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The Sampling Distribution of the Maximum Likelihood Estimators for the Parameters of Beta-Binomial Distribution

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Abstract

In this paper sampling distributions for the maximum likelihood estimators of the Beta- Binomial model are obtained numerically using approximate random numbers from the Beta-Binomial distribution. STATGRAPHICS package is used to obtain the best fitted distributions using Chi-square and Kolmogorov-Smirnov tests.

Keywords: Beta-Binomial distribution (BBD), Maximum likelihood estimation (MLE), Sampling distribution, Chi-square and Kolmogorov-Smirnov tests

1. Introduction

The Beta-Binomial model is one of the oldest discrete probability mixture models and is widely applied in the social, physical, and health sciences, it was formally proposed by Skellam (1948), although the idea was suggested earlier by Pearson (1925) in an experimental investigation of Bayes theorem. A familiar model in many applications is to assume that an observed set of counts is from a Binomial distribution with unit specific probabilities governed by a Beta distribution (the so-called Beta-Binomial model; BBM).

Consider a population in which for each member some event occurs as the outcome of a Bernoulli trial with fixed probability P , given $0 < P < 1$, the number of occurrences for x in n trials has the Binomial distribution with density mass function given by

$$f\,(x\,;n,P\,)=\binom{n}{x}P^{x}\,(1-P\,)^{n-x}\quad ,\ x\,=0,1,...,n \tag{1.1}$$

Suppose that P is a random variable which follows a Beta distribution,

$$f\,(P;\alpha,\beta)=\frac{1}{B\,(\alpha,\beta)}\,P^{\alpha-1}\,(1-P\,)^{\beta-1},\quad \alpha,\beta>0,\ \ 0<P<1 \tag{1.2}$$

where α,β are unknown shape parameters and $B\,(\alpha,\beta)$ is the complete Beta function. Consider the following simple mixture model

$$p\,(x\,;n\,,\alpha\,,\beta)=\int_0^1 f\,(x\,;n\,,P\,)\;f\,(P\,;\alpha,\beta)\;\;dP\,.$$

Using equations (1.1) and (1.2), the Beta-Binomial model will be

$$p(x\,;n,\alpha,\beta)=\binom{n}{x}\frac{B\,(\alpha+x\,,\beta+n-x\,)}{B\,(\alpha,\beta)},\quad x\,=0,1,2,...,n,\ \alpha,\beta>0 \tag{1.3}$$

The cumulative distribution function c.d.f of X , can be written as follows:

$$F(x\,)=\sum_{j=0}^{x}\binom{n}{j}\frac{B\,(\alpha+j\,,\beta+n-j\,)}{B\,(\alpha,\beta)},$$

$$=\sum_{j=0}^{x}\binom{n}{j}\frac{B(\beta+j\,,\alpha+n-j\,)}{(n+1)B(n-j\,+1,j\,+1)B(\alpha,\beta)},\qquad x\,=0,1,2,...,n,\ \alpha,\beta>0.$$

where $B\,(\alpha,\beta)$ is the complete Beta function, n is non-negative integer, and α and β are shape parameters.

Currently, there are little studies for estimation of the unknown parameters of the Beta-Binomial distribution. Griffiths (1973) estimated the unknown parameters of Beta-Binomial distribution using maximum likelihood estimation and applied his results to the household distribution of the total numbers of a disease, Lee and Sabavala (1987) developed a Bayesian procedures for the Beta-Binomial model and, used a suitable reparameterization, establishes a conjugate-type property for a Beta family of priors. The transformed parameters have interesting interpretations, especially in marketing applications, and are likely to be more stable proposed a Bayesian estimation and prediction for the Beta-Binomial Model, Tripathi et al. (1994) introduced some alternative methods for estimating the parameters in the Beta Binomial and truncated Beta-Binomial models, Lee and Lio (1999) introduced a note on Bayesian estimation and prediction for the Beta-Binomial model. The main purpose of the paper is to extend the study of Lee and Sabavala (1987) by numerical integration. This method can be used for the general case of trials. When the number of trials is two, the results are similar to those from Lee and Sabavala (1987). Quintana and Wing-Kuen (1996) introduced a Bayesian estimation of Beta-Binomial model by simulating posterior densities and Everson and Braldlow (2002) presented a Bayesian inference for the Beta-Binomial distribution via polynomial expansions.

