 1. If B is an $n \times n$ symmetric matrix and $\tilde{B}$ is an $r \times r$ principal submatrix of B with eigenvalues $\lambda_1(B) \geq \dots \geq \lambda_n(B)$ and $\lambda_1(\tilde{B}) \geq \dots \geq \lambda_r(\tilde{B})$, respectively, then the eigenvalues of B interlace those of $\tilde{B}$, that is

$$\lambda_i(B) \geq \lambda_i(\tilde{B}) \geq \lambda_{n-r+i}(B) \quad \text{for } i=1, \dots, r.$$

In the sequel, we assume that G is a triangle-free graph.

For a vertex $v \in V$, let $w_1, w_2, \dots, w_k$ be the vertices adjacent to v , where $k = d_v$, and let L_v be the $(k+1) \times (k+1)$ principal submatrix of L corresponding to the vertices $v, w_1, w_2, \dots, w_k$. Since G is triangle-free, then it follows that

$$L_v = \begin{bmatrix} d_v & -1 & -1 & \dots & -1 \\ -1 & d_{w_1} & 0 & \dots & 0 \\ -1 & 0 & d_{w_2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -1 & 0 & \dots & 0 & d_{w_k} \end{bmatrix}.$$

let L_v be written in the partitioned form

$$L_v = \begin{bmatrix} d_v & e^t \\ e & D_v \end{bmatrix},$$

where

$$e = [-1, -1, \dots, -1]^t \quad \text{and} \quad D_v = \text{diag}[d_{w_1}, d_{w_2}, \dots, d_{w_k}].$$

Let the smallest eigenvalue of L_v be denoted by μ_v , i.e., $\mu_v = \lambda_{k+1}(L_v)$ and let δ_v be the smallest eigenvalue of D_v , i.e., $\delta_v = \min_{1 \leq i \leq k} \{d_{w_i}\}$. Then by applying

Lemma 1 to L and its principal submatrix L_v , we have

$$\lambda_{k+1}(L) \geq \mu_v \geq \lambda_n(L) = 0, \quad (3)$$

and applying Lemma 1 to L_v and its principal submatrix D_v yields

$$\mu_v \leq \delta_v . \quad (4)$$

Also, by the ordering of the eigenvalues of L , we have that

$$\lambda_{k+1}(L) \geq a(G) = \lambda_{n-1} > \lambda_n(L) = 0. \quad (5)$$

So, from (3) and (5), we note that for each $v \in V$, both μ_v and $a(G)$ lie between $\lambda_n(L)$ and $\lambda_{k+1}(L)$.

It can be shown [6, p. 411] that the characteristic polynomial of L_v can be written as

$$\det(L_v - \mu I) = (d_v - \mu) \det(D_v - \mu I) - e^t [\text{adj}(D_v - \mu I)] e .$$

where $\text{adj}(D_v - \mu I)$ is the adjugate (classical adjoint) of $(D_v - \mu I)$.

We first consider the case in (4) where $\mu_v < \delta_v$, then for $\mu \leq \mu_v$, $(D_v - \mu I)$ is invertible and so, the characteristic polynomial of L_v takes the form

$$\det(L_v - \mu I) = \left[d_v - \mu - \sum_{i=1}^k \frac{1}{d_{w_i} - \mu} \right] \prod_{i=1}^k (d_{w_i} - \mu).$$

Thus, μ_v is the minimal zero of the function

$$f_v(\mu) = d_v - \mu - \sum_{i=1}^k \frac{1}{d_{w_i} - \mu} . \quad (6)$$

Theorem 1. Let g_v be the function defined for $\mu \leq \mu_v$ by

$$g_v(\mu) = d_v - \mu - \frac{d_v}{(\delta_v - \mu)} . \quad (7)$$

Then, the minimal zero of g_v , say η_v , is less than or equal to the minimal zero of f_v . That is

$$\mu_v \geq \eta_v . \quad (8)$$

Proof. From (6), we have that for all $\mu \leq \mu_v$

$$\begin{aligned}
f_v(\mu) &= d_v - \mu - \sum_{i=1}^k \frac{1}{d_{w_i} - \mu} \\
&\geq d_v - \mu - \sum_{i=1}^k \frac{1}{\delta_v - \mu} \\
&= d_v - \mu - \frac{k}{\delta_v - \mu} \\
&= g_v(\mu).
\end{aligned}$$

Since μ_v is a root of f_v , then $g_v(\mu_v) \leq f_v(\mu_v) = 0$ and since $\lim_{\mu \rightarrow -\infty} g_v(\mu) = \infty$, then g_v has its minimal zero in the interval $(-\infty, \mu_v]$, which implies (8). $\square$

From (7), the equation $g_v(\mu) = 0$ can be written in the polynomial form $q_v(\mu) = 0$ Where

$$q_v(\mu) = \mu^2 - (d_v + \delta_v)\mu + (d_v\delta_v - d_v).$$

and, therefore,

$$\eta_v = \frac{1}{2} (d_v + \delta_v - \sqrt{(d_v - \delta_v)^2 + 4d_v}).$$

It is noted that the results above are proved under the assumption that $\mu_v < \delta_v$. In the the following lemma, we show that the inequality (8) also holds when $\mu_v = \delta_v$.