In this paper we introduced sampling distributions for the maximum likelihood estimators of the Beta-Binomial model numerically using approximate random numbers from the Beta-Binomial. STATGRAPHICS package is used to obtain the best fitted distributions using Chi-square and Kolmogorov-Smirnov tests.

2. Maximum Likelihood Estimators (MLE)

Lee and Sabavala (1987), used the maximum likelihood method to estimate the unknown parameters for the Beta-Binomial distribution when $n = 2$ and Lee and Lio (1999) discussed some estimation problem to estimate the unknown reparametrized parameters (π, θ) when $n \geq 2$.

(a) Maximum Likelihood estimators for the unknown parameters of the Beta-Binomial distribution when $n = 2$.

Lee and Sabavala (1987) estimated the unknown parameters for Beta-Binomial distribution (α, β) , using the following likelihood function

$$L(\alpha,\beta) = \prod_{x=0}^{n} [p(x;n,\alpha,\beta)]^{f_x}$$

$$L(\alpha,\beta) = \prod_{x=0}^{n}\left(\binom{n}{x}\frac{(\alpha+x-1)(\alpha+x-1)...\alpha(n+\beta-x-1)(n+\beta-x-1)...\beta}{(\alpha+\beta+n-1)(\alpha+\beta+n-2)...(\alpha+\beta)}\right)^{f_x} \quad (2.1)$$

where f_x is the number of the units with x occurrences has the Binomial distribution and $n = \sum_{x=0}^{n} f_x$, the log likelihood is given by

$$\ln L = c + \sum_{i=0}^{n} f_x \ln[(\alpha+x-1)(\alpha+x-2)...\alpha.(n+\beta-x-1)(n+\beta-x-2)...\beta]$$

$$-\sum_{i=0}^{n} f_x \ln[(\alpha+\beta+n-1)(\alpha+\beta+n-2)...(\alpha+\beta)]. \quad (2.2)$$

The first derivatives of (2.2) with respect to α and β are given by

$$\frac{\partial \ln L}{\partial \alpha} = \sum_{x=1}^{n-1} f_x\, A(\alpha, x) - f\ \ A(\alpha+\beta, n) \quad \text{and}$$

$$\frac{\partial \ln L}{\partial \beta} = \sum_{x=1}^{n-1} f_{n-x}\, A(\beta, x) - f\ \ A(\alpha+\beta, n) \quad (2.3)$$

where $A(\beta,x) = 1/(\beta) + 1/(\beta+1) + ... + 1/(\beta+x-1)$. Equating (2.3) to zero, the maximum likelihood estimates $\hat{\alpha}$ and $\hat{\beta}$ for α and β respectively can be obtained as the solution of the following equations

$$\sum_{x=1}^{n} f_x\, A(\hat{\alpha}, x) - f\ \ A(\hat{\alpha}+\hat{\beta}, n) = 0 \quad \text{and}$$

$$\sum_{x=1}^{n} f_{n-x}\, A(\hat{\beta}, x) - f\ \ A(\hat{\alpha}+\hat{\beta}, n) = 0$$

These may be solved numerically for obtaining the estimates $\hat{\alpha}$ and $\hat{\beta}$.

(b)Maximum Likelihood estimators for the unknown reparametrized parameters of the Beta-Binomial distribution when $n \geq 2$.

Lee and Lio (1999) estimated the unknown reparametrized parameters (π,θ) of the Beta-Binomial distribution when $n \geq 2$, the likelihood function is given by

$$L(\alpha,\beta) = \prod_{x=0}^{n} p(x;n,\alpha,\beta)^{f_x}$$

they assumed that $\alpha = \frac{\pi}{\theta} - \pi$, $\beta = (\frac{1}{\theta} - 1).(1 - \pi)$, $A_i = (1-\pi)(1-\theta) + i\,\theta$, $B_i = \pi(1-\theta) + i\,\theta$ and $C_i = (1-\theta) + i\,\theta$, the reparametrized likelihood function of π and θ is

$$L(\pi,\theta/n,\{f_x\}) = (\prod_{i=1}^{n-1}(\frac{A_i}{C_i})^{f_0}) \times (\prod_{i=1}^{n-1}(((\prod_{i=0}^{x-1}B_i)(\prod_{i=0}^{n-x-1}A_i))/(\prod_{i=0}^{n-1}C_i))^{f_x}) \cdot (\prod_{i=1}^{n-1}(\frac{B_i}{C_i})^{f_n}) \quad (2.4)$$

Taking the first derivatives of the natural logarithm of the likelihood function (2.4) with respect to π and θ, equating the results to zero and solving these results numerically, the estimates $\hat{\pi}$ and $\hat{\theta}$ can be obtained.

3. Numerical Illustration and Sampling Distribution for the Maximum Likelihood estimators.

A numerical illustration for the maximum likelihood estimators for the Beta-Binomial distribution using Binomial and Poisson approximation will be obtained. MATHCAD program is used to evaluate the ML estimators and to obtain Pearson Coefficient and Pearson family of distribution. A trail will be made to obtain the best fitted distribution to maximum likelihood estimators using STATGRAPHICS package.

(a) Binomial and Poisson Approximation

The Beta-Binomial distribution is a combination of Binomial distribution with probability of Success P .Suppose P is a random variable that follows Beta distribution with shape parameters α and β, respectively, the Beta-Binomial distribution may be obtained from a Binomial distribution. Teerapabolarn (2008) obtained an upper bound on Binomial approximation to the Beta-Binomial distribution when $P_{\alpha,\beta} = \frac{\alpha}{\alpha+\beta}$ as follows:

$$Bin_{\alpha,\beta} = (1 - P(\alpha,\beta)^{n-1} - q(\alpha,\beta)^{n-1})\frac{n(n-1)}{(n+1)(1+\alpha+\beta)} \quad (3.1)$$

and showed that the results gave a good Binomial approximation to the Beta-Binomial distribution if $P_{\alpha,\beta} = \frac{\alpha}{\alpha+\beta}$ or $\frac{n}{\beta}$ is small.

Teerapabolarn (2009) presented a Poisson approximation to the Beta-Binomial distribution. He concluded that if $\frac{\alpha}{\alpha+\beta}$ is small, then the Binomial distribution with distribution parameters n and

$P_{\alpha,\beta} = \frac{\alpha}{\alpha+\beta}$ can be approximated by the Poisson distribution with mean

$\lambda_{(\alpha,\beta,n)} = n\,P_{\alpha,\beta} = \frac{n\alpha}{\alpha+\beta}$.

He obtained a different upper bound on Poisson approximation to the Beta-Binomial distribution as follows.

- If $\alpha \geq (n-1)$ and $\lambda = \frac{n\alpha}{\alpha+\beta}$, the upper bound on Poisson approximation to the Beta-Binomial distribution will be

$$Pois_{\alpha,\beta} = (1-e^{-\lambda(\alpha,\beta,n)})\frac{(\alpha+1)(\alpha+\beta)-n\beta}{(\alpha+\beta)(1+\alpha+\beta)} . \tag{3.2}$$

- If $\alpha = (n-1)$ and $\lambda = \frac{n\alpha}{\alpha+\beta}$, the upper bound on Poisson approximation to the Beta-Binomial distribution will be

$$Pois_{\alpha,\beta} = (1-e^{-\lambda(\alpha,\beta,n)})\frac{n\alpha}{(\alpha+\beta)(1+\alpha+\beta)} .$$

- A result of this approximation, without the condition $\alpha \geq (n-1)$, can be obtained by the following fact

$$Pois_{\alpha,\beta} = (1-(\frac{\alpha}{\alpha+\beta})^{n+1}-(\frac{\beta}{\alpha+\beta})^{n+1})\frac{n-1}{(1+\alpha+\beta)}(1-e^{-\lambda(\alpha,\beta,n)})\frac{\alpha}{(\alpha+\beta)},$$

which gives a good approximation if $\frac{n-1}{(1+\alpha+\beta)}$ and $\frac{\alpha}{(\alpha+\beta)}$ are small.