Lemma 2. If $\mu_v = \delta_v$ then the inequality (8) holds.

Proof. If $\mu_v = \delta_v$ then

$$q_v(\mu_v) = q_v(\delta_v) = -d_v < 0$$

and since $\lim_{\mu \rightarrow -\infty} q_v(\mu) = \infty$ then we must have $\mu_v > \eta_v$, that is, the inequality in (8) holds. $\square$

Let

$$\bar{\eta} = \frac{1}{n} \sum_{v \in V} \eta_v. \quad (9)$$

then in view of (3), (5) and (8) one can consider $\bar{\eta}$ as an estimation for the

algebraic connectivity $a(G)$.

Corollary 1. For a star graph G we have $\bar{\eta}$ is a lower bound for the algebraic connectivity and $\lim_{n \rightarrow \infty} \bar{\eta} = 1 = a(G)$.

Proof . For any star graph G , we have

$$\bar{\eta} = \frac{(n-1)}{2n} \left(n - \sqrt{(n-2)^2 + 4} \right). \quad (10)$$

It can be easily shown that $\bar{\eta} \leq 1 = a(G)$ and that $\bar{\eta} \rightarrow 1$ as $n \rightarrow \infty$. $\square$

Corollary 2. For star graphs with $n \geq 4$, the lower bound $\bar{\eta}$ is larger than the lower bounds in (1) and (2).

Proof . For a star graph, it is easy to see that the lower bounds in (1) and (2) are $\frac{2}{n}$ and $\frac{n}{1+(n-1)(n-2)}$, respectively. Now, one can easily verify that for $n \geq 4$, $\bar{\eta}$, as in (10), is larger than the lower bounds in (1) and (2). $\square$

Corollary 3. Let G be a k -regular triangle-free graph, then an estimation for the algebraic connectivity is $a(G) \approx k - \sqrt{k}$.

Proof . It can be easily shown that for a k -regular triangle-free graph, $\bar{\eta} = k - \sqrt{k}$. $\square$

3. Examples

We now give some examples to illustrate our results. Consider the two triangle-free graphs shown in the figure below.

Fig. 1

The actual value of the algebraic connectivity, $a(G)$, the lower bounds in (1) and (2) and the estimation $\bar{\eta}$ in (9) are shown in the table below.

Table 1

Graph G	$a(G)$	(1)	(2)	$\bar{\eta}$
G_1	0.52	0.27	0.26	0.31
G_2	1.44	0.22	0.27	0.80

4. Remarks

- (1) Table 1 shows that in some cases, $\bar{\eta}$ is a better estimation for the algebraic connectivity than the lower bounds in (1) and (2).
- (2) For star graphs, it can be easily seen that as $n \rightarrow \infty$, both the lower bounds in (1) and (2) get further from 1 while, as stated in Corollary 1, our estimation $\bar{\eta} \rightarrow 1 = a(G)$.
- (3) Note that Corollary 3 applies for the Petersen graph, which is a 3-regular graph and for which $a(G) = 2$, applying (9), we get $a(G) \approx 1.27$. However, applying (1) and (2), we have $a(G) \geq 0.2$ and $a(G) \geq 0.16$, respectively. This adds an example where our estimation in (9) is better than some known lower bounds for the algebraic connectivity.
- (4) It is known [4] that random k -regular graphs have the algebraic connectivity bounded away from zero. This shows that the lower bounds in (1) and (2) are poor for regular graphs. In fact, the authors in [4] proved that for a random k -regular graph G , $a(G) = k - O(\sqrt{k})$ with probability $\rightarrow 1$ as n grows. Hence, the result of Corollary 3 is good in some sense.
- (5) Finally, we would like to note that $\bar{\eta}$ is just an estimation for the algebraic connectivity which may be far from the true value for some graphs. For example, consider the k -cube which is bipartite (hence triangle-free), k -regular, and has algebraic connectivity equal to 2 regardless of the value of k , so corollary 3 does not apply for k -cubes with large k .

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