A MATHCAD program (2) is written to evaluate (3.1) and (3.2) for $n = 5,10$, $\alpha = 2...10$ and $\beta = 2...10$, for Poisson approximate α must be greater than $n-1$. The following relative values between the upper bound on Binomial approximation and the upper bound on Poisson approximation

$$R_{\alpha,\beta} = \frac{Bin_{\alpha,\beta}}{Pois_{\alpha,\beta}}$$

are obtained, to show which one is the best approximation, all results are listed in program (2), from these results we conclude that

- If $n = 5$, most relative values are less than one, then the Binomial approximation gave more accurate results than Poisson approximation.
- If $n = 10$, most relative values are more than one, then the Poisson approximation gave more accurate results than the Binomial approximation.

(b) Maximum Likelihood Illustration and suggested Pearson Family of distributions

A numerical illustration for the maximum likelihood estimators for the Beta-Binomial distribution using Binomial and Poisson approximation will be obtained. MATHCAD program 3 and 4 are used to evaluate the ML estimators using equations (1.3) and (2.1) respectively using the following steps

Step 1: For $\alpha = 30$, 35 and $\beta = 65$, 70, 100, random samples of size $n = 5$, 10 and 15 form approximation Beta-Binomial distribution are generated, that is, for Binomial approximation with parameters $n' = 100$ and $p = \dfrac{\alpha}{\alpha + \beta}$ and Poisson approximation with parameters $n' = 100$ and $\lambda = \dfrac{n\alpha}{\alpha + \beta}$.

Step 2: A Chi-square goodness-of-fit test for every generated sample (Binomial, Poisson) approximation, is carried out using the following hypotheses

H_0: Data follows the Beta-Binomial Distribution.

H_1: Data does not follow the Beta-Binomial Distribution.

Step 3: Using $\alpha = 0.05$, if P-values for χ^2 tests > 0.05, do not reject H_0 and go further for obtaining maximum likelihood estimates. If P-values for χ^2 tests < 0.05, stop and generate another set of random numbers.

Step 4: Repeated (1), (2) and (3) 100 times and compute square root of mean square error (MSE), Skewness β_1, Kurtosis β_2, Pearson coefficient K_p and Pearson type for maximum likelihood estimators.

For different n, α and β all results are displayed in tables 1 and 2, and we conclude the following results

1. For $\alpha = 30, 35$ and $\beta = 65, 70$ maximum likelihood estimators have the minimum $\sqrt{\text{MSE}}$ for most sample sizes.

2. As the sample size increases, $\sqrt{\text{MSE}}$ of the estimated parameters decreases.

3. Generally, we observe that the sampling distribution of the maximum likelihood estimators $\hat{\alpha}$ and $\hat{\beta}$ have Pearson type I distribution with the following form

$$f(x) = G(x - a_1)^{m_1}(a_2 - x)^{m_2}$$

where

$$m_1 = \frac{a + a_1}{c_2(a_2 - a_1)} \quad , \quad m_2 = -\frac{a + a_2}{c_2(a_2 - a_1)}$$

for both $(x - a_1)$ and $(a_2 - x)$ to be positive it must to be $a_1 < x < a_2$. Where a_1 and a_2 is the root of equation (2-14) with $a_1 < 0 < a_2$, $a_1 = c_1$ from equation (2.15) and G is a constant, for different samples size n.

(c) Best Fitted Distribution using Statgraphics package.

Using STATGRAPHICS package and generated random numbers in section 3, the fitted distributions for the maximum likelihood estimators $\hat{\alpha}$ and $\hat{\beta}$ are obtained, the fitted distributions will be tested using Kolmogrov Smirnov and Chi-square goodness-of-fit tests to determine the best fit. Goodness-of-fit test for the maximum likelihood estimators for the Beta-Binomial distribution are obtained using Binomial and Poisson approximation for different n, α and β. We conclude that the maximum likelihood estimators of Beta Binomial distribution using Poisson and Binomial approximation have normally densities when $n = 5$, 10 and 15.

STATGRAPHICS package is used to obtain sampling distribution for the maximum likelihood estimators, all results are listed in table 3 and 4. From these tables we conclude that the sampling distributions for the maximum likelihood estimators of the Beta-Binomial distribution have normal densities.

Table (1): Maximum likelihood estimates, standard deviation Square root of mean square error (MSE), Skewness β_1, Kurtosis β_2, Pearson coefficient and Pearson type, for Beta-Binomial distribution by using Binomial approximation with $\alpha = 30{,}35$, $\beta = 65{,}70$ and different samples size $n = 5{,}10$ and 15. Number of repetitions $N = 100$.

Binomial Parameters				ML estimates		Square root of MSE	
n	α	β	p	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
5	35	65	0.35	37.302	65.387	2.937	1.458
	35	70	0.333	35.519	66.908	2.115	3.752
	30	65	0.316	32.876	66.767	3.902	2.934
	30	70	0.3	33.171	69.392	3.463	2.179
10	35	65	0.35	31.886	69.055	3.224	4.112
	35	70	0.333	32.289	69.774	3.226	1.69
	30	65	0.316	29.055	66.961	2.642	4.127
	30	70	0.3	29.296	71.328	1.762	1.866
15	35	65	0.35	33.015	66.615	3.178	2.227
	35	70	0.333	33.667	68.141	2.823	2.832
	30	65	0.316	30.852	67.031	1.422	2.337
	30	70	0.3	30.565	68.131	1.602	2.496

Table (1) (continued): Maximum likelihood estimates, standard deviation Square root of mean square error (MSE), Skewness β_1, Kurtosis β_2, Pearson coefficient and Pearson type, for Beta-Binomial distribution by using Binomial approximation with $\alpha = 30{,}35$, $\beta = 65{,}70$ and different samples size $n = 5{,}10$ and 15. Number of repetitions $N = 100$.

Skewness β_1		**Kurtosis** β_2		**Pearson coefficient**		**Pearson type**	
$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
0.936	0.407	3.12	2.731	-0.3529	-0.19581	I	I
0.074	.00002	2.521	2.589	-0.04861	-0.00002	I	I
.0001	0.866	3.238	3.6	0.00019	-0.57181	IV	I
0.224	1.016	7.413	3.929	0.02568	-0.8086	IV	I
0.468	0.116	6.871	3.069	0.06891	-0.42501	IV	I
5.378	8.322	8.039	12.4	-1.68865	-3.24854	I	I
1.57	0.099	4.447	1.465	-0.91662	-0.02644	I	I
0.123	0.141	5.897	4.195	0.01935	0.05692	IV	IV
1.29	5.583	3.07	8.922	-0.37875	-2.13542	I	I
0.332	2.867	2.164	4.976	-0.10825	-0.8676	I	I
0.21	0.505	3.438	4.88	0.68001	0.1939	IV	IV
2.269	2.615	6.07	5.077	-4.00912	-0.9269	I	I

Table (2): Maximum likelihood estimates, standard deviation, Square root of mean square error (MSE), Skewness β_1, Kurtosis β_2, Pearson coefficient and Pearson type, for Beta-Binomial distribution by using Binomial approximation with $\alpha = 30,35$, $\beta = 65,70$ and different samples size $n = 5,10$ and 15. Number of repetitions $N = 100$.

Poisson Parameters				**ML estimates**		**Square root of MSE**	
n	α	β	λ	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
5	35	65	1.75	35.655	65.894	2.664	3.062
	35	70	1.667	33.742	69.657	2.698	2.516
	30	65	1.579	32.276	68.643	3.56	4.749
	30	70	1.5	31.549	71.232	2.827	2.569
10	35	65	3.5	34.932	66.198	0.643	1.291
	35	70	3.333	33.551	68.254	2.182	2.725
	30	65	3.158	30.499	66.587	2.437	3.574
	30	70	3	29.725	70.529	3.059	3.019
15	35	65	5.25	33.800	64.729	2.095	1.727
	35	70	5	34.280	70.374	1.474	2.485
	30	65	4.737	31.732	65.825	2.304	1.42
	30	70	4.5	29.314	69.249	2.428	1.865

Table (2) (continued): Maximum likelihood estimates, standard deviation, Square root of mean square error (MSE), Skewness β_1, Kurtosis β_2, Pearson coefficient and Pearson type, for Beta-Binomial distribution by using Binomial approximation with $\alpha = 30,35$, $\beta = 65,70$ and different samples size $n = 5,10$ and 15. Number of repetitions $N = 100$.

Skewness β_1		**Kurtosis** β_2		**Pearson coefficient**		**Pearson type**	
$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
.0006	.006	2.067	1.804	-0.00025	-0.002	I	I
.0043	.00003	2.532	2.558	-0.0035	-0.00029	I	I
.0001	0.513	2.602	2.685	-0.0001	-0.208	I	I
.0001	0.105	2.533	3.253	-0.002	0.42574	I	IV
8.938	11.421	12.59	17.16	-3.024	-5.692	I	I
1.072	1.152	5.373	6.625	0.673	0.305	IV	IV
.0004	0.05	2.063	1.767	-0.002	-0.016	I	I
1.409	1.522	5.497	4.496	1.863	-1.011	VI	I
3.342	4.623	19.10	12.38	0.277	1.568	IV	VI
.0009	0.72	4.534	3.231	0.00024	-0.383	IV	I
0.782	.0047	3.613	5.05	-0.631	0.001	I	IV
0.163	.0073	2.398	2.351	-0.005	-0.073	I	I

Table (3): Sampling distribution for the maximum likelihood estimators for Beta-Binomial distribution using Binomial approximation with $\alpha = 30, 35,\ \beta = 65, 70$ and different samples size $n = 5$, 10 and 15.

Binomial Parameters				Binomial approximation			
				$\hat{\alpha}$		$\hat{\beta}$	
n	α	β	p	**Distribution under null hypothesis**	**P- value**	**Distribution under null hypothesis**	**P- value**
5	35	65	0.35	Weibull	0.158196	Weibull (3-Parameter)	0.172185
	35	70	0.333	Normal	0.477264	Normal	0.9086133
	30	65	0.316	Normal	0.688424	Normal	0.5417574
	30	70	0.3	Normal	0. 06347	Normal	0.0598431
10	35	65	0.35	Normal	0.052811	Normal	0.0856961
	35	70	0.333	Cauchy	0.666474	Loglogistic (3-Parameter)	0.415679
	30	65	0.316	Weibull (3-Parameter)	0.580657	Loglogistic (3-Parameter)	0.305599
	30	70	0.3	Normal	0.280671	Normal	0.1016224
15	35	65	0.35	Lognormal (3-Parameter)	0.217899	Cauchy	0.453673
	35	70	0.333	Lognormal (3-Parameter)	0.22274	Lognormal (3-Parameter)	0.198876
	30	65	0.316	Normal	0.129433	Normal	0.3566183
	30	70	0.3	Cauchy	0.522177	Lognormal (3-Parameter)	0.257388

Table (4): Sampling distribution for the maximum likelihood estimators for Beta-Binomial distribution using Poisson approximation with $\alpha = 30, 35,\ \beta = 65, 70$ and different samples size $n = 5$, 10 and 15.

Poisson Parameters				**Poisson approximation**			
				$\hat{\alpha}$		$\hat{\beta}$	
n	α	β	λ	**Distribution under null hypothesis**	**P- value**	**Distribution under null hypothesis**	**P- value**
5	35	65	1.75	**Distribution under null hypothesis**	0.4147891	**Distribution under null hypothesis**	0.1701621
	35	70	1.667	Normal	0.4061171	Normal	0.6529034
	30	65	1.579	Normal	0.9982694	Normal	0.2269423
	30	70	1.5	Normal	0.9128981	Normal	0.7373810
10	35	65	3.5	Normal	0.132561	Normal	0.279345
	35	70	3.333	Normal	0.0873657	Normal	0.100649
	30	65	3.158	Lognormal (3-Parameter)	0.7790992	Loglogistic (3-Parameter)	0.3294981
	30	70	3	Normal	0.3398363	Normal	0.1496264
15	35	65	5.25	Normal	0.0417944	Normal	0.1259748
	35	70	5	Normal	0.2266425	Normal	0.2195759
	30	65	4.737	Normal	0.2089496	Normal	0.7784910
	30	70	4.5	Normal	0.7242227	Normal	0.4418723

